Chin. Ann. Math.
24B:1(2003),35-44.

ON STABILITY AND REGULARIZATION FOR BACKWARD HEAT EQUATION***

LIU Jijun* LUO Dingjun**

Abstract

Consider a backward heat equation in a bounded domain $\Omega \subset R^2$ with the noisy data in the initial time geometry. The aim is to find the temperature for $0 < \varepsilon < t < T$. For this ill-posed problem, the authors give a continuous dependence estimate of the solution. Moreover, the convergence rate of the approximate solution is also given.

Keywords Heat equation, Inverse problem, Stability, Regularization
2000 MR Subject Classification 35R25, 35R30
Chinese Library Classification O175.26, O175.29 **Document Code** A
Article ID 0252-9599(2003)01-0035-10

§1. Introduction

Consider a backward heat equation in a bounded domain $\Omega \subset R^2$ with piece wise smooth boundary $\partial\Omega$. We assume there is neither heat source nor heat sink within the homogeneous media, and the boundary condition on $\partial\Omega$ is of the Robin type. If the initial temperature at time $t = 0$ is also given, then the temperature field $u(x,t)$ in $0 < t < T$ is governed by

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = -\Delta u, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial u(x,t)}{\partial \nu(x)} + h(x)u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where both the nonnegative boundary impedance $h(x)$ and the outward normal direction $\nu(x)$ on $\partial\Omega \times (0,T)$ are time independent. For some initial function $u_0(x)$, there exists a unique $u(x,t)$ in $0 < t < T$ for this backward system. It is well-known that the above system is ill-posed. That is, the solution may not exist provided that there are errors in the initial time geometry. By initial time geometry errors we mean the following ones.

Manuscript received January 14, 2002.

*Department of Mathematics, Nanjing Normal University, Nanjing 210097, China and Department of Mathematics, Southeast University, Nanjing 210096, China.
E-mail: jjliu@ seu.edu.cn
**Department of Mathematics, Nanjing Normal University, Nanjing 210097, China.
E-mail djluo@ njnu.edu.cn

***Project supported by China Postdoctoral Science Foundation (No.2002031224) and the Science Foundation of Southeast University (No. 9207011148).

Firstly, the initial data may not be given at the exact initial time $t = 0$, but at some initial curve $t = \varepsilon f(x)$ for small $\varepsilon > 0$ and $|f(x)| \le 1$. In this case, we should solve $v(x,t)$ in $\{(x,t) : \varepsilon f(x) \le t \le T, x \in \Omega\}$ by

$$\begin{cases} \frac{\partial v(x,t)}{\partial t} = -\Delta v, & (x,t) \in \Omega \times (\varepsilon f(x), T), \\ \frac{\partial v(x,t)}{\partial \nu(x)} + h(x)v(x,t) = 0, & (x,t) \in \partial\Omega \times (\varepsilon f(x)|_{\partial\Omega}, T), \\ v(x, \varepsilon f(x)) = v_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

For initial value $v_0(x)$ given appropriately, we assume that there exists a unique solution to this problem. However, if we are given two initial values $u_0(x)$ and $v_0(x)$ with a priori error

$$\|u_0 - v_0\|_{L^2(\Omega)} \le \beta(\varepsilon), \tag{1.3}$$

where $\beta(\varepsilon) \to 0$ as $\varepsilon \to 0$, then one interesting problem is

(A) What is the error of $u(x,t) - v(x,t)$ in $\{(x,t) : \varepsilon f(x) \le t \le T, x \in \Omega\}$?

Secondly, the initial data $v_0(x)$ are generally obtained by measurement, hence we can only get the approximate data $\hat{v}_0(x)$ up to an error level

$$\|v_0(\cdot) - \hat{v}_0(\cdot)\|_{L^2(\Omega)} \le \delta. \tag{1.4}$$

In this case another problem appears, namely,

(B) How can we get the approximation of $v(x,t)$ from the noisy data $\hat{v}_0(x)$?

It is well-known that the following backward heat problem

$$\begin{cases} \frac{\partial v(x,t)}{\partial t} = -\Delta v, & (x,t) \in \Omega \times (\varepsilon f(x), T), \\ \frac{\partial v(x,t)}{\partial \nu(x)} + h(x)v(x,t) = 0, & (x,t) \in \partial\Omega \times (\varepsilon f(x)|_{\partial\Omega}, T), \\ v(x, \varepsilon f(x)) = \hat{v}_0(x), & x \in \Omega, \end{cases} \tag{1.5}$$

does not have classical solution generally, so we can not get the approximation of $v(x,t)$ from this problem directly. We must seek some regularizing solution from the noisy data $\hat{v}_0(x)$. Also, in this case, the construction of the regularizing scheme such as the choice of regularizing parameter, together with the convergence rate of the regularizing solution as $\delta \to 0$, should be studied. It has been found that the conditional stability can be applied to construct the regularizing scheme for the ill-posed problem[2,7], both in the numerical inversions and in the theoretical analysis. Furthermore, it seems that this method is more efficient than the filtering method proposed in [10], which is essentially to filter out the small singular value for the compact operator[5].

The backward heat problems have been studied for a long time. For some backgrounds and mathematical treatments on this topic, we refer to [1,3,8,9] and the references therein. However, the research seems far away from satisfactory.

Thus, our final problem is to get the approximation of $u(x,t)$ in $\{(x,t) : \varepsilon \le t \le T, x \in \Omega\}$ from the noisy data $\hat{v}_0(x)$ according to (1.1)–(1.4). This problem depends entirely on the solutions to problems (A) and (B).

In this paper, we establish the conditional stability firstly. Then, we give a new method to get the approximate regularizing solution to the backward problem with the error both at the initial time and at the data themselves. Finally, motivated by the new idea in [2], a criterion for the choice of the regularizing parameter α is proposed to this problem. This work generalizes the known research result on the backward heat problems in [4].

§2. Conditional Stability

For backward heat problem with initial date given at $t = \varepsilon f(x)$, if the date is appropriate,

then there exists a unique solution. Now we establish a continuous dependence for the solution, which is important for our regularizing method. We always assume $|f(x)| \leq 1$ and $f(x) = 0$ on $\partial\Omega$ in this paper.

Theorem 2.1. *Let $u(x,t)$ and $v(x,t)$ be the solutions to* (1.1) *and* (1.2) *respectively with the initial value satisfying* (1.3). *If $u(x,T)$ and $v(x,T)$ satisfy*

$$\|u(T)\| \leq M_0, \qquad \|v(T)\| \leq M_0 \tag{2.1}$$

for some known constant $M_0 > 0$, then it follows that

$$\|(u-v)(t)\| \leq 4M_0 \left[\|u(\varepsilon) - u_0\| + \|v(\varepsilon) - v_0\| + \beta(\varepsilon)\right]^{\frac{T-t}{T-\varepsilon}} \tag{2.2}$$

for any $t \in [\varepsilon, T]$, where $\|u(t)\| = \|u(\cdot,t)\|_{L^2(\Omega)}$.

Proof. Let $p(x,t) = u(x,t) - v(x,t)$ and define $F(t) = \|p(t)\|^2$ for any $t \in [\varepsilon, T]$. Then the convexity of function $\ln F(t)$ leads to

$$\|p(t)\| \leq \|p(\varepsilon)\|^{\frac{T-t}{T-\varepsilon}} \|p(T)\|^{\frac{t-\varepsilon}{T-\varepsilon}}. \tag{2.3}$$

On the other hand,

$$\|p(\varepsilon)\|^2 \leq 3\left[\|u(\varepsilon) - u_0\| + \|v(\varepsilon) - v_0\| + \beta(\varepsilon)\right]^2 \tag{2.4}$$

due to the triangle inequality and (1.3). Now (2.3) and (2.4) generate (2.2).

Remark 2.1. This result shows that it is possible for us to estimate $\|(u-v)(t)\|$ for $\varepsilon \leq t \leq T$ by ε provided $\|u(T)\|, \|v(T)\|$ are bounded. For this purpose, we need to bound $\|u(\varepsilon) - u_0\|$ and $\|v(\varepsilon) - v_0\|$ respectively, since $\beta(\varepsilon)$ is the known error between initial data. It is enough for us to estimate $\|v(\varepsilon) - v_0\|$.

Theorem 2.2. *If we assume that the final value of $v(x,t)$ satisfies*

$$\|v(T)\|_{H^3(\Omega)} \leq M \tag{2.5}$$

with some known constant M and $f(x)|_{\partial\Omega} = 0, |f(x)| \leq 1$ for the initial curve $t = \varepsilon f(x)$, then there exists a constant $C = C(M, T, \|h\|_{L^\infty(\partial\Omega)}) > 0$ such that

$$\|v(\varepsilon) - v_0\| \leq C\left[1 + \|\nabla f\|_{L^\infty(\Omega)} + \|v_0\|_{L^2(\partial\Omega)} + \|\nabla v_0\|_{L^2(\Omega)} + \frac{1}{\sqrt{T-\varepsilon}}\right]\varepsilon^{1/2}. \tag{2.6}$$

Remark 2.2. From our procedure, it is easy to see that (2.6) without $\|v_0\|_{L^2(\partial\Omega)}$ still holds for the Dirichlet boundary condition.

Proof. Since the backward problem for $v(x,t)$ with the given initial value $v_0(x)$ generates a final value $v(\cdot,T) \in H^3(\Omega)$, the correspondent direct heat problem with the initial value $v(x,T)$ defines a function $\tilde{v}(x,t) \in W_2^{1,0}(Q_T)$, where $Q_T = \{(x,t) : x \in \Omega, 0 \leq t \leq T\}$. Due to the unique solvability of the direct heat problem in $W_2^{1,0}(Q_T)$ (see [6, Charpt 4]), it follows that $v(x,t) = \tilde{v}(x,t)$ in $Q_T \setminus Q(f)$, where $Q(f) = \{(x,t) : 0 \leq t \leq \varepsilon f(x), x \in \Omega\}$. Furthermore, if we extend $v(x,t)$ to Q_T by defining $v(x,t) = \tilde{v}(x,t)$ in $Q(f)$, then $v(x,t) \in W_2^{1,0}(Q_T)$ satisfies

$$\begin{cases} \frac{\partial v(x,t)}{\partial t} = -\Delta v, & (x,t) \in \Omega \times (0,T), \\ \frac{\partial v(x,t)}{\partial \nu(x)} + h(x)v(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T), \\ v(x, \varepsilon f(x)) = v_0(x), & x \in \Omega \end{cases} \tag{2.7}$$

with $v(x,T)$ satisfying (2.5). So we get from $v(x,\varepsilon) - v_0(x) = \int_{\varepsilon f(x)}^{\varepsilon} \partial_s v(x,s) ds$ that

$$\|v(\varepsilon) - v_0\|^2 = -\int_{Q_\varepsilon \setminus Q(f)} [v(x,\varepsilon) - v_0(x)] \Delta v(x,s) ds dx. \tag{2.8}$$

Noticing $f(x) = 0$ on $\partial\Omega$ and integrating the second integral by parts with respect to x yields

$$\|v(\varepsilon) - v_0\|^2 = B_1 + B_2 + B_3, \tag{2.9}$$

where

$$B_1 = \int_0^\varepsilon ds \int_{\partial\Omega} [v(x,\varepsilon) - v_0(x)]h(x)v(x,s)dl(x),$$
$$B_2 = -\varepsilon \int_\Omega [v(x,\varepsilon) - v_0(x)] \bigtriangledown v(x,s)|_{s=\varepsilon f(x)} \cdot \bigtriangledown f(x)dx,$$
$$B_3 = \int_\Omega \bigtriangledown_x (v(x,\varepsilon) - v_0(x))\Big(\int_{\varepsilon f(x)}^\varepsilon \bigtriangledown_x v(x,s)ds\Big)dx.$$

Now the estimates on B_1, B_2, B_3 given in the following section complete the proof.

The estimate on $\|u(\varepsilon) - u_0\|$ can be obtained from Theorem 2.3 immediately by taking $f(x) = 0$ and $v_0(x) = u_0(x)$. That is,

Theorem 2.3. *If we assume*

$$\|u(T)\|_{H^3(\Omega)} \le M \tag{2.10}$$

for the final value of $u(x,t)$ with some known positive constant M, then there exists a constant $C > 0$ depending only on $M, T, \|h\|_{L^\infty(\partial\Omega)}$ such that

$$\|u(\varepsilon) - u_0\| \le C\Big[\|u_0\|_{L^2(\partial\Omega)} + \|\bigtriangledown u_0\|_{L^2(\Omega)} + \frac{1}{\sqrt{T-\varepsilon}} + 1\Big]\varepsilon^{1/2}. \tag{2.11}$$

Now we can give an answer to our problem A, based on the above theorems.

Theorem 2.4. *Let $|f(x)| \le 1$ for $x \in \Omega$ and $f(x) = 0$ on $\partial\Omega$. If $u(x,t)$ and $v(x,t)$ are the solutions of* (1.1) *and* (1.2) *satisfying* (2.5) *and* (2.10) *respectively, then there exists a constant $C = C(M, T, \|h\|) > 0$ such that*

$$\|(u-v)(t)\| \le C\left[(1 + \|\bigtriangledown f\|_{L^\infty})\varepsilon^{1/2} + \beta(\varepsilon)\right]^{\frac{T-t}{T-\varepsilon}}, \quad \varepsilon \le t \le T. \tag{2.12}$$

Proof. It follows from Theorems 2.1, 2.2, 2.3 that

$$\|(u-v)(t)\| \le C_0\Big[\Big(E(u_0, v_0, f) + \frac{2}{\sqrt{T-\varepsilon}} + 2\Big)\varepsilon^{1/2} + \beta(\varepsilon)\Big]^{\frac{T-t}{T-\varepsilon}}, \tag{2.13}$$

where

$$E(u_0, v_0, f) = \|\bigtriangledown u_0\|_{L^2(\Omega)} + \|u_0\|_{L^2(\partial\Omega)} + \|\bigtriangledown v_0\|_{L^2(\Omega)} + \|v_0\|_{L^2(\partial\Omega)} + \|\bigtriangledown f\|_{L^\infty}.$$

Since $u(x,T), v(x,T)$ satisfy the bounded condition (2.5) and (2.10), the application of semigroup theory to the direct heat equation about $u(x,t)$ and $v(x,t)$ says that there exists a constant $C > 0$ such that

$$\|u_0\|_{H^3(\Omega)}, \|v(\cdot,t)\|_{H^3(\Omega)}, \|v_t(\cdot,t)\|_{H^1(\Omega)} \le CM_0 \tag{2.14}$$

for all $t \in [0,T]$. Moreover, the Sobolev trace theorem implies that

$$\|u_0\|_{L^2(\partial\Omega)} \le CM. \tag{2.15}$$

Now we consider $v_0(x) = v(x, \varepsilon f(x))$. Since

$$\bigtriangledown v_0(x) - \bigtriangledown v(x,\varepsilon) = -\int_{\varepsilon f(x)}^\varepsilon \bigtriangledown \frac{\partial v(x,s)}{\partial s} ds + \int_{\varepsilon f(x)}^\varepsilon \frac{\partial v(x,s)}{\partial s} ds\varepsilon \bigtriangledown f(x),$$

we get

$$\frac{|\bigtriangledown v_0(x)|^2}{3} \le |\bigtriangledown v(x,\varepsilon)|^2 + T\int_0^T \Big|\bigtriangledown \frac{\partial v(x,s)}{\partial s}\Big|^2 ds + T\varepsilon^2 |\bigtriangledown f(x)|^2 \int_0^T \Big|\frac{\partial v(x,s)}{\partial s}\Big|^2 ds.$$

Integrating this inequality in Ω says from (2.14) that

$$\|\bigtriangledown v_0\|^2_{L^2(\Omega)} \le C(1+\|\bigtriangledown f\|^2_{L^\infty(\Omega)})M_0^2 \le C(1+\|\bigtriangledown f\|_{L^\infty(\Omega)})^2 M_0^2, \tag{2.16}$$

where we have used the bound of $\|\bigtriangledown v(\varepsilon)\|$ (see (3.4) in the next section). As for $\|v_0\|_{L^2(\partial\Omega)}$, it follows that

$$\|v_0\|^2_{L^2(\partial\Omega)} = \int_{\partial\Omega} v^2(x,\varepsilon f(x))dl(x) = \int_{\partial\Omega} v^2(x,0)dl(x) \tag{2.17}$$

due to $f(x)=0$ on $\partial\Omega$. Since $v(\cdot,0)\in H^3(\Omega)$, the Sobolev trace theorem tells us

$$\|v_0\|_{L^2(\partial\Omega)} \le C. \tag{2.18}$$

Inserting these estimates into $E(u_0,v_0,f)$ leads to (2.12) from (2.13) immediately.

§3. Estimates on B_1, B_2 and B_3

Firstly, we estimate B_3.

Theorem 3.1. *If $\|v(T)\| \le M$, then there exists a constant $C>0$ such that*

$$|B_3| \le \frac{1}{4}\|v(\varepsilon)-v_0\|^2 + C\Big(\frac{1}{T-\varepsilon} + \|\bigtriangledown v_0\|^2\Big)\varepsilon. \tag{3.1}$$

Proof. It is easy to see

$$\begin{aligned} B_3 &\le \sqrt{2\varepsilon}\Big[\int_\Omega |\bigtriangledown (v(x,\varepsilon)-v_0(x))|^2 dx\Big]^{1/2}\Big[\int_\Omega\int_{\varepsilon f(x)}^{\varepsilon} |\bigtriangledown v(x,s)|^2 ds dx\Big]^{1/2} \\ &\le 2\sqrt{\varepsilon}(\|\bigtriangledown v(\varepsilon)\| + \|\bigtriangledown v_0\|)\Big[\int_\Omega\int_{\varepsilon f(x)}^{\varepsilon} |\bigtriangledown_x v(x,s)|^2 ds dx\Big]^{1/2} \end{aligned} \tag{3.2}$$

due to $|f(x)|\le 1$. Define

$$J_1(\varepsilon) = \|\bigtriangledown v(\varepsilon)\|, \qquad J_2(\varepsilon) = \Big[\int_\Omega\int_{\varepsilon f(x)}^{\varepsilon} |\bigtriangledown_x v(x,s)|^2 ds dx\Big]^{1/2}.$$

It is necessary to estimate J_1 and J_2. From the divergence theorem, we get

$$J_1^2(\varepsilon) = -\int_\varepsilon^T \frac{\partial}{\partial s}\left[\frac{T-s}{T-\varepsilon}\int_\Omega |\bigtriangledown_x v(x,s)|^2 dx\right] ds = \sum_{j=1}^3 A_j(\varepsilon), \tag{3.3}$$

$$A_1(\varepsilon) = \frac{1}{T-\varepsilon}\int_\varepsilon^T \|\bigtriangledown_x v(s)\|^2 ds,$$

$$A_2(\varepsilon) = -2\int_\varepsilon^T \frac{T-s}{T-\varepsilon}\int_{\partial\Omega} \frac{\partial v(x,s)}{\partial \nu(x)}\frac{\partial v(x,s)}{\partial s} dx ds,$$

$$A_3(\varepsilon) = 2\int_\varepsilon^T \frac{T-s}{T-\varepsilon}\int_\Omega \Delta v(x,s)\frac{\partial v(x,s)}{\partial s} dx ds \le 0.$$

Now it follows from the boundary condition for $v(x,t)$ that

$$A_1(\varepsilon) = -\frac{1}{T-\varepsilon}\int_\varepsilon^T\int_{\partial\Omega} h(x)v^2(x,s)dx ds + \frac{1}{T-\varepsilon}\int_\varepsilon^T\int_\Omega v\frac{\partial v}{\partial s}dx ds,$$

$$A_2(\varepsilon) \le \frac{1}{T-\varepsilon}\int_\varepsilon^T\int_{\partial\Omega} h(x)w^2(x,s)dx ds.$$

Inserting the above expressions to (3.3) generates

$$J_1^2(\varepsilon) \le \frac{1}{T-\varepsilon}\int_\varepsilon^T \int_\Omega v\frac{\partial v}{\partial s}dxds \le \frac{1}{2(T-\varepsilon)}\int_\Omega v^2(x,T)dx,$$

therefore it follows from (2.5) that

$$J_1(\varepsilon) \le \frac{M}{\sqrt{2(T-\varepsilon)}}. \tag{3.4}$$

Now we estimate $J_2(\varepsilon)$. Firstly, rewrite $J_2^2(\varepsilon)$ as

$$J_2^2(\varepsilon) = \int_\Omega \int_{\varepsilon f(x)}^{\varepsilon} \bigtriangledown v \cdot \bigtriangledown (v - v_0)dsdx + \int_\Omega \int_{\varepsilon f(x)}^{\varepsilon} \bigtriangledown v \cdot \bigtriangledown v_0 dsdx = J_2^{(1)} + J_2^{(2)}. \tag{3.5}$$

Noticing $v(x, \varepsilon f(x)) = v_0(x)$, by exchanging the order of integration and integrating by parts with respect to x, we get

$$J_2^{(1)} = -\int_\Omega \int_{\varepsilon f(x)}^{\varepsilon} \Delta v(x,s)[v(x,s) - v_0(x)]dsdx = \frac{1}{2}\left\|v(\varepsilon) - v_0\right\|^2.$$

However, it follows that $J_2^{(2)} \le [2\varepsilon \left\|\bigtriangledown v_0\right\|^2 + J_2^2(\varepsilon)]/2$ from the Cauchy-Schwartz inequality due to $|f(x)| \le 1$, so we get from (3.5) that

$$J_2(\varepsilon) \le \left\|v(\varepsilon) - v_0\right\| + 2\sqrt{\varepsilon}\left\|\bigtriangledown v_0\right\|. \tag{3.6}$$

Now inserting (3.4) and (3.6) into (3.2) says

$$\begin{aligned} B_3 \le{}& \frac{1}{4}\left\|v(\varepsilon) - v_0\right\|^2 + 4\varepsilon\Big(\frac{M^2}{2(T-\varepsilon)} + \left\|\bigtriangledown v_0\right\|^2\Big) \\ &+ 2\sqrt{2}\varepsilon\left\|\bigtriangledown v_0\right\|^2 + \sqrt{2}\varepsilon\Big(\frac{M^2}{2(T-\varepsilon)} + \left\|\bigtriangledown v_0\right\|^2\Big), \end{aligned}$$

which completes the proof of Theorem 3.1 immediately.

For the estimates on B_1 and B_2, we need the following result[6].

Lemma 3.1. *Let $\phi(x) \in W_2^1(\Omega)$. Then there exists a constant $C = C(\Omega) > 0$ such that*

$$\int_{\partial\Omega} \phi^2(x)dx \le \int_\Omega \Big[\gamma|\bigtriangledown \phi(x)|^2 + C(\Omega)\Big(\frac{C(\Omega)}{4\gamma} + 1\Big)\phi^2(x)\Big]dx, \quad \forall \gamma > 0. \tag{3.7}$$

For simplicity, we will set $C(\Omega) = 1$ in the sequel.

Theorem 3.2. *There exists a constant $C_1 = C_1(M, \Omega, \|h\|_{L^\infty(\partial\Omega)}, T) > 0$ such that*

$$|B_1| \le C_1\Big(\int_{\partial\Omega} v_0^2 dl(x) + \int_\Omega |\bigtriangledown v_0|^2 dx + \frac{1}{T-\varepsilon} + 1\Big)\varepsilon + \frac{1}{4}\left\|v(\varepsilon) - v_0\right\|^2. \tag{3.8}$$

Proof. Let $\|h\|_{L^\infty(\partial\Omega)}^2 \le C_0$. It follows from the expression of B_1 that

$$\begin{aligned} |B_1| &\le \gamma_1\varepsilon\int_{\partial\Omega}[v(x,\varepsilon) - v_0(x)]^2 dl(x) + \frac{C_0}{4\gamma_1}\int_0^\varepsilon ds\int_{\partial\Omega} v^2(x,s)dl(x) \\ &= B_{11}(\gamma_1) + B_{12}(\gamma_1). \end{aligned} \tag{3.9}$$

For the first term, it follows from Lemma 3.1 that

$$\begin{aligned} B_{11}(\gamma_1) &\le \gamma_1\varepsilon\Big[\varepsilon\int_\Omega |\bigtriangledown (v(x,\varepsilon) - v_0(x))|^2 dx + \frac{1+4\varepsilon}{4\varepsilon}\int_\Omega (v(x,\varepsilon) - v_0(x))^2 dx\Big] \\ &\le 2\gamma_1\varepsilon^2(\left\|\bigtriangledown v(\varepsilon)\right\|^2 + \left\|\bigtriangledown v_0\right\|^2) + \gamma_1\Big(\frac{1}{4} + T\Big)\left\|v(\varepsilon) - v_0\right\|^2 \end{aligned} \tag{3.10}$$

with any constant $\gamma_1 > 0$. For the second term, it yields

$$\Big(\frac{C_0}{4\gamma_1}\Big)^{-1} B_{12}(\gamma_1) \le \gamma_2 \int_0^\varepsilon ds \int_\Omega |\bigtriangledown v(x,s)|^2 dx + \Big(\frac{1}{4\gamma_2}+1\Big)\int_0^\varepsilon ds \int_\Omega v^2(x,s)dx \tag{3.11}$$

with any constant $\gamma_2 > 0$. On one hand, since $v(x,t)$ solves (2.7), it follows from integrating by parts that $\int_\Omega v^2(x,s)dx$ is increasing for $s \in (0,T)$, which implies

$$\int_0^\varepsilon ds \int_\Omega v^2(x,s)dx \le M^2\varepsilon. \tag{3.12}$$

On the other hand, integrating by parts leads to

$$\begin{aligned}
&\int_0^\varepsilon ds \int_\Omega |\bigtriangledown v(x,s)|^2 dx = \int_0^\varepsilon ds \int_\Omega \bigtriangledown(v-v_0)\cdot\bigtriangledown v dx + \int_0^\varepsilon ds \int_\Omega \bigtriangledown v\cdot\bigtriangledown v_0 dx\\
&= -\int_0^\varepsilon ds \int_{\partial\Omega} [v(x,s)-v_0(x)]h(x)v(x,s)dl(x)\\
&\quad + \int_0^\varepsilon ds \int_\Omega [v(x,s)-v_0(x)]\frac{\partial v(x,s)}{\partial s}dx + \int_0^\varepsilon ds \int_\Omega \bigtriangledown v(x,s)\cdot\bigtriangledown v_0(x)dx\\
&\le \gamma_3 \int_0^\varepsilon ds \int_{\partial\Omega} v^2(x,s)dl(x) + \frac{1}{4\gamma_3}\int_0^\varepsilon ds \int_{\partial\Omega} h^2(x)v_0^2(x)dl(x)\\
&\quad + \frac{1}{2}\int_\Omega [v(x,\varepsilon)-v_0(x)]^2 dx + \int_0^\varepsilon ds \int_\Omega \bigtriangledown v(x,s)\cdot\bigtriangledown v_0(x)dx\\
&\le \gamma_3 \int_0^\varepsilon ds \int_{\partial\Omega} v^2(x,s)dl(x) + \frac{C_0\varepsilon}{4\gamma_3}\int_{\partial\Omega} v_0^2(x)dl(x) + \frac{1}{2}\|v(\varepsilon)-g\|^2\\
&\quad + \gamma_4 \int_0^\varepsilon ds \int_\Omega |\bigtriangledown v(x,s)|^2 dx + \frac{1}{4\gamma_4}\int_0^\varepsilon ds \int_\Omega |\bigtriangledown v_0(x)|^2 dx\\
&\le \gamma_3 \int_0^\varepsilon ds \int_{\partial\Omega} v^2(x,s)dl(x) + \gamma_4 \int_0^\varepsilon ds \int_\Omega |\bigtriangledown v(x,s)|^2 dx\\
&\quad + \frac{1}{2}\|v(\varepsilon)-v_0\|^2 + C(\gamma_3,\gamma_4,v_0)\varepsilon
\end{aligned}$$

with the function

$$C(\gamma_3,\gamma_4,v_0) = \frac{C_0}{4\gamma_3}\int_{\partial\Omega} v_0^2(x)dl(x) + \frac{1}{4\gamma_4}\int_\Omega |\bigtriangledown v_0(x)|^2 dx$$

and constants $\gamma_3, \gamma_4 > 0$. Now taking

$$0 < \gamma_4 < 1 \tag{3.13}$$

in this estimate leads to

$$\int_0^\varepsilon ds \int_\Omega |\bigtriangledown v(x,s)|^2 dx \le \frac{\gamma_3}{1-\gamma_4}\frac{4\gamma_1}{C_0}B_{12}(\gamma_1) + \frac{\|v(\varepsilon)-v_0\|^2}{2(1-\gamma_4)} + \frac{C(\gamma_3,\gamma_4,v_0)}{1-\gamma_4}\varepsilon.$$

Inserting (3.12) and this estimate to (3.11) and choosing $\gamma_2, \gamma_3, \gamma_4$ such that

$$1 - \frac{\gamma_2\gamma_3}{1-\gamma_4} > 0, \tag{3.14}$$

then we get

$$B_{12}(\gamma_1) \le \frac{C_0}{4\gamma_1}\Big[\frac{\gamma_2 C(\gamma_3,\gamma_4,v_0)}{1-\gamma_4-\gamma_2\gamma_3}\varepsilon + \frac{\gamma_2\|v(\varepsilon)-v_0\|^2}{2(1-\gamma_4-\gamma_2\gamma_3)} + \frac{1+4\gamma_2}{4\gamma_2}M^2\varepsilon\Big]. \tag{3.15}$$

Now inserting (3.10) and (3.15) into (3.9) leads to

$$|B_1| \le \Big[\frac{C_0}{4\gamma_1}\Big(\frac{\gamma_2 C(\gamma_3,\gamma_4,v_0)}{1-\gamma_4-\gamma_2\gamma_3}+\frac{1+4\gamma_2}{4\gamma_2}M^2\Big)+2\gamma_1 T(\|\bigtriangledown v(\varepsilon)\|^2+\|\bigtriangledown v_0\|^2)\Big]\varepsilon+$$
$$+\Big[\gamma_1\Big(\frac{1}{4}+T\Big)+\frac{C_0\gamma_2}{8\gamma_1(1-\gamma_4-\gamma_2\gamma_3)}\Big]\|v(\varepsilon)-v_0\|^2.$$

Now applying (3.4) we get

$$|B_1| \le C(\gamma_j,v_0,\varepsilon)\varepsilon+\Big[\gamma_1\Big(\frac{1}{4}+T\Big)+\frac{C_0\gamma_2}{8\gamma_1(1-\gamma_4-\gamma_2\gamma_3)}\Big]\|v(\varepsilon)-v_0\|^2, \tag{3.16}$$

where

$$C(\gamma_j,v_0,\varepsilon)=\frac{C_0}{4\gamma_1}\Big(\frac{\gamma_2 C(\gamma_3,\gamma_4,v_0)}{1-\gamma_4-\gamma_2\gamma_3}+\frac{1+4\gamma_2}{4\gamma_2}M^2\Big)+2\gamma_1 T\Big(\frac{M^2}{2(T-\varepsilon)}+\|\bigtriangledown v_0\|^2\Big).$$

Notice that $C(\gamma_j,v_0,\varepsilon)$ is bounded uniformly as $\varepsilon\to 0$ from this expression. Now we can take $\gamma_1>0,\gamma_2>0$ such that

$$\gamma_1\Big(\frac{1}{4}+T\Big)\le\frac{1}{8},\quad \frac{C_0\gamma_2}{8\gamma_1(1-\gamma_4-\gamma_2\gamma_3)}\le\frac{1}{8}. \tag{3.17}$$

Then (3.16) leads to (3.8) immediately.

Theorem 3.3. *If the final value $v(x,T)$ satisfies*

$$\|v(T)\|_{H^3(\Omega)}\le M, \tag{3.18}$$

then there exists a constant $C_2>0$ such that

$$|B_2|\le\frac{1}{4}\|v(\varepsilon)-v_0\|^2+C_2\|\bigtriangledown f\|^2_{L^\infty(\Omega)}\varepsilon. \tag{3.19}$$

Proof. From the expression of B_2 and Lemma 3.1, we get

$$|B_2|\le\eta_1\|v(\varepsilon)-v_0\|^2+\frac{\varepsilon^2}{4\eta_1}\int_\Omega\big|\bigtriangledown v(x,s)_{s=\varepsilon f(x)}\cdot\bigtriangledown f(x)\big|^2dx. \tag{3.20}$$

On the other hand,

$$\bigtriangledown v(x,s)\big|_{s=\varepsilon f(x)}-\bigtriangledown v(x,s)\,|_{s=\varepsilon}=-\int_{\varepsilon f(x)}^{\varepsilon}\partial_s(\bigtriangledown v(x,s))ds$$

tells us

$$\begin{aligned}&\big|\bigtriangledown v(x,s)_{s=\varepsilon f(x)}\cdot\bigtriangledown f(x)\big|^2\\ &\le\Big(|\bigtriangledown v(x,s)_{s=\varepsilon}\cdot\bigtriangledown f(x)|+|\bigtriangledown f(x)|\int_{\varepsilon f(x)}^{\varepsilon}|\partial_s\bigtriangledown v(x,s)|ds\Big)^2\\ &\le 2\Big(|\bigtriangledown v(x,s)_{s=\varepsilon}\cdot\bigtriangledown f(x)|^2+2|\bigtriangledown f(x)|^2\int_{\varepsilon f(x)}^{\varepsilon}|\bigtriangledown(\partial_s v(x,s))|^2ds\Big),\end{aligned}$$

which implies that

$$\begin{aligned}&\int_\Omega\big|\bigtriangledown v(x,s)_{s=\varepsilon f(x)}\cdot\bigtriangledown f(x)\big|^2dx\\ &\le\|\bigtriangledown f\|^2_{L^\infty(\Omega)}\Big[\int_\Omega|\bigtriangledown v(x,\varepsilon)|^2dx+2\int_0^T ds\int_\Omega|\bigtriangledown(\partial_s v(x,s))|^2dx\Big]\\ &\le\|\bigtriangledown f\|^2_{L^\infty(\Omega)}\Big[\frac{M^2}{2(T-\varepsilon)}+\int_0^T\|v_s(\cdot,s)\|^2_{H^1(\Omega)}ds\Big]\le C\|\bigtriangledown f\|^2_{L^\infty(\Omega)}\end{aligned} \tag{3.21}$$

from (2.14) and (3.4). Inserting this estimate into (3.20) and taking $\eta_1=\frac{1}{4}$ lead to (3.19) immediately.

§4. Regularization Method

This section is devoted to the solution of our problem (B), i.e, how to get the approximate solution from the noisy data $\hat{v}_0(x)$. For the exact initial data $v_0(x)$ given at the curve $t = \varepsilon f(x)$, we assume that there exists a unique solution $v(x,t)$ in $\{(x,t) : \varepsilon f(x) \le t \le T, x \in \Omega\}$ to (1.2). We also assume this solution satisfies

$$\|v(\cdot,T)\|_{H^3(\Omega)} \le M. \tag{4.1}$$

Noticing (2.16) and (2.18), we see that the following is a direct result of Theorem 2.3.

Corollary 4.1. *If we assume that the final value of $v(x,t)$ satisfies*

$$\|v(T)\|_{H^3(\Omega)} \le M \tag{4.2}$$

with some known constant M and $f(x)|_{\partial\Omega} = 0, |f(x)| \le 1$ for the initial curve $t = \varepsilon f(x)$, then there exists a constant $C = C(M, T, \|h\|_{L^\infty(\partial\Omega)}) > 0$ such that

$$\|v(\varepsilon) - v_0\| \le C\left[1 + \|\bigtriangledown f\|_{L^\infty(\Omega)}\right]\varepsilon^{1/2}. \tag{4.3}$$

Now, for $p(x) \in H^3(\Omega)$, define a map $K : p(x) \to V(x, \varepsilon f(x))$, where $V(x,t)$ is given by the direct problem

$$\begin{cases} \frac{\partial V}{\partial t} = -\Delta V, & (x,t) \in \Omega \times (0,t_0), \\ \frac{\partial V}{\partial \nu(x)} + h(x)V = 0, & (x,t) \in \Omega \times (0,t_0), \\ V(x,T) = p(x) & x \in \Omega. \end{cases} \tag{4.4}$$

Corresponding to $v_0(x) = v(x, \varepsilon f(x))$, denote by $p_0(x)$ the final value $v(x,T)$ which implies $K \cdot p_0 = v_0$. For the noisy data $\hat{v}_0(x)$ satisfying (1.4), we first construct the regularizing solution $p_\delta(x)$ to the equation $K \cdot p = \hat{v}$, then construct the approximate solution $v_\delta(x,t)$ from $p_\delta(x)$ by solving direct problem (4.4). We can also give an estimate on the convergence rate of approximate solution. Introduce the functional

$$F_\alpha^\delta(p) = \|Kp - \hat{v}\|_{L^2(\Omega)}^2 + \alpha \|p\|_{H^3(\Omega)}^2 \tag{4.5}$$

over the admissible set $\mu_M = \{p(x) : \|p\|_{H^3(\Omega)} \le M\}$.

Theorem 4.1. *For any $C_0^2 > M^2 + 1$, there exists an approximate minimizer $p_\delta(x)$ for functional $F_{\delta^2}^\delta(p(\cdot))$ over μ_{C_0} which satisfies*

$$F_{\delta^2}^\delta(p_\delta) \le C_0^2\delta^2, \tag{4.6}$$

$$\|K \cdot p_\delta - K \cdot p_0\|_{L^2(\Omega)} \le (C_0 + 1)\delta. \tag{4.7}$$

Proof. Firstly, it is easy to know that

$$\begin{aligned} F_{\delta^2}^\delta(p_0) &= \|K \cdot p_0 - \hat{v}\|_{L^2}^2 + \delta^2 \|p_0\|_{H^3}^2 = \|v_0 - \hat{v}\|_{L^2}^2 + \delta^2 \|p_0\|_{H^3}^2 \\ &\le \delta^2 + M^2\delta^2 = (M^2+1)\delta^2 \le C_0^2\delta^2, \end{aligned} \tag{4.8}$$

which implies $\{p : F_{\delta^2}^\delta(p) \le C_0^2\delta^2\} \ne \emptyset$. Hence (4.6) is proven. From this inequality we also know

$$\|p_\delta\|_{H^3} \le C_0, \tag{4.9}$$

$$\|K \cdot p_\delta - \hat{v}\|_{L^2} \le C_0\delta. \tag{4.10}$$

Therefore we get

$$\|K \cdot p_\delta - K \cdot p_0\| \le \|K \cdot p_\delta - \hat{v}\| + \|\hat{v} - K \cdot p_0\| \le (C_0 + 1)\delta.$$

For $p_\delta(x)$ constructed in this theorem, denote by $v_\delta(x,t)$ the solution to direct heat problem (4.4) corresponding to $p = p_\delta$. Now we estimate the error between $v_\delta(x,t)$ and $v(x,t)$ for $\varepsilon \le t \le T$.

Theorem 4.2. *There exists a constant $C = C(C_0, M, \|h\|) > 0$ such that*

$$\|(v_\delta - v)(t)\| \le C\left[(1 + \|\bigtriangledown f\|_{L^\infty})\varepsilon^{1/2} + \delta\right]^{\frac{T-t}{T-\varepsilon}}. \tag{4.11}$$

Proof. Following the way in the proof of Theorem 2.1, it is easy to show that

$$\begin{aligned}\|(v_\delta - v)(t)\|_{L^2} &\le \|(v_\delta - v)(\varepsilon)\|_{L^2}^{\frac{T-t}{T-\varepsilon}} \|(v_\delta - v)(T)\|_{L^2}^{\frac{T-\varepsilon}{T-\varepsilon}} \\ &\le \|(v_\delta - v)(\varepsilon)\|_{L^2}^{\frac{T-t}{T-\varepsilon}} \|p_\delta - p_0\|_{H^3}^{\frac{T-\varepsilon}{T-\varepsilon}} \\ &\le 2C_0 \|(v_\delta - v)(\varepsilon)\|_{L^2}^{\frac{T-t}{T-\varepsilon}}. \end{aligned} \tag{4.12}$$

On the other hand,

$$\begin{aligned}\|(v_\delta - v)(\varepsilon)\| &\le \|v_\delta(\varepsilon) - K\cdot p_\delta\| + \|K\cdot p_0 - v(\varepsilon)\| + \|K\cdot p_\delta - K\cdot p_0\| \\ &\le \|v_\delta(\varepsilon) - K\cdot p_\delta\| + \|K\cdot p_0 - v(\varepsilon)\| + (C_0+1)\delta \\ &\le C\left[(1 + \|\bigtriangledown f\|_{L^\infty})\varepsilon^{1/2} + \delta\right] \end{aligned} \tag{4.13}$$

from Corollary 4.1 and (4.7), which leads to (4.11) immediately.

References

[1] Ames, K. A. & Payne, L. E., Stablizing the backward heat equation against errors in the initial time geometry, Inequalities and Applications, **3**, World Scientific, 1994, 47–52.

[2] Cheng, J. & Yamamoto, M., One new strategy for the choice of the regularized parameter in Tikhonove regularization, *Inverse Problems*, **16:**3(2000).

[3] Huang, C. H. & Chin, S. C., A two-dimensional inverse problem in imaging the thermal conductivity of a non-homogeneous media, *International Journal of Heat and Mass Transfer,* **43**(2000), 4061–4071.

[4] Isakov, V., Inverse problems for partial differential equations, Springer-Verlag, 1998.

[5] Kirsh, A., An introduction to the mathematical theory of inverse problems, Springer-Verlag, 1996.

[6] Ladyzhenskaya, O. A., The boundary value problems of mathematical physics, Springer-Verlag, New York, 1985.

[7] Liu, J. J., On the numerical solution for the forward and backward heat equation, to appear in *J. Comput. Appl. Maths.*.

[8] Liu, J. J., Determination of temperature field for backward heat transfer, *Comm. Korea Maths Socity,* **16:**3(2001), 385–397.

[9] Payne, L. E., Improperly posed problems in partial differential equations, Regional Conference Series in Applied mathematics, SIAM, Philadelphia, 1975.

[10] Seidman, T. I., Optimal filtering for the backward heat equation, *SIAM J. Numer. Anal.,* **33:**1(1998), 162–170.