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THE SPACE-FRACTIONAL TELEGRAPH EQUATION AND THE RELATED FRACTIONAL TELEGRAPH PROCESS***

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Abstract

The space-fractional telegraph equation is analyzed and the Fourier transform of its fundamental solution is obtained and discussed.

A symmetric process with discontinuous trajectories, whose transition function satisfies the space-fractional telegraph equation, is presented. Its limiting behaviour and the connection with symmetric stable processes is also examined.

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§1. Introduction

It is well known that Brownian motion B is the limit, in some sense, of the telegrapher's process T (see [3]).

The transition function $p_B(x,t) = p_B$ of B is the fundamental solution of

$$\begin{cases} \frac{\partial p}{\partial t} = \frac{1}{2}\frac{\partial^2 p}{\partial x^2}, & x \in \mathbb{R},\ t > 0, \\ p(x,0) = \delta(x), \end{cases} \tag{1.1}$$

while the transition function $p_T(x,t) = p_T$ of T is the fundamental solution of

$$\begin{cases} \frac{\partial^2 p}{\partial t^2} + 2\lambda\frac{\partial p}{\partial t} = c^2\frac{\partial^2 p}{\partial x^2}, x \in \mathbb{R},\ t > 0, \\ p(x,0) = \delta(x), \\ p_t(x,0) = 0, \end{cases} \tag{1.2}$$

δ being the Dirac's delta function.

It was discovered by Riesz[5] that the transition functions of symmetric stable processes with characteristic functions

$$U(\gamma,t) = e^{-|\gamma|^\alpha t/2} \tag{1.3}$$

of degree $0 < \alpha \le 2$ are solutions of the fractional diffusion equation

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^\alpha u}{\partial |x|^\alpha}, & x \in \mathbb{R},\ t > 0, \\ u(x,0) = \delta(x), \\ u_t(x,0) = 0, \end{cases} \tag{1.4}$$

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where the last condition must be considered only when $1 < \alpha \le 2$.

By $\frac{\partial^\alpha}{\partial|x|^\alpha}$ we mean the pseudo-differential operator defined as follows:

$$\frac{\partial^\alpha}{\partial|x|^\alpha} = -\frac{1}{2\cos\frac{\alpha\pi}{2}}\left[W_+^\alpha + W_-^\alpha\right], \tag{1.5}$$

where W_+^α, W_-^α are the Weyl's fractional derivatives (see [7, p. 109])

$$\begin{cases} (W_+^\alpha f)(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_{-\infty}^x \frac{f(t)dt}{(x-t)^{\alpha+1-m}}, \\ (W_-^\alpha f)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_x^\infty \frac{f(t)dt}{(t-x)^{\alpha+1-m}}, \end{cases} \tag{1.6}$$

with $m-1<\alpha<m,\ m\in\mathbb{N}$.

Heat and wave fractional equations have been considered by other authors (in the integral form by Fujita[2] and by Schneider and Wyss[9] or with fractional derivatives in both members by Saichev and Zaslavsky[6]).

An extension of the operator (1.5) has been introduced by Feller[1] and the Fourier transform of the solution of (1.4) has been proved to be, in this case, the characteristic function of an asymmetric stable process (for a deep and complete analysis see [8]).

Our aim here is to study the solutions of the space-fractional telegraph equation, namely

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + 2\lambda\frac{\partial u}{\partial t} = c^2\frac{\partial^\alpha u}{\partial|x|^\alpha}, \quad 0<\alpha<2,\ \alpha\neq 1, \\ u(x,0)=\delta(x), \\ u_t(x,0)=0, \end{cases} \tag{1.7}$$

where c, λ are non-negative constants.

The fundamental solution of (1.7) is the transition function of some type of process whose characteristic function has the form

$$\begin{aligned} U(\gamma,t) = \frac{e^{-\lambda t}}{2}\Big[\Big(1+\frac{\lambda}{\sqrt{\lambda^2-c^2|\gamma|^\alpha}}\Big)e^{t\sqrt{\lambda^2-c^2|\gamma|^\alpha}} \\ + \Big(1-\frac{\lambda}{\sqrt{\lambda^2-c^2|\gamma|^\alpha}}\Big)e^{-t\sqrt{\lambda^2-c^2|\gamma|^\alpha}}\Big], \end{aligned} \tag{1.8}$$

for $0<\alpha<2,\ \alpha\neq 1$.

It can be easily ascertained that, for $\alpha=2$, (1.8) coincides with the characteristic function of the telegrapher's process (see [4]). We note that in this case equation (1.7) coincides with the classical telegraph equation (1.2) since, by definition, $\frac{\partial^\alpha}{\partial|x|^\alpha} = \frac{\partial^2}{\partial x^2}$ for $\alpha=2$ (see [1]).

Furthermore, as $c,\lambda\to\infty$, (1.8) converges to the characteristic function (1.3) of the stable process and the fractional telegraph equation (1.7) tends to the fractional heat equation (1.4) (provided that $c^2/\lambda\to 1$).

In a certain sense the process governed by the fractional telegraph equation (1.7) (we call it fractional telegraph process, FTP) is in the same position, with respect to symmetric stable processes, as the classical telegrapher's process with respect to Brownian motion.

We can present the interactions among the processes mentioned above in the following table:

TELEGRAPH EQUATION	$\Longrightarrow$	HEAT EQUATION
TELEGRAPHER'S PROCESS	$\Longrightarrow$	BROWNIAN MOTION
FRACTIONAL TEL. EQ.	$\Longrightarrow$	FRACTIONAL HEAT. EQ.
FRACTIONAL TEL. PROC.	$\Longrightarrow$	SYMMETRIC STABLE PROC.

One of the basic results of this paper is the construction of a process (the FTP), whose characteristic function coincides with (1.8). Since inverting the Fourier transform (1.8) is a task which overcomes our ability, we content ourselves in describing some qualitative features of the FTP. Our approach to its construction is inspired by the following considerations.

The operator defined by formula (1.5) does not satisfy the semigroup property. In particular, we shall prove that

$$\frac{\partial^\alpha}{\partial|x|^\alpha} = -\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}, \quad 1<\alpha<2. \tag{1.9}$$

For this reason we can decompose the fractional equation (1.7) into the linear differential system

$$\begin{cases} \frac{\partial f}{\partial t} = -ic\frac{\partial^{\alpha/2} f}{\partial|x|^{\alpha/2}} + \lambda(b-f), \\ \frac{\partial b}{\partial t} = ic\frac{\partial^{\alpha/2} b}{\partial|x|^{\alpha/2}} + \lambda(f-b). \end{cases} \tag{1.10}$$

In the study of the classical telegrapher's process appears a system similar to (1.10) (see [3]) with the substantial difference that the space derivatives are here replaced by the operator $\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}$ defined by formula (1.5).

We shall see that the FTP (whose distribution is related to (1.10)) possesses discontinuous trajectories. The distribution of the length of jumps (taking values in $(0,\infty)$) is essentially concentrated on small values and is also connected with the degree α of the operator (1.5). The FTP differs substantially from the classical telegraph process in that it spreads instantaneously on the line and has discontinuous sample paths (these properties are shared by the limiting stable process).

Another important difference with respect to the classical telegraph process is that it develops on the imaginary axis and the role of the related Poisson process is to invert the direction of values on the line.

§2. On the Solution of the Fractional Telegraph Equation

We consider the fractional telegraph equation

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} + 2\lambda\frac{\partial u}{\partial t} = c^2\frac{\partial^\alpha u}{\partial|x|^\alpha}, \quad 1<\alpha<2, \\ u(x,0)=\delta(x), \\ u_t(x,0)=0, \end{cases} \tag{2.1}$$

where $\frac{\partial^\alpha u}{\partial|x|^\alpha}$ is the Riesz fractional derivative (see [1]), to be understood as the inverse of the Riesz potential

$$(I^\alpha f)(x) = \frac{1}{2\Gamma(\alpha)\cos\frac{\alpha\pi}{2}}\int_{-\infty}^{\infty}|x-t|^{\alpha-1}f(t)dt. \tag{2.2}$$

The explicit representation of the Riesz fractional derivative is

$$\frac{\partial^\alpha f}{\partial|x|^\alpha} = -I^{-\alpha}f = -\frac{1}{2\cos\frac{\alpha\pi}{2}}\left[W_+^\alpha f + W_-^\alpha f\right], \tag{2.3}$$

where

$$\begin{cases} (W_+^\alpha f)(x) = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_{-\infty}^{x}\frac{f(t)dt}{(x-t)^{\alpha+1-m}}, \\ (W_-^\alpha f)(x) = \frac{(-1)^m}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_{x}^{\infty}\frac{f(t)dt}{(t-x)^{\alpha+1-m}} \end{cases} \tag{2.4}$$

(for $m=\lfloor\alpha\rfloor+1$) are the Weyl's derivatives. These are related to the Weyl's integrals

$$\begin{cases} (W_+^{-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)}\int_{-\infty}^{x}(x-t)^{\alpha-1}f(t)dt, \\ (W_-^{-\alpha} f)(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}f(t)dt, \end{cases} \tag{2.5}$$

by the following relationships

$$W_\pm^\alpha = (\pm 1)^m D^m W_\pm^{-(m-\alpha)}, \tag{2.6}$$

where D^m is the usual m-th derivative.

The Fourier transform of the solution to problem (2.1) can be obtained explicitly and its expression is presented in next theorem.

Theorem 2.1. *The Fourier transform*

$$U(\gamma,t)=\int_{-\infty}^{+\infty}e^{i\gamma x}u(x,t)dx \tag{2.7}$$

of the solution to (2.1) *is given by*

$$\begin{aligned}U(\gamma,t)=&\frac{e^{-\lambda t}}{2}\Big[\Big(1+\frac{\lambda}{\sqrt{\lambda^2-c^2|\gamma|^\alpha}}\Big)e^{t\sqrt{\lambda^2-c^2|\gamma|^\alpha}}\\&+\Big(1-\frac{\lambda}{\sqrt{\lambda^2-c^2|\gamma|^\alpha}}\Big)e^{-t\sqrt{\lambda^2-c^2|\gamma|^\alpha}}\Big].\end{aligned} \tag{2.8}$$

Proof. In view of (2.3) and (2.6) we have

$$\begin{aligned}&\int_{-\infty}^{+\infty}e^{i\gamma x}\frac{\partial^\alpha}{\partial|x|^\alpha}u(x,t)dx\\&=\int_{-\infty}^{+\infty}e^{i\gamma x}\Big\{-\frac{1}{2\cos\frac{\alpha\pi}{2}}\left[W_+^\alpha+W_-^\alpha\right]u(x,t)\Big\}dx\\&=-\frac{1}{2\cos\frac{\alpha\pi}{2}}\int_{-\infty}^{+\infty}e^{i\gamma x}\Big[D^2W_+^{-(2-\alpha)}+D^2W_-^{-(2-\alpha)}\Big]u(x,t)dx\\&=-\frac{(-i\gamma)^2}{2\cos\frac{\alpha\pi}{2}}\int_{-\infty}^{+\infty}e^{i\gamma x}\Big[W_+^{-(2-\alpha)}+W_-^{-(2-\alpha)}\Big]u(x,t)dx.\end{aligned} \tag{2.9}$$

It is easy to realize that

$$\begin{aligned}W_+^{-(2-\alpha)}u(x,t)&=\frac{1}{\Gamma(2-\alpha)}\int_{-\infty}^{x}(x-y)^{1-\alpha}u(y,t)dy\\&=\frac{1}{\Gamma(2-\alpha)}\int_{-\infty}^{+\infty}(x-y)^{1-\alpha}1_{(-\infty,x)}(y)u(y,t)dy\\&=\frac{1}{\Gamma(2-\alpha)}\int_{-\infty}^{+\infty}(x-y)^{1-\alpha}1_{(0,\infty)}(x-y)u(y,t)dy.\end{aligned} \tag{2.10a}$$

Analogously

$$\begin{aligned}W_-^{-(2-\alpha)}u(x,t)&=\frac{1}{\Gamma(2-\alpha)}\int_{x}^{+\infty}(y-x)^{1-\alpha}u(y,t)dy\\&=\frac{1}{\Gamma(2-\alpha)}\int_{-\infty}^{+\infty}(y-x)^{1-\alpha}1_{(x,\infty)}(y)u(y,t)dy\\&=\frac{1}{\Gamma(2-\alpha)}\int_{-\infty}^{+\infty}(y-x)^{1-\alpha}1_{(0,\infty)}(y-x)u(y,t)dy.\end{aligned} \tag{2.10b}$$

Therefore

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} e^{i\gamma x} W_+^{-(2-\alpha)} u(x,t)dx \\
&= \frac{1}{\Gamma(2-\alpha)} \int_{-\infty}^{+\infty} e^{i\gamma x} \Big[\int_{-\infty}^{+\infty} (x-y)^{1-\alpha} 1_{(0,\infty)}(x-y) u(y,t) dy \Big] dx \\
(x-y=w)& \\
&= \frac{1}{\Gamma(2-\alpha)} \int_{-\infty}^{+\infty} e^{i\gamma y} u(y,t)dy \int_{-\infty}^{+\infty} e^{i\gamma w} w^{1-\alpha} 1_{(0,\infty)}(w)dw \\
&= \frac{1}{\Gamma(2-\alpha)} U(\gamma,t) \int_0^{+\infty} w^{1-\alpha} e^{i\gamma w} dw \\
(\gamma w = iy)& \\
&= \frac{1}{\Gamma(2-\alpha)} U(\gamma,t) \int_0^{+\infty} \Big(\frac{iy}{\gamma}\Big)^{1-\alpha} e^{-y} \frac{i}{\gamma} dy \\
&= \Big(\frac{i}{\gamma}\Big)^{2-\alpha} U(\gamma,t) = (-i\gamma)^{-(2-\alpha)} U(\gamma,t). \qquad (2.11a)
\end{aligned}
$$

In the same way we obtain

$$
\begin{aligned}
&\int_{-\infty}^{+\infty} e^{i\gamma x} W_-^{-(2-\alpha)} u(x,t)dx \\
&= \frac{1}{\Gamma(2-\alpha)} \int_{-\infty}^{+\infty} e^{i\gamma x} \Big[\int_{-\infty}^{+\infty} (y-x)^{1-\alpha} 1_{(0,\infty)}(y-x) u(y,t) dy \Big] dx \\
(y-x=w)& \\
&= \frac{1}{\Gamma(2-\alpha)} \int_{-\infty}^{+\infty} e^{i\gamma z} u(z,t)dz \int_0^{+\infty} e^{-i\gamma w} w^{1-\alpha} dw \\
(\gamma w = -iy)& \\
&= \frac{1}{\Gamma(2-\alpha)} U(\gamma,t) \int_0^{+\infty} \Big(-\frac{iy}{\gamma}\Big)^{1-\alpha} e^{-y} \frac{1}{i\gamma} dy \\
&= (i\gamma)^{-(2-\alpha)} U(\gamma,t). \qquad (2.11b)
\end{aligned}
$$

All these calculations permit us to conclude that $U(\gamma,t)$ solves the initial-value problem

$$
\begin{aligned}
\frac{\partial^2 U}{\partial t^2} + 2\lambda \frac{\partial U}{\partial t} &= -\frac{c^2}{2\cos\frac{\alpha\pi}{2}} (-i\gamma)^2 \Big[(-i\gamma)^{-(2-\alpha)} + (i\gamma)^{-(2-\alpha)} \Big] \\
&= -\frac{c^2}{2\cos\frac{\alpha\pi}{2}} \big[(-i\gamma)^\alpha + (i\gamma)^\alpha \big] \\
&= -\frac{c^2}{2\cos\frac{\alpha\pi}{2}} \Big[(|\gamma| e^{i\frac{\pi}{2}})^\alpha + (|\gamma| e^{-i\frac{\pi}{2}})^\alpha \Big] \\
&= -c^2 |\gamma|^\alpha, \\
U(\gamma,0) &= 1, \\
U_t(\gamma,0) &= 0.
\end{aligned}
$$

It is now a simple matter to obtain the characteristic function (2.8).

Remark 2.1. The characteristic function $U = U(\gamma,t)$ can be written conveniently as follows:

$$U(\gamma,t)=\frac{e^{-\lambda t}}{2}\Big\{\lambda\Big[\frac{e^{t\sqrt{\lambda^2-c^2|\gamma|^\alpha}}-e^{-t\sqrt{\lambda^2-c^2|\gamma|^\alpha}}}{\sqrt{\lambda^2-c^2|\gamma|^\alpha}}\Big] + \frac{\partial}{\partial t}\Big[\frac{e^{t\sqrt{\lambda^2-c^2|\gamma|^\alpha}}-e^{-t\sqrt{\lambda^2-c^2|\gamma|^\alpha}}}{\sqrt{\lambda^2-c^2|\gamma|^\alpha}}\Big]\Big\}. \tag{2.12}$$

The problem of finding the inverse Fourier transform

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\gamma x}\Big[\frac{e^{t\sqrt{\lambda^2-c^2|\gamma|^\alpha}}-e^{-t\sqrt{\lambda^2-c^2|\gamma|^\alpha}}}{\sqrt{\lambda^2-c^2|\gamma|^\alpha}}\Big]d\gamma \tag{2.13}$$

seems possible only for the special case where $\alpha = 2$. In this case it is proved explicitly in [4] that

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\gamma x}\Big[\frac{e^{t\sqrt{\lambda^2-c^2\gamma^2}}-e^{-t\sqrt{\lambda^2-c^2\gamma^2}}}{\sqrt{\lambda^2-c^2\gamma^2}}\Big]d\gamma=\frac{1}{c}I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\Big). \tag{2.14}$$

Because of (2.14) and considering the representation (2.12) it is possible to write down the distribution of the classical telegrapher's process in all its components, that is,

$$p_T(x,t)=\frac{e^{-\lambda t}}{2c}\Big[\lambda I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\Big)+\frac{\partial}{\partial t}I_0\Big(\frac{\lambda}{c}\sqrt{c^2t^2-x^2}\Big)\Big] + \frac{e^{-\lambda t}}{2}\left[\delta(x-ct)+\delta(x+ct)\right]. \tag{2.15}$$

Nothing similar is possible in the fractional case, for $1<\alpha<2$. However some qualitative features of the process, whose characteristic function coincides with (2.8), can be obtained by a completely different approach. Our idea is to construct a process whose distribution $u(x,t)=\Pr\{X(t)\in dx\}$ is a solution of the fractional differential equation (2.1) and whose related joint distributions, defined as

$$\begin{aligned} f(x,t)dx &= \Pr\{X(t)\in dx, N(t) \text{ is even}\},\\ b(x,t)dx &= \Pr\{X(t)\in dx, N(t) \text{ is odd}\} \end{aligned} \tag{2.16}$$

($X(t)$ being the current position of a randomly moving particle and $N(t)$ the number of events of a Poisson process), are solutions of the fractional differential system (1.10), into which equation (2.1) can be split up.

The main difficulty is that the pseudo-differential operator (2.3) does not satisfy the semigroup property (while the Riesz potential (2.2) does). In the next section we shall examine this fact and obtain the announced differential system.

§3. Decomposing the Fractional Telegraph Equation

We want to show here that, for $1<\alpha<2$, the decomposition (1.9) is possible, where the basic and substantial novelty lies in the sign appearing in it. This has far-reaching implications, as the reader will realize.

Theorem 3.1. *For $1<\alpha<2$ the following relationship holds:*

$$\Big(\frac{\partial^\alpha}{\partial|x|^\alpha}f\Big)(x)=-\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}\Big(\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}f\Big)(x) \tag{3.1}$$

where f is a function vanishing at infinity as $|x|^{\frac{\alpha}{2}-1-\varepsilon}$, $\varepsilon>0$.

Proof. It is convenient to write $\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}$ as follows:

$$\begin{aligned}
&\Big(\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}f\Big)(x)\\
&=-\frac{1}{2\cos\frac{\alpha\pi}{4}}\Big[\frac{1}{\Gamma(1-\frac{\alpha}{2})}\frac{d}{dx}\int_{-\infty}^{x}\frac{f(t)dt}{(x-t)^{\frac{\alpha}{2}}}-\frac{1}{\Gamma(1-\frac{\alpha}{2})}\frac{d}{dx}\int_{x}^{\infty}\frac{f(t)dt}{(t-x)^{\frac{\alpha}{2}}}\Big]\\
&=-\frac{1}{2\cos\frac{\alpha\pi}{4}\Gamma(1-\frac{\alpha}{2})}\frac{d}{dx}\int_0^\infty\frac{f(x-y)-f(x+y)}{y^{\frac{\alpha}{2}}}dy\\
&=C\frac{d}{dx}\int_0^\infty\frac{f(x-y)-f(x+y)}{y^{\frac{\alpha}{2}}}dy. \qquad (3.2)
\end{aligned}$$

We can write

$$\begin{aligned}
&\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}\Big\{\frac{\partial^{\alpha/2}f}{\partial|x|^{\alpha/2}}\Big\}(x)\\
&=C\frac{d}{dx}\int_0^\infty\frac{1}{w^{\frac{\alpha}{2}}}\Big\{\frac{\partial^{\alpha/2}f}{\partial|u|^{\alpha/2}}(u)\Big|^{u=x-w}-\frac{\partial^{\alpha/2}f}{\partial|u|^{\alpha/2}}(u)\Big|^{u=x+w}\Big\}dw\\
&=C^2\frac{d}{dx}\int_0^\infty\frac{1}{w^{\frac{\alpha}{2}}}\Big\{\Big[\frac{d}{du}\int_0^\infty\frac{f(u-y)-f(u+y)}{y^{\frac{\alpha}{2}}}dy\Big]^{u=x-w}\\
&\quad-\Big[\frac{d}{du}\int_0^\infty\frac{f(u-y)-f(u+y)}{y^{\frac{\alpha}{2}}}dy\Big]^{u=x+w}\Big\}dw\\
&=C^2\frac{d}{dx}\int_0^\infty\frac{dw}{w^{\frac{\alpha}{2}}}\int_0^\infty\frac{f_x(x-w-y)-f_x(x-w+y)-f_x(x+w-y)+f_x(x+w+y)}{y^{\frac{\alpha}{2}}}dy. \qquad (3.3)
\end{aligned}$$

We concentrate now our attention on

$$I=\int_0^\infty\frac{dw}{w^{\frac{\alpha}{2}}}\int_0^\infty\frac{f_x(x-w-y)+f_x(x+w+y)}{y^{\frac{\alpha}{2}}}dy, \qquad (3.4)$$

which can be evaluated by performing the transformation $w+y=v$, $w-y=u$.

Therefore we have, after some successive substitutions,

$$\begin{aligned}
I&=\frac{2^\alpha}{2}\int_0^\infty dv\,[f_x(x-v)+f_x(x+v)]\int_{-v}^{v}\frac{du}{(v^2-u^2)^{\alpha/2}}\\
&=2^{\alpha-1}\int_0^\infty\frac{dv}{v^{\alpha-1}}\,[f_x(x-v)+f_x(x+v)]\int_0^1 z^{\frac{1}{2}-1}(1-z)^{1-\frac{\alpha}{2}-1}dz\\
&=2^{\alpha-1}\frac{\Gamma(\frac{1}{2})\Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{3}{2}-\frac{\alpha}{2})}\int_0^\infty\frac{dv}{v^{\alpha-1}}\,[f_x(x-v)+f_x(x+v)]\,. \qquad (3.5)
\end{aligned}$$

By means of the same transformation as above we have

$$
\begin{aligned}
J &= \int_0^\infty \frac{dw}{w^{\frac{\alpha}{2}}} \int_0^\infty \frac{f_x(x-w+y)+f_x(x+w-y)}{y^{\frac{\alpha}{2}}}dy \\
&= \frac{2^\alpha}{2}\int_0^\infty dv \int_{-v}^{v} \frac{f_x(x-u)+f_x(x+u)du}{(v^2-u^2)^{\alpha/2}} \\
&= 2^\alpha \int_0^\infty [f_x(x-u)+f_x(x+u)]\,du \int_u^\infty \frac{dv}{(v^2-u^2)^{\alpha/2}} \\
&= 2^{\alpha-1}\int_0^\infty \frac{f_x(x-u)+f_x(x+u)}{u^{\alpha-1}}du \int_0^1 (1-w)^{-\frac{1}{2}+\frac{\alpha}{2}-1} w^{1-\frac{\alpha}{2}-1}dw \\
&= 2^{\alpha-1}\frac{\Gamma(\frac{\alpha}{2}-\frac{1}{2})\Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{1}{2})}\int_0^\infty \frac{du}{u^{\alpha-1}}\left[f_x(x-u)+f_x(x+u)\right]. \qquad (3.6)
\end{aligned}
$$

Now, taking into account the reflection formula

$$
\Gamma(z)\Gamma(1-z)=\frac{\pi}{\sin\pi z}\,, \qquad z\neq 0,\pm1,\pm2,\cdots
$$

and the Euler's duplication formula

$$
\Gamma(z+\frac{1}{2})=2^{1-2z}\Gamma(\frac{1}{2})\frac{\Gamma(2z)}{\Gamma(z)}\,, \qquad 2z\neq 0,-1,-2,\cdots
$$

we readily have

$$
\begin{aligned}
&2^{\alpha-1}\frac{\Gamma(\frac{1}{2})\Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{3}{2}-\frac{\alpha}{2})} - 2^{\alpha-1}\frac{\Gamma(\frac{\alpha}{2}-\frac{1}{2})\Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{1}{2})} \\
&= 2^{\alpha-1}\Gamma(1-\frac{\alpha}{2})\Big[\frac{\Gamma^2(\frac{1}{2})-\frac{\pi}{\sin(\frac{\alpha}{2}-\frac{1}{2})\pi}}{\Gamma(\frac{1}{2})\Gamma(\frac{3}{2}-\frac{\alpha}{2})}\Big] \\
&= 2^{\alpha-1}\Gamma(1-\frac{\alpha}{2})\Big(\pi+\frac{\pi}{\cos\frac{\alpha\pi}{2}}\Big)\frac{1}{\Gamma(\frac{1}{2})}\frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(\frac{1}{2})2^{1-2(1-\alpha/2)}\Gamma(2-\alpha)} \\
&= \frac{\Gamma^2(1-\frac{\alpha}{2})}{\Gamma(2-\alpha)}\frac{1+\cos\frac{\alpha\pi}{2}}{\cos\frac{\alpha\pi}{2}}\,. \qquad (3.7)
\end{aligned}
$$

Formula (3.3) can be written as

$$
\begin{aligned}
&C^2\frac{d}{dx}\left[I-J\right] \\
&= \frac{1}{2^2\cos^2\frac{\alpha\pi}{4}\Gamma^2(1-\frac{\alpha}{2})}\frac{\Gamma^2(1-\frac{\alpha}{2})}{\Gamma(2-\alpha)}\frac{1+\cos\frac{\alpha\pi}{2}}{\cos\frac{\alpha\pi}{2}}\frac{d}{dx}\int_0^\infty \frac{f_x(x-u)+f_x(x+u)}{u^{\alpha-1}}du \\
&= \frac{1}{2\cos\frac{\alpha\pi}{2}\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_0^\infty \frac{f(x-u)+f(x+u)}{u^{\alpha-1}}du \\
&= \Big(-\frac{\partial^\alpha f}{\partial|x|^\alpha}\Big)(x). \qquad (3.8)
\end{aligned}
$$

In the last step we have considered the definition (2.3) for $1<\alpha<2$.

Remark 3.1. We note that, for arbitrary real values p and q, it is true that

$$
\frac{\partial^p}{\partial|x|^p}\frac{\partial^q}{\partial|x|^q}\neq\frac{\partial^{p+q}}{\partial|x|^{p+q}}
$$

and a relationship as simple as that of Theorem 3.1 does not hold in general.

For the case $0 < \alpha < 1$, by performing substantially the same calculations as above, we have

$$\begin{aligned}
&\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}\Big\{\frac{\partial^{\alpha/2} f}{\partial|x|^{\alpha/2}}\Big\}(x) \\
&= \frac{1}{4\cos^2\frac{\alpha\pi}{4}\Gamma(2-\alpha)}\frac{d^2}{dx^2}\int_0^\infty \frac{f(x-u)+f(x+u)}{u^{\alpha-1}}du \\
&\neq -\frac{1}{2\cos\frac{\alpha\pi}{2}\Gamma(1-\alpha)}\frac{d}{dx}\int_0^\infty \frac{f(x-u)-f(x+u)}{u^{\alpha}}du \\
&= \Big(\frac{\partial^\alpha f}{\partial|x|^\alpha}\Big)(x).
\end{aligned}$$

Remark 3.2. The fractional telegraph equation, in view of Theorem 3.1, can be written as

$$\frac{\partial^2}{\partial t^2} + 2\lambda\frac{\partial}{\partial t} = c^2\frac{\partial^\alpha}{\partial|x|^\alpha} = -c^2\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}$$

and thus

$$0 = \Big(\frac{\partial}{\partial t} - ic\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}\Big)\Big(\frac{\partial}{\partial t} + ic\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}\Big)f + 2\lambda\frac{\partial f}{\partial t}\ . \tag{3.9}$$

Assuming that

$$\Big(\frac{\partial}{\partial t} + ic\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}\Big)f = \lambda b - \lambda f, \tag{3.10}$$

from (1.9) we obtain

$$\Big(\frac{\partial}{\partial t} - ic\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}\Big)(\lambda b - \lambda f) + 2\lambda\frac{\partial f}{\partial t} = 0 \tag{3.11}$$

and, replacing

$$ic\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}f = \lambda b - \lambda f - \frac{\partial}{\partial t}$$

into (3.11), we get

$$\Big(\frac{\partial}{\partial t} - ic\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}\Big)b = \lambda f - \lambda b. \tag{3.12}$$

§4. A Process Related to the Fractional System

We consider in this section the one-dimensional motion of a particle which moves forward and backward performing jumps of random amplitude Y, with $Y > 0$.

We first assume that during every time interval $[t, t + \Delta t)$ a particle can either make a jump in the positive direction (with probability 1/2) or a jump in the negative direction (with the same probability).

We also assume that the distribution of Y is the following one:

$$p(y, \Delta t) = \begin{cases} \frac{\alpha}{2}\frac{\Delta t}{y^{1+\frac{\alpha}{2}}} & \text{for } (\Delta t)^{\frac{2}{\alpha}} < y < \infty, \\ 0 & \text{for } y < (\Delta t)^{\frac{2}{\alpha}}. \end{cases} \tag{4.1}$$

It is clear that (4.1) assigns higher probability to small-valued jump lengths.

For example,

$$\Pr\Big\{(\Delta t)^{\frac{2}{\alpha}} \le Y \le 2^{\frac{2}{\alpha}}(\Delta t)^{\frac{2}{\alpha}}\Big\} = \frac{1}{2} \tag{4.2}$$

and, for $1 < \alpha < 2$, the right end point of the interval in (4.2) is located in $(2(\Delta t)^{\frac{2}{\alpha}}, 4(\Delta t)^{\frac{2}{\alpha}})$.

If we denote by $N(t)$ the number of events of a homogeneous Poisson process of parameter $\lambda > 0$ and by $X = X(t), t > 0$ the current position of the particle, our task here is to derive the equations governing the following probabilities

$$\begin{aligned} f(x,t)dx &= \Pr\{X(t) \in dx, N(t) \text{ is even}\}, \\ b(x,t)dx &= \Pr\{X(t) \in dx, N(t) \text{ is odd}\}. \end{aligned} \tag{4.3}$$

Theorem 4.1. *The integro-differential system governing* (4.3) *is*

$$\begin{cases} \dfrac{\partial f}{\partial t} = \dfrac{1}{2}\dfrac{\alpha}{2}\displaystyle\int_0^\infty [f(x-y,t) - f(x,t)]\frac{dy}{y^{1+\alpha/2}} \\ \qquad + \dfrac{1}{2}\dfrac{\alpha}{2}\displaystyle\int_0^\infty [f(x+y,t) - f(x,t)]\frac{dy}{y^{1+\alpha/2}} + \lambda(b-f), \\ \dfrac{\partial b}{\partial t} = \dfrac{1}{2}\dfrac{\alpha}{2}\displaystyle\int_0^\infty [b(x-y,t) - b(x,t)]\frac{dy}{y^{1+\alpha/2}} \\ \qquad + \dfrac{1}{2}\dfrac{\alpha}{2}\displaystyle\int_0^\infty [b(x+y,t) - b(x,t)]\frac{dy}{y^{1+\alpha/2}} + \lambda(f-b). \end{cases} \tag{4.4}$$

Proof. We derive only the first equation since the other one follows in the same way. We suppose, for the sake of definiteness, that we evaluate

$$f(x, t+\Delta t)dx = \Pr\{X(t+\Delta t) \in dx, N(t+\Delta t) \text{ is even}\}.$$

In the case where no Poisson event happens in the interval of time $[t, t+\Delta t)$ and $N(t)$ is an even number, a point x can be reached, at time $t + \Delta t$, if either a jump upward (with probability 1/2) or a jump downward (with the same probability) occurs. Another case is where the particle is located around x and a Poisson event happens during $[t, t+\Delta t)$ when the cumulative number $N(t)$ was odd at time t. For the random movement occurring every Δt instants we can therefore write

$$\begin{aligned} f(x,t+\Delta t) &= (1-\lambda\Delta t)\Big\{\frac{1}{2}\int_{(\Delta t)^{2/\alpha}}^\infty f(x-y,t)\frac{\alpha}{2}\frac{\Delta t}{y^{1+\alpha/2}}dy \\ &\quad + \frac{1}{2}\int_{(\Delta t)^{2/\alpha}}^\infty f(x+y,t)\frac{\alpha}{2}\frac{\Delta t}{y^{1+\alpha/2}}dy\Big\} + \lambda\Delta t b + o(\Delta t) \\ &= (1-\lambda\Delta t)\{\frac{1}{2}\int_{(\Delta t)^{2/\alpha}}^\infty [f(x-y,t) - f(x,t)]\frac{\alpha}{2}\frac{\Delta t}{y^{1+\alpha/2}}dy \\ &\quad + \frac{1}{2}\int_{(\Delta t)^{2/\alpha}}^\infty [f(x+y,t) - f(x,t)]\frac{\alpha}{2}\frac{\Delta t}{y^{1+\alpha/2}}dy \\ &\quad + \int_{(\Delta t)^{2/\alpha}}^\infty f(x,t)\frac{\alpha}{2}\frac{\Delta t}{y^{1+\alpha/2}}dy\} + \lambda\Delta t b(x,t) + o(\Delta t) \\ &= \frac{1}{2}\frac{\alpha}{2}\Delta t\Big\{\int_{(\Delta t)^{2/\alpha}}^\infty [f(x-y,t) - f(x,t)]\frac{dy}{y^{1+\alpha/2}} \\ &\quad + \int_{(\Delta t)^{2/\alpha}}^\infty [f(x+y,t) - f(x,t)]\frac{dy}{y^{1+\alpha/2}}\Big\} \\ &\quad + (1-\lambda\Delta t)f(x,t) + \lambda\Delta t b(x,t) + o(\Delta t). \end{aligned} \tag{4.5}$$

Now expanding $f(x, t+\Delta t)$, simplifying in both members and letting $\Delta t \to 0$ we obtain the first equation in (4.4).

Remark 4.1. We recall the relationships between the right-handed and left-handed

Marchaud's and Weyl's derivatives:

$$
\begin{aligned}
{}^{M}D_{+}^{\alpha/2} f(x,t) &= \frac{\alpha}{2}\frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_0^\infty \frac{f(x,t)-f(x-y,t)}{y^{1+\alpha/2}}dy \\
&= \frac{1}{\Gamma(1-\frac{\alpha}{2})}\frac{d}{dx}\int_{-\infty}^{x} \frac{f(y,t)}{(x-y)^{\alpha/2}}dy = W_{+}^{\alpha/2} f(x,t), \qquad (4.6a) \\
{}^{M}D_{-}^{\alpha/2} f(x,t) &= \frac{\alpha}{2}\frac{1}{\Gamma(1-\frac{\alpha}{2})}\int_0^\infty \frac{f(x,t)-f(x+y,t)}{y^{1+\alpha/2}}dy \\
&= -\frac{1}{\Gamma(1-\frac{\alpha}{2})}\frac{d}{dx}\int_{x}^{\infty} \frac{f(y,t)}{(y-x)^{\alpha/2}}dy = W_{-}^{\alpha/2} f(x,t), \qquad (4.6b)
\end{aligned}
$$

for $0 < \alpha/2 < 1$.

It is now transparent that the integrals appearing in (4.4) can be expressed in terms of Marchaud's derivatives and thus

$$
\begin{aligned}
&\frac{\alpha}{4}\int_0^\infty [f(x-y,t)-f(x,t)]\frac{dy}{y^{1+\alpha/2}} + \frac{\alpha}{4}\int_0^\infty [f(x+y,t)-f(x,t)]\frac{dy}{y^{1+\alpha/2}} \\
&= -\frac{\Gamma(1-\frac{\alpha}{2})}{2}[W_{+}^{\alpha/2}+W_{-}^{\alpha/2}]f(x,t) \\
&= \Gamma(1-\frac{\alpha}{2})\cos\frac{\alpha\pi}{4}\frac{\partial^{\alpha/2}}{\partial|x|^{\alpha/2}}f(x,t)\ .
\end{aligned}
$$

In conclusion the system (4.4) can be rewritten in the following manner

$$
\begin{cases}
\frac{\partial f}{\partial t} = \Gamma(1-\frac{\alpha}{2})\cos\frac{\alpha\pi}{4}\frac{\partial^{\alpha/2} f}{\partial|x|^{\alpha/2}} + \lambda(b-f), \\
\frac{\partial b}{\partial t} = \Gamma(1-\frac{\alpha}{2})\cos\frac{\alpha\pi}{4}\frac{\partial^{\alpha/2} b}{\partial|x|^{\alpha/2}} + \lambda(f-b).
\end{cases} \qquad (4.7)
$$

Remark 4.2. Consider now the process

$$
X'(t) = -\frac{ic}{\Gamma(1-\frac{\alpha}{2})\cos\frac{\alpha\pi}{4}}(-1)^{N(t)}X(t), \qquad (4.8)
$$

where $c > 0$. Clearly $X'(t)$ takes values on the imaginary axis, but develops according to the rules governing the evolution of $X(t), t > 0$. Furthermore, if $N(t)$ is even,

$$
x' = -\frac{icx}{\Gamma(1-\frac{\alpha}{2})\cos\frac{\alpha\pi}{4}}
$$

and, if $N(t)$ is odd,

$$
x' = \frac{icx}{\Gamma(1-\frac{\alpha}{2})\cos\frac{\alpha\pi}{4}}\ ,
$$

so that, in terms of the coordinate x', the system (4.7) can be rewritten as

$$
\begin{cases}
\frac{\partial f}{\partial t} = -ic\frac{\partial^{\alpha/2} f}{\partial|x'|^{\alpha/2}} + \lambda(b-f), \\
\frac{\partial b}{\partial t} = ic\frac{\partial^{\alpha/2} b}{\partial|x'|^{\alpha/2}} + \lambda(f-b).
\end{cases} \qquad (4.9)
$$

In deriving (4.9) we have taken into account that Weyl's derivative can be written down as follows:

$$
\begin{aligned}
W_{+}^{\alpha/2} f(x,t) &= \frac{1}{\Gamma(1-\frac{\alpha}{2})}\frac{d}{dx}\int_0^\infty \frac{f(x-w,t)}{w^{\alpha/2}}dw, \\
W_{-}^{\alpha/2} f(x,t) &= \frac{1}{\Gamma(1-\frac{\alpha}{2})}\frac{d}{dx}\int_0^\infty \frac{f(x+w,t)}{w^{\alpha/2}}dw.
\end{aligned}
$$

Remark 4.3. We can show that the process whose transition function is the fundamental solution of the fractional telegraph equation converges in the limit to the symmetric stable process with characteristic function (1.3). Our idea is to consider the Laplace transform of (2.8) and take the limit as $c \to \infty$, $\lambda \to \infty$, in such a way that $c^2/\lambda \to 1$:

$$\begin{aligned}\int_0^\infty e^{-\mu t} U(\gamma, t) dt &= \frac{1}{2}[(\mu + \lambda + \sqrt{\lambda^2 - c^2|\gamma|^\alpha})(\lambda + \sqrt{\lambda^2 - c^2|\gamma|^\alpha}) \\ &\quad + (\mu + \lambda - \sqrt{\lambda^2 - c^2|\gamma|^\alpha})(\sqrt{\lambda^2 - c^2|\gamma|^\alpha} - \lambda)] \\ &\quad \cdot \frac{1}{\sqrt{\lambda^2 - c^2|\gamma|^\alpha}[(\lambda + \mu)^2 - (\lambda^2 - c^2|\gamma|^\alpha)]} \\ &= \frac{\mu + 2\lambda}{\mu^2 + 2\lambda\mu + c^2|\gamma|^\alpha} \, . \end{aligned} \tag{4.10}$$

It is now a simple matter to observe that

$$\lim_{\lambda, c \to \infty, c^2/\lambda \to 1} \int_0^\infty e^{-\mu t} U(\gamma, t) dt = \frac{2}{2\mu + |\gamma|^\alpha} = \int_0^\infty e^{-\mu t} e^{-|\gamma|^\alpha t/2} dt.$$

This result corresponds to the fact that the fractional telegraph equation (1.2) converges, as $\lambda \to \infty$, $c \to \infty$, to the fractional heat-wave equation appearing in (1.4).

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References

[1] Feller, W., On a generalization of Marcel Riesz' potential and semigroups generated by them, *Meddelanden Universitets Matematiska Siminarium, Lund,* **21**(1952), 73–81.

[2] Fujita, Y., Integrodifferential equation which interpolates the heat and wave equations, *Osaka Journal of Mathematics,* **27**(1990), 309–321; 797–804.

[3] Orsingher, E., Probability law, flow functions, maximum distributions of wave-governed random motions and their connections with Kirchoff's law, *Stochastic Processes and Their Applications,* **34**(1990), 49–66.

[4] Orsingher, E., Motions with reflecting and absorbing barriers driven by the telegraph equation, *Random Operators and Stochastic Equations,* **3**:1(1995), 9–21.

[5] Riesz, M., L'integrale de Riemann-Liouville et le problème de Cauchy, *Acta Mathematica, Lund,* **81**(1948), 1–223.

[6] Saichev, A. I. & Zaslavsky, G. M., Fractional kinetic equations: solutions and applications, *Chaos*, **7**(1997), 753–764.

[7] Samko, S. G., Kilbas, A. A. & Marichev, O. I., Fractional integrals and derivatives, Gordon and Breach Science Publishers, Amsterdam, 1993.

[8] Samorodnitsky, G. & Taqqu, M.S., Stable non-Gaussian random processes, Chapman and Hall, New York 1994.

[9] Schneider, W. R. & Wyss, W., Fractional diffusion and wave equations, *Journal of Mathematical Physics,* **30**(1989), 134–144.