Chin. Ann. Math.
24B:1(2003),57-64.

GEOMETRY AND DIMENSION OF SELF-SIMILAR SET****

YIN Yongcheng* JIANG Haiyi** SUN Yeshun***

Abstract

The authors show that the self-similar set for a finite family of contractive similitudes (similarities, i.e., $|f_i(x) - f_i(y)| = a_i|x - y|$, $x, y \in \mathbf{R}^N$, where $0 < a_i < 1$) is uniformly perfect except the case that it is a singleton. As a corollary, it is proved that this self-similar set has positive Hausdorff dimension provided that it is not a singleton. And a lower bound of the upper box dimension of the uniformly perfect sets is given. Meanwhile the uniformly perfect set with Hausdorff measure zero in its Hausdorff dimension is given.

Keywords Self-similar set, Uniformly perfect set, Hausdorff dimension
2000 MR Subject Classification 28A80, 37C45
Chinese Library Classification O174 **Document Code** A
Article ID 0252-9599(2003)01-0057-08

§1. Introduction

Let $\{f_1, f_2, \cdots, f_m\}$, $m \geq 2$, be a finite family of contractive similitudes in N dimensional Euclidean space $\mathbf{R}^N$, that is, $f_j(x) = a_j g_j(x) + b_j$, where $0 < a_j < 1$, $b_j \in \mathbf{R}^N$, and $g_j \in O(N), 1 \leq j \leq m$.

Denote

$$G_k = \{f_{j_1} \circ \cdots \circ f_{j_k} | 1 \leq j_i \leq m, 1 \leq i \leq k\},$$

then $G = \bigcup\limits_{k \geq 1} G_k$ is the semi-group generated by $\{f_1, f_2, \cdots, f_m\}$.

Let $\mathcal{E}$ be the collection of all non-empty compact subsets of $\mathbf{R}^N$. For $X_1, X_2 \in \mathcal{E}$,

$$d_H(X_1, X_2) = \max\{d(x, X_2), d(X_1, y) | x \in X_1, y \in X_2\}$$

is the Hausdorff distance between X_1 and X_2, where $d(x, X_2)$ is the Euclidean distance from x to X_2 and $d(X_1, y)$ is the Euclidean distance from X_1 to y. It is well-known that the space $(\mathcal{E}, d_H)$ is complete.

Manuscript received November 11, 2001.
*Department of Mathematics, Zhejiang University, Hangzhou 310027, China.
E-mail: yin@ math.zju.edu.cn
**Department of Mathematics, Zhejiang University, Hangzhou 310027, China.
E-mail: hyjiang@ math.zju.edu.cn
***Department of Mathematics, Zhejiang University, Hangzhou 310027, China.
E-mail: sun@ math.zju.edu.cn
****Project supported by the National Natural Science Foundation of China (No.10171090, No.10231040).

Define a mapping from $(\mathcal{E}, d_H)$ to itself by

$$T(X) = \bigcup_{j=1}^{m} f_j(X).$$

Then

$$d_H(T(X_1), T(X_2)) \leq \overline{s} \cdot d_H(X_1, X_2)$$

for any $X_1, X_2 \in \mathcal{E}$, where $0 < \overline{s} = \max_{1 \leq j \leq m} a_j < 1$. We also denote $\min_{1 \leq j \leq m} a_j$ by $\underline{s}$.

By contraction mapping theorem, there exists a unique $X \in \mathcal{E}$ such that

$$T(X) = \bigcup_{j=1}^{m} f_j(X) = X.$$

Moreover, $T^k(X_0) \to X$ in Hausdorff metric as $k \to \infty$ for any initial $X_0 \in \mathcal{E}$, where T^k is the k-th iteration of T. The unique fixed point X of T is called the self-similar set for the family $\{f_1, f_2, \cdots, f_m\}$. For more details, see [1] and [2].

A compact subset E of $\mathbf{R}^N$ is uniformly perfect if there is a constant $0 < c \leq 1$ such that for any point $x_0 \in E$ and $0 < r < \mathrm{diam}(E)$, the Euclidean annulus $\{x | cr \leq |x - x_0| \leq r\}$ meets E. Uniformly perfect sets were introduced by A.F.Beardon and Ch.Pommerenke (1979) in the complex plane, who showed that the compact set $E \subset \mathbf{C}$ is uniformly perfect if and only if the hyperbolic metric of $\overline{\mathbf{C}} \backslash E$ is comparable to the reciprocal of the distance to the boundary. There are many other characterizations of uniformly perfect plannar sets (see [3]).

The main result of this note is

Theorem 1.1 (Main Theorem). *The self-similar set X for a finite family of contractive similitudes is a uniformly perfect set or a singleton.*

The following statement is an interesting corollary.

Corollary 1.1. *The self-similar set X for a finite family of contractive similitudes has positive Hausdorff dimension except it is a singleton.*

Furthermore, for uniformly perfect sets, we have the following results.

Theorem 1.2. *The upper box dimension of the uniformly perfect set $E \subset \mathbf{R}^N$ with uniform constant $0 < c \leq 1$ has the following inequality,*

$$\frac{\log \frac{1}{2}}{\log \frac{c}{2}} \leq \overline{\dim}_B(E) \leq N.$$

In Section 4, the uniformly perfect set with Hausdorff measure zero in its Hausdorff dimension is also given.

§2. Uniform Perfectness of X

For the semi-group G, the discontinuous set $\Omega \subset \mathbf{R}^N$ of G consists of these points x, so that there is an open ball B centered at x such that there are only finitely many $g \in G$ satisfying $gB \bigcap B \neq \emptyset$. Its complement $\Lambda = \mathbf{R}^N \backslash \Omega$ is called the limit set of G.

Our first result is

Theorem 2.1. $\Lambda = X$.

Proof. For any point $x \in \Omega$, there is an open ball B centered at x such that there are only finitely many $g \in G$ satisfying $gB \bigcap B \neq \emptyset$.

Denote $X_0 = \overline{B}, X_k = \bigcup_{g\in G_k} g(X_0) = T^k(X_0)$. Then $X_k \to X$ in the Hausdorff metric as $k \to \infty$. We conclude that $X_k \bigcap B = \emptyset$ for large k. This implies $x \notin X$.

On the other hand, if $x \notin X$, there is an open ball B centered at x such that B is disjoint with an ε_0-neighborhood $N_{\varepsilon_0}(X)$ of X for some $\varepsilon_0 > 0$. Denote $X_0 = \overline{B}, X_k = T^k(X_0) = \bigcup_{g\in G_k} g(X_0)$. Then $X_k \subset N_{\varepsilon_0}(X)$ for large k. This yields $X_k \bigcap B = \emptyset$ for large k and $\sharp\{g \in G | gB \bigcap B \neq \emptyset\} < \infty$. Hence $x \in \Omega$.

We proved that $\Omega = \mathbf{R}^N \backslash X$, i.e. $\Lambda = X$.

From definitions, it is clear that $gX \subset X, \Omega \subset g\Omega$ for any $g \in G$ and X contains fixed points of mappings $\{f_j | 1 \leq j \leq m\}$. When mappings $\{f_j | 1 \leq j \leq m\}$ have a common fixed point x, take $X_0 = \{x\}$, then $X_k = T(X_{k-1}) = \{x\}$ for all $k \geq 1$ and the self-similar set $X = \{x\}$ is a singleton. Hence X is a singleton if and only if these mappings $\{f_j | 1 \leq j \leq m\}$ have a common fixed point.

The next theorem provides a topological property of X.

Theorem 2.2. *The self-similar set is either a perfect set or a singleton.*

Proof. Suppose that X has an isolated point x. Choose an open ball $B(x, r)$ centered at x with radius $r > 0$ such that $B(x, r) \bigcap X = \{x\}$. From Theorem 2.1, there exists an element $g \in G$ such that $g(B(x, \frac{1}{2}r)) \bigcap B(x, \frac{1}{2}r) \neq \emptyset$. Since $g(B(x, \frac{1}{2}r))$ is also a ball centered at $g(x) \in X$ with radius less than $\frac{1}{2}r$, we have $g(x) \in B(x, r)$ and $g(x) = x$. For large k, $g^{-k}(B(x, r)) = B(x, c^{-k}r) \supset X$, where $0 < c = |g\prime(x)| < 1$. This yields $g^k(X) \subset B(x, r) \bigcap X = \{x\}$. Hence $X = \{x\}$.

The following statement is useful in the proof of our main theorem.

Theorem 2.3. *Suppose that U is an open set intersecting the self-similar set X. Then there exists $g_k \in G_k$ such that $g_k^{-1}U \supset X$ for every sufficiently large k.*

Proof. Let x_0 be a point in $X \bigcap U$. Take a small $r > 0$ such that $B(x_0, r) \subset U$. There is $g_k \in G_k$ such that $g_k^{-1}x_0 \in X$ for every $k \geq 1$. Look at the preimage $g_k^{-1}B(x_0, r)$, it is a ball centered at $g_k^{-1}x_0 \in X$ with radius at least $\overline{s}^{-k}r$. Choose k_0 such that $\overline{s}^{-k_0} > \mathrm{diam}X$, then $g_{k_0}^{-1}B(x_0, r) \supset X$. For every $k \geq k_0$, the element $g_k = g_{k_0} \circ f_{j_1} \circ \cdots \circ f_{j_{k-k_0}} \in G_k$ satisfies

$$g_k^{-1}U = f_{j_{k-k_0}}^{-1} \circ \cdots \circ f_{j_1}^{-1} \circ g_{k_0}^{-1}U \supset f_{j_{k-k_0}}^{-1} \circ \cdots \circ f_{j_1}^{-1}X \supset X,$$

where $1 \leq j_1, \cdots, j_{k-k_0} \leq m$.

In the remainder of this section, we give the proof of our main theorem.

Proof of the Main Theorem. We assume X is not a singleton. Suppose X is not uniformly perfect. Then there is a sequence of round annuli $\{A_n\}$ in $\mathbf{R}^N \backslash X = \Omega$, $A_n = \{x | r_n \leq |x - x_n| \leq R_n\}$ with center x_n in X, separating X such that $\frac{R_n}{r_n} \to +\infty$ as $n \to +\infty$. The condition $R_n \leq \mathrm{diam}X < +\infty$ implies that r_n tends to 0.

From Theorem 2.2, X contains uncountably many points. Fix two points of X which is of distance greater than a given $\delta > 0$. For every $g_k \in G_k$,

$$g_k^{-1}B(x_n, r_n) = B(g_k^{-1}x_n, |(g_k^{-1})'(x_n)| r_n),$$

where $\overline{s}^{-k} \leq |(g_k^{-1})'(x_n)| \leq \underline{s}^{-k}$. From Theorem 2.3 and its proof, we can choose the first integer k_n and an element $g_{k_n} \in G_{k_n}$ for large n such that the diameter of $g_{k_n}^{-1}B(x_n, r_n)$ exceeds $\underline{s}\delta$ and $g_{k_n}^{-1}x_n \in X$. The diameter of $g_{k_n}^{-1}B(x_n, r_n)$ is at most δ.

Denote $\widetilde{A_n} = g_{k_n}^{-1}A_n = \{x | \widetilde{r_n} \leq |x - \widetilde{x_n}| \leq \widetilde{R_n}\}$. Then $\widetilde{A_n} \subset \Omega = \mathbf{R}^N \backslash X$, $\widetilde{x_n} = g_{k_n}^{-1}x_n \in X$ and $\frac{1}{2}\underline{s}\delta \leq \widetilde{r_n} \leq \frac{1}{2}\delta$. Hence $\{x || x - \widetilde{x_n}| \leq \widetilde{R_n}\}$ contains at most one of these two (fixed) points for large n. Consequently $\{x||x-\widetilde{x_n}|>\widetilde{R_n}\}$ intersects X and $\widetilde{R_n}\le \mathrm{diam}X<+\infty$. Since $\frac{\widetilde{R_n}}{\widetilde{r_n}}=\frac{R_n}{r_n}$ tends to ∞, we conclude that $\widetilde{r_n}$ tends to 0. It contradicts with $\widetilde{r_n}\ge\frac{1}{2}\underline{s}\delta$.

This completes the proof of the main theorem.

§3. Hausdorff Dimension of Self-Similar Sets

Let A be a non-empty bounded subset of $\mathbf{R}^N$, and $0\le s\le N$. For each $\delta>0$ let

$$\mathcal{H}^s_\delta(A)=\inf\Big\{\sum_i(\mathrm{diam}(U_i))^s : A \text{ is covered by sets } U_i \text{ with } 0<\mathrm{diam}(U_i)\le\delta\Big\},$$

where the infimum is taken over all coverings of A by a (finite or countable) collection of sets with diameters at most δ. We may define

$$\mathcal{H}^s(A)=\lim_{\delta\to0}\mathcal{H}^s_\delta(A).$$

We call $\mathcal{H}^s(A)$ the s-dimensional Hausdorff measure of A.

It is easy to see that there is a number s at which $\mathcal{H}^s(A)$ jumps from ∞ to 0; we call this number s the Hausdorff (or Hausdorff-Besicovitch) dimension of A which we denote by $\dim_H(A)$. Thus

$$\dim_H(A)=\sup\{s:\mathcal{H}^s(A)=\infty\}=\inf\{s:\mathcal{H}^s(A)=0\}.$$

In this section, we want to prove Corollary 1.1.

Before proving this corollary, we construct a Cantor set which has positive Hausdorff dimension and this Cantor set is a subset of a given uniformly perfect set. This implies the following lemma is true.

Lemma 3.1. *A non-empty uniformly perfect set X has positive Hausdorff dimension.*

Proof. Let c be the constant given by the definition of the uniform perfectness of X. The following is a general observation.

Let x be any point in X, $0<r<\mathrm{diam}(X)$. We divide the radius of $B(x,r)$ into $m=\lfloor 3/c\rfloor+2$ equal segments. For $B(x,\frac{m-1}{m}r)$, it follows from the definition that the closed annulus $A=\{y|c\frac{m-1}{m}r\le|y-x|\le\frac{m-1}{m}r\}$ meets X. Take any point y in $A\cap X$. Then $B(y,\frac{1}{m}r)$ is contained in $B(x,r)$ and the distance $d(B(y,\frac1m r),B(x,\frac1m r))>\frac1m r$ since $c\frac{m-1}{m}r>\frac3m r$.

Now we are going to construct a Cantor set C in X. Denote by d the half of the diameter of X. We start with a point $x_{0,0}\in X$ and take $r=d$. Let $E_0=B(x_{0,0},d)$. Making use of the above observation we find two disjoint balls $B(x_{1,1},\frac1m d)$ and $B(x_{1,2},\frac1m d)$, where $x_{1,1}=x_{0,0}$. The distance between them is greater than $\frac1m d$. Let $E_1=B(x_{1,1},\frac1m d)\cup B(x_{1,2},\frac1m d)$. Then $E_1\subset E_0$. Inductively, we can find E_k in E_{k-1} which is a union of 2^k disjoint balls with centers in X and radii of $m^{-k}d$. And the distance between any two of these balls is greater than $m^{-k}d$. With this construction, if we set $C=\bigcap\limits_{k=0}^{\infty}E_k$, then C is a Cantor set in X.

Take a unit mass on E_0, split it equally between the two balls of E_1, split the mass on each of these equally between the two corresponding balls of E_2, and so on, to get a mass distribution μ on C. Each ball in E_k has mass 2^{-k}. Let U be a subset of C with $\mathrm{diam}(U)<d$, and let k be the integer such that $m^{-(k+1)}d\le\mathrm{diam}(U)<m^{-k}d$. Then U

intersects at most one ball of E_k. Hence

$$\mu(U) \leq 2^{-k} = 2(\frac{1}{d})^{\frac{\log 2}{\log m}}(m^{-(k+1)}d)^{\frac{\log 2}{\log m}} \leq 2(\frac{1}{d})^{\frac{\log 2}{\log m}}(\mathrm{diam}(U))^{\frac{\log 2}{\log m}}$$

for $\mathrm{diam}(U) < d$.

Since $\mu(C) = 1$, the mass distribution principle gives

$$\dim_H(C) \geq \frac{\log 2}{\log m} > 0.$$

This completes the proof.

Proof of Corollary 1.1. From Theorem 1.1 and the above Lemma 3.1, we know this corollary is true.

§4. Hausdorff Measure and Upper Box Dimension of Uniformly Perfect Sets

We now define another frequently used definition of dimension.

Let F be a bounded subset of $\mathbf{R}^N$, and $0 \leq s \leq N$. For $\delta > 0$, let $N_\delta(F)$ be the least number of sets of diameter at most δ that can cover F. We define the lower and upper box-counting dimensions of F as

$$\underline{\dim}_B(F) = \underline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}, \quad \overline{\dim}_B(F) = \overline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}.$$

If these are equal, we call the common value the box-counting dimension, abbreviated to box dimension,

$$\dim_B(F) = \lim_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}.$$

[Note that this is the case if $N_\delta(F) \sim \delta^{-\dim_B(F)}$]. Box dimension has also been called metric dimension, capacity, logarithmic density, entropy dimension, $\cdots$.

We get precisely the same answer if we take $N_\delta(F)$ to be the following:

(a) the least number of (closed) balls of radius δ that cover F;

(b) the least number of sets of diameter at most δ that cover F;

(c) the least number of cubes of side δ that cover F;

(d) the number of cubes of the lattice of side δ that intersect F;

(e) the largest number of disjoint balls of radius δ centred in F.

In the next, we will prove Theorem 1.2.

Proof of Theorem 1.2. Let $N_\delta(E)$ be the largest number of disjoint balls of radius δ centred in E. Then the upper box dimension of E is $\overline{\dim}_B(E) = \overline{\lim}_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta}$.

Let c be the constant in the definition of the uniform perfectness of E. The following is a general observation.

Let x be any point in E, $0 < r < \mathrm{diam}(E)$. For $B(x,r)$, it follows from the definition that the closed annulus $A = \{y | cr \leq |y-x| \leq r\}$ meets E. Take any point y in $A \cap E$. Then $B(x, \frac{c}{2}r) \cap B(y, \frac{c}{2}r) = \emptyset$.

Thus we have $N_{(\frac{1}{2}c)\delta}(E) \geq 2N_\delta(E)$. Inductively, we have $N_{(\frac{1}{2}c)^n\delta}(E) \geq 2^n N_\delta(E)$ for all $n \geq 1$. Fix a $\delta_0 > 0$, we have

$$\overline{\lim_{n\to\infty}} \frac{\log N_{(\frac{1}{2}c)^n\delta_0}(E)}{-\log(\frac{1}{2}c)^n\delta_0} \geq \lim_{n\to\infty} \frac{\log(2^n N_{\delta_0}(E))}{-\log((\frac{1}{2}c)^n\delta_0)} = \frac{\log\frac{1}{2}}{\log\frac{c}{2}}.$$

That is to say, there is a subsequence $\{(\frac{1}{2}c)^n\delta_0\}_{n\geq 1}$ such that

$$\overline{\lim_{n\to\infty}}\frac{\log N_{(\frac{1}{2}c)^n\delta_0}(E)}{-\log(\frac{1}{2}c)^n\delta_0}\geq \frac{\log\frac{1}{2}}{\log\frac{c}{2}}.$$

Then

$$\overline{\dim}_B(E)=\overline{\lim_{\delta\to 0}}\frac{\log N_\delta(E)}{-\log\delta}\geq\frac{\log\frac{1}{2}}{\log\frac{c}{2}}.$$

This completes the proof.

Evidently, when the uniformly perfect set is a segment with $c=1$ in $\mathbf{R}^1$, in the inequality of Theorem 1.2, the equality holds.

In the above section, we have proved that the Hausdorff dimension of the uniformly perfect set $E\subset\mathbf{R}^N$ with the uniform constant $0<c\leq 1$ has the following inequality,

$$\frac{\log 2}{\log\left([\frac{3}{c}]+2\right)}\leq \dim_H(E)\leq N.$$

It is to say that the Hausdorff dimension of a uniformly perfect set is positive. In the next, we will prove there is a uniformly perfect set whose Hausdorff measure is zero.

Self-similar sets are among the most important and the most typical fractals, which were first considered by Moran[4] and systematically studied by Hutchinson[5]. For self-similar sets, the Hausdorff dimension and Upper Box dimension coincide.

Let Δ be the one-dimensional Sierpinski gasket as in [6, Fig.5.1]. In [7, p.214] one finds a method due to Kahane to prove that $\mathcal{H}^1(P_L\Delta)=0$ for $\gamma_{2,1}$ almost all $L\in G(2,1)$. However, it seems to be difficult to decide for which lines L this holds. Kenyon[8] showed that $\mathcal{H}^1(P_L\Delta)=0$ if the angle between L and the x-axis is irrational.

Applying Corollary 9.4 and Theorem 18.1 in [6] to self-similar sets such as Δ one obtains self-similar subsets K of $\mathbf{R}$ with $\dim_H(K)=1$ and $\mathcal{L}^1(K)=0=\mathcal{H}^1(K)$.

Meanwhile, we know the similar set generated by $\{f_1,f_2,\cdots,f_m\}$ is either a uniformly perfect set or a singleton. Since $\dim_H(K)=1$, we know K is not a singleton. This uniformly perfect set K has zero Hausdorff measure.

In the next, we give some exact examples about uniformly perfect sets with zero Hausdorff measure.

Let the set S be a self-similar set in $\mathbf{R}^2$ for the three contracting linear maps

$$f_1:(x,y)\mapsto\Big(\frac{x}{3},\frac{y}{3}\Big), f_2:(x,y)\mapsto\Big(\frac{x+1}{3},\frac{y}{3}\Big), f_3:(x,y)\mapsto\Big(\frac{x}{3},\frac{y+1}{3}\Big).$$

It is easy to see S is the set of points in $\mathbf{R}^2$ with an expansion in base 3 using negative powers of the base and digits $\{(0,0),(1,0),(0,1)\}$, that is,

$$S=\Big\{\sum_{i=1}^{\infty}\alpha_i 3^{-i}|\alpha_i\in\{(0,0),(1,0),(0,1)\}\Big\}.$$

See Figure 1.

Fig. 1. The set S

Since the set S is self-similar and satisfies the open set condition[5], the Hausdorff dimension of S is one. We have called S the one-dimensional Sierpinski gasket. In [8], Kenyon defined S_u to be the linear projection of S onto the x-axis, $S_u = \pi_u(S)$, where π_u sends $(0,1)$ to the point $(u,0)$, that is,

$$\pi_u = \begin{pmatrix} 1 & u \\ 0 & 0 \end{pmatrix}.$$

See Figure 2. For example, S_0 is the usual "middle third" Cantor set on the interval $[0,1/2]$, $S_{\frac{1}{2}}$ is the interval $[0,1/2]$.

Fig. 2

Kenyon proved S_u has one-dimensional Lebesgue measure zero when the number u is irrational. And he also proved that if u is irrational and $\{\frac{p_i}{q_i}\}_{i\geq 1}$ a sequence of rationals such that $p_i+q_i \equiv 0 \bmod 3$, $q_i \to \infty$, and there exist constants $C, \alpha > 0$ for which $|u - \frac{p_i}{q_i}| < \frac{C}{q_i^{\alpha}}$, then $\dim_{\mathcal{H}}(S_u) \geq 1 - \frac{1}{\alpha}$.

Let M be a positive integer,

$$X_M = \Big\{ u \text{ is irrational in } \mathcal{R} | \text{There is } \Big\{\frac{p_i}{q_i}\Big\}_{i\geq 1} \text{ a sequence of rationals such that } p_i+q_i \equiv 0 \bmod 3,\ q_i \to \infty, \text{ and there exist constants } C, M > 0 \text{ for which } |u - \frac{p_i}{q_i}| < \frac{C}{q_i^M} \Big\},$$

and let $X = \bigcap\limits_{k=1}^{\infty} X_k$. Then by the results of Kenyon, for every $u \in X$, $\dim_{\mathcal{H}}(S_u) = 1$. In the next we will prove X is not empty.

Let $u = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \cdots + \frac{1}{10^{n!}} + \cdots$ and $\frac{p_i}{q_i} = \frac{1}{10^{1!}} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \cdots + \frac{1}{10^{i!}}$. We have

$$\left|u-\frac{p_i}{q_i}\right|=\frac{1}{10^{(i+1)!}}+\cdots\le 2\times\frac{1}{10^{(i+1)!}}=\frac{2}{(10^{i!})^{i+1}}\le\frac{2}{q_i^M}$$

for all $i+1\ge M$, where $q_i=10^{i!}$. It is easy to take a subsequence $\{\frac{p_{i_k}}{q_{i_k}}\}_{k\ge1}$ such that $p_{i_k}+q_{i_k}\equiv 0 \bmod 3$. This means $u\in X$.

Let $X_u\subset\mathbf{R}$ be a self-similar set for the three linear maps

$$x\mapsto\frac{x}{3},\quad x\mapsto\frac{x+1}{3},\quad x\mapsto\frac{x+u}{3}.$$

It is easy to see that it is the set of real numbers which have an expansion in base 3 using negative powers of 3 and digits $\{0,1,u\}$. It is enough to prove $S_u=X_u$. For $y\in S_u$, there is an $x\in S$ such that $y=\pi_u(x)=\pi_u\Big(\sum\limits_{i=1}^{\infty}\alpha_i3^{-i}\Big)=\sum\limits_{i=1}^{\infty}\pi_u(\alpha_i)3^{-i}=\sum\limits_{i=1}^{\infty}\beta_i3^{-i}$, where $\alpha_i\in\{(0,0),(1,0),(0,1)\},\beta_i\in\{(0,0),(1,0),(u,0)\}$, so $y\in X_u$. Conversely it is also true.

Thus if $u\in X$, for example, $u=\frac{1}{10^{1!}}+\frac{1}{10^{2!}}+\frac{1}{10^{3!}}+\cdots+\frac{1}{10^{n!}}+\cdots$, then S_u is a uniformly perfect set with Hausdorff measure zero in its Hausdorff dimension.

§5. A Counterexample

A mapping $f:\mathbf{R}^N\to\mathbf{R}^N$ is called contractive if there exists a constant $0<c<1$ such that $|f(x_1)-f(x_2)|\le c\cdot|x_1-x_2|$ for all $x_1,x_2\in\mathbf{R}^N$.

For a family of contractive mappings $\{f_1,f_2,\cdots,f_m\}$, which are not necessarily similitudes in $\mathbf{R}^N$, there is also an attractor X such that $X=\bigcup\limits_{j=1}^{m}f_j(X)$. But Theorem 2.2 and the main theorem are not true generally. Now we construct a counterexample in $\mathbf{R}$.

Define a function $f\in C^1(\mathbf{R})$ satisfying $f(x)=x(x-1)$ in $[0,1]$, $f(x)=1$ in $(-\infty,-1]\cup[2,+\infty)$, and $f(x)>0$ in $(-1,0)\cup(1,2)$. Let $f_1(x)=\frac{1}{2M}f(x), f_2(x)=f_1(x)+1$, where $M=\max\limits_{x\in\mathbf{R}}|f'(x)|$. Then f_1 and f_2 are contractive, and $f_1(0)=f_1(1)=0, f_2(0)=f_2(1)=1$. Denote $X_0=\{0,1\}$. Then $X_k=f_1(X_{k-1})\cup f_2(X_{k-1})=\{0,1\}$ for all $k\ge1$. Hence the invariant set X of $\{f_1,f_2\}$ consists of two points.

References

[1] Barnslay, M. F., Fractals everywhere, Academic Press, 1988.
[2] Falconer, K. J., Fractal geometry: Mathematical foundations and applications, John Wiley & Sons, 1990.
[3] Pommerenke, Ch., Uniformly perfect sets and the Poincaré metric, *Arch. Math.,* **32**(1979), 192–199.
[4] Moran, P. A. P., Additve functions of intervals and Hausdorff measure, *Proc. Camb. Philos. Soc.,* **42**(1946), 15–23.
[5] Hutchinson, J. E., Fractal and self-similarity, *Indiana Univ. Math. J.*, **30**(1981), 713–747.
[6] Mattila, P., Geometry of sets and measures in Euclidean spaces (Fractals and rectifiability), Cambridge University Press, 1995.
[7] de Guzmán, M., Real variable methods in Fourier analysis, North-Holland, 1981.
[8] Kenyon, R., Projecting the one-dimensional Sierpinski gasket, *Isr. J. of Math.*, **97**(1997), 221–238.