Chin. Ann. Math.
24B:1(2003),65-72.

BOUNDEDNESS OF HIGH ORDER RIESZ-BESSEL TRANSFORMATIONS GENERATED BY GENERALIZED SHIFT OPERATOR ON Ba SPACES

A. ŞERBETÇI*   İ. EKINCIOĞLU**

Abstract

The boundedness of high order Riesz-Bessel transformations generated by generalized shift operator on Ba Spaces $(L_{p_{m,\nu}}(\mathbb{R}_n^+), a_m)$ is examined.

Keywords Generalized shift operator, Riesz-Bessel transformations, Ba spaces
2000 MR Subject Classification 47G10, 45E10, 47B37
Chinese Library Classification O177.6, O175.5   **Document Code** A
Article ID 0252-9599(2003)01-0065-08

§1. Introduction

A new class of function spaces, denoted by Ba, was introduced by X.X. Ding in [4]. This class of spaces is a very natural generalization of the classical L_p spaces and also includes some important Orlicz spaces, Orlicz-Sobolev spaces, etc.

It is well known that the kernel of classical Riesz transformations R_j depends on ordinary shift operator and the operator R_j is bounded on $L_{p,\nu}$ spaces[9]. The kernel of high order Riesz-Bessel transformations R_{B_j} depends on generalized shift operator and the operator R_{B_j} is bounded on L_{p_ν} weighted spaces[7]. The boundedness of classical Riesz transformations in Ba spaces were investigated by W.D. Chang[3]. The aim of this paper is to prove the boundedness of the high order Riesz-Bessel transformations R_{B_j}generated by a generalized shift operator on Ba spaces.

§2. Ba Spaces and High Order Riesz-Bessel Transformations Generated By a Generalized Shift Operator

Let $B = \{ B_1, \dots , B_m, \cdots \}$ be a sequence of Banach function spaces and $a = \{a_1, a_2, \cdots ,$ $a_m, \cdots \}$ be a sequence of non-negative real numbers. Let $\phi(z) = \sum\limits_{m=1}^{\infty} a_m z^m$ be an entire

Manuscript received October 15, 2001.
*Department of Mathematics, Ankara University, 06100 Tandogan-Ankara, Turkey.
E-mail: serbetci@science.ankara.edu.tr
**Department of Mathematics, Dumlupınar University, Kütahya, Turkey.
E-mail: ekinci@dumlupinar.edu.tr

function. For $f \in \bigcap\limits_{m=1}^{\infty} B_m$, we form a power series as follows

$$I(f,\alpha) = \sum_{m=1}^{\infty} a_m \, \|f\|_{B_m}^m \, \alpha^m,$$

where $\|.\|_{B_m}$ is the B_m-norm of f. Let R_f denote the radius of convergence of the series $I(f,\alpha)$ and Ba the following function set

$$\mathrm{B}a = \Big\{ f: \ f \in \bigcap_{m=1}^{\infty} B_m, R_f > 0 \Big\}.$$

The set Ba is proved to be a Banach space when we define the norm of an element $f \in \mathrm{B}a$ by

$$\|f\|_{\mathrm{B}a} = \inf_{\alpha>0} \Big\{ \frac{1}{\alpha}: \ I(f,\alpha) \le 1 \Big\}$$

(see [3,5]).

In this paper we will restrict the Banach spaces

$$B_m = L_{p_m,\nu}(\mathbb{R}_n^+) = \Big\{ f: \ \|f\|_{L_{p_m,\nu}} \equiv \Big(\int_{\mathbb{R}_n^+} |f(x)|^{p_m} x_n^{2\nu} dx \Big)^{\frac{1}{p_m}} < \infty \Big\},$$

where $\nu > 0$ is a fixed parameter, $\mathbb{R}_n^+ = \{x: \ x = (x_1, x_2, \cdots, x_n), \ x_n > 0\}$, and $1 < p_m < \infty$, $m = 1, 2, \cdots$. For simplicity, we will denote $\|.\|_{Ba}$ by $\|.\|$ and $\|.\|_{L_{p_m,\nu}}$ by $\|.\|_{p_m,\nu}$.

Denote by T^y the generalized shift operator acting according to the law

$$T^y \ f(x) = \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu)\Gamma(\frac{1}{2})} \int_0^{\pi} f \ \Big[x' - y', \ \sqrt{x_n^2 + y_n^2 - 2x_n y_n \cos\alpha} \Big] \sin^{2\nu-1} \alpha d\alpha,$$

where $x = (x', x_n)$ and $y = (y', y_n)$ with $x', y' \in \mathbb{R}_{n-1}$. We remark that T^y is closely connected with the Bessel differential operator

$$B_{x_n} = \frac{\partial^2}{\partial x_n^2} + \frac{2\nu}{x_n} \frac{\partial}{\partial x_n}, \quad x_n > 0$$

and Δ_B, the Laplacean-Bessel operator,

$$\Delta_B = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2} + \frac{\partial^2}{\partial {x_n}^2} + \frac{2\nu}{x_n} \frac{\partial}{\partial x_n}, \qquad \nu > 0$$

(see [8]).

The shift T^y generates the corresponding convolution (B-convolution)

$$(f_1 * f_2)(y) = \int_{\mathbb{R}_n^+} f_1(x) \big[T^x f_2(y) \big] x_n^{2\nu} \ dx.$$

We note the property of the "B-convolution": $(f_1 * f_2) = (f_2 * f_1)$.

The goal of this paper is to have the proof of boundedness of the singular integral operators generated by the generalized shift operator on Ba:

$$R_{B_j} \ f(x) \ = \ c_1 \ (n,\nu) \ \lim_{\varepsilon \to 0} \int\limits_{\substack{0<\varepsilon<|y| \\ 0<y_n<\infty}} \frac{y_j}{|y|^{n+1+2\nu}} \ T^y \ f(x) y_n^{2\nu} \ dy,$$

where

$$c_1(n,\nu) = 2^{\frac{n+2\nu}{2}} \ \Gamma\Big(\frac{n+1+2\nu}{2} \Big) \frac{1}{\sqrt{\pi}}$$

(see [1,2,6] for details). Boundedness of singular integrals generated by generalized shift operator on $L_{p,\nu}(\mathbb{R}_n^+)$, denoted by R_{B_j}, were investigated by Ekincioğlu and Serbetci in [7] that

$$\left\|R_{B_j}(f)\right\|_{p,\nu} \le A_p \|f\|_{p,\nu}, \quad f \in L_{p,\nu}.$$

In addition,

$$A_p = \mathcal{O}((p-1)^{\frac{3}{2}}) \qquad \text{as } p \to 1^+,\ A_p = \mathcal{O}(p) \qquad \text{as } p \to \infty.$$

We first prove two lemmas needed to facilitate the computations.

Lemma 2.1.

$$\int_{\{s\in\mathbb{R}_n^+:\ |s|>\epsilon\}} \frac{s_j}{|s|^{n+1+2\nu}} \left[T^s(f)(x)\right] s_n^{2\nu} ds = \frac{\Gamma(\nu+\frac{1}{2})}{2\Gamma(\nu)\Gamma(\frac{1}{2})} \int_{|\tilde{x}-\tilde{y}|>\epsilon} \frac{x_j - y_j}{|\tilde{x}-\tilde{y}|^{n+1+2\nu}} \times f(y', \sqrt{y_n^2 + y_{n+1}^2})\ y_{n+1}^{2\nu-1} d\tilde{y},$$

where $\tilde{x} = (x', x_n, 0),\ x' = (x_1, \cdots, x_{n-1}),\ \tilde{y} = (y', y_n, y_{n+1})$ *and* $d\tilde{y} = dy_1 \cdots dy_n dy_{n+1}$.

Proof.

$$\begin{aligned} I &= \int_{\{s\in\mathbb{R}_n^+:\ |s|>\epsilon\}} \frac{s_j}{|s|^{n+1+2\nu}} \left[T^s(f)(x)\right] s_n^{2\nu} ds \\ &= \int_{|s|>\epsilon} \frac{s_j}{|s|^{n+1+2\nu}} \Big\{ \frac{\Gamma(\nu+\frac{1}{2})}{2\Gamma(\nu)\Gamma(\frac{1}{2})} \\ &\quad \times \int_0^\pi f\ \left[x' - s',\ \sqrt{x_n^2 + s_n^2 - 2x_n s_n \cos\alpha}\right] \sin^{2\nu-1}\alpha d\alpha \Big\} s_n^{2\nu} ds \\ &= \int_{|s|>\epsilon} \frac{s_j}{|s|^{n+1+2\nu}} \\ &\quad \times \Big\{ \frac{\Gamma(\nu+\frac{1}{2})}{2\Gamma(\nu)\Gamma(\frac{1}{2})} \int_0^\pi f\ (x' - s',\ \sqrt{(x_n - s_n\cos\alpha)^2 + (s_n \sin\alpha)^2}) \sin^{2\nu-1}\alpha d\alpha \Big\} s_n^{2\nu} ds, \end{aligned}$$

we pass to the new variables $\widetilde{x} = (x_1, x_2, \cdots, x_n, 0),\ \widetilde{y} = (y', \cdots, y_n, y_{n+1})$: $x' - s' = y',\ y_n = x_n - s_n\cos\alpha,\ y_{n+1} = s_n \sin\alpha,\ 0 \le \alpha < \pi$. Since the Jacobian of the transformation is equal to s_n^{-1}, we have

$$I = \frac{\Gamma(\nu+\frac{1}{2})}{2\Gamma(\nu)\Gamma(\frac{1}{2})} \int_{|\tilde{x}-\tilde{y}|>\epsilon} \frac{x_j - y_j}{|\tilde{x}-\tilde{y}|^{n+1+2\nu}} f\big(y', \sqrt{y_n^2 + y_{n+1}^2}\big)\ y_{n+1}^{2\nu-1} d\tilde{y}.$$

Lemma 2.2.

$$\int_{\mathbb{R}_n^+} f(s) s_n^{2\nu} ds = \frac{\Gamma(\nu+\frac{1}{2})}{\Gamma(\nu)\Gamma(\frac{1}{2})} \int_{\mathbb{R}_{n+1}^+} f\Big(y', \sqrt{y_n^2 + y_{n+1}^2}\Big)\ y_{n+1}^{2\nu-1} dy\ dy_{n+1}.$$

Proof. The proof is obtained by means of the substitution $y' = s',\ y_n = s_n\cos\alpha,\ y_{n+1} = s_n \sin\alpha,\ 0 \le \alpha < \pi,\ s_n > 0$.

§3. The Boundedness of R_{B_j} in B*a* Spaces

It is natural to inquire about Riesz-Bessel transforms actions on $(L_{p_{m,\nu}}, a_m)$. Our question is: for what sequences $\{p_m\}$ and $\{a_m\}$, is the operator R_{B_j} bounded in $(L_{p_{m,\nu}}, a_m)$? That is, when do we have the inequality

$$\left\|R_{B_j}(f)\right\| \le A_p \|f\|, \qquad f \in \mathrm{B}a?$$

It turns out that the boundedness of R_{B_j} depends only on the sequence $\{p_m\}$, not on $\{a_m\}$. Therefore, we have

Theorem 3.1. *The Riesz-Bessel transformations R_{B_j} $j = 1, 2, \cdots, n-1$ are bounded operators on Ba spaces if and only if there exist constants α and β such that the sequence $\{p_m\}$ satisfies*

$$1 < \alpha < p_m < \beta, \qquad \text{for all } a_m \neq 0. \tag{3.1}$$

Proof. Suppose that (3.1) holds. Then by continuity, constants $A_p(\alpha)$ and $A_p(\beta)$ are both finite and there exists a positive number K such that $A_p(\alpha) \le K$. Therefore, if (3.1) is satisfied, we have $A_{p_m}(\alpha) \le K,\ m = 1, 2, \cdots$. Now let $f \in Ba$. By the definition of the Ba-norm of f, we have

$$I\Big(f, \frac{1}{\|f\|}\Big) = \sum_{m=1}^{\infty} \frac{a_m \|f\|^m_{p_{m,\nu}}}{\|f\|^m} \le 1,$$

and so

$$I\Big(R_{B_j}(f), \frac{1}{K\|f\|}\Big) = \sum_{m=1}^{\infty} \frac{a_m \left\|R_{B_j}(f)\right\|^m_{p_{m,\nu}}}{(K\|f\|)^m} \le \sum_{m=1}^{\infty} \frac{a_m A^m_{p_m} \|f\|^m_{p_{m,\nu}}}{(K\|f\|)^m} \le 1.$$

This implies that

$$\left\|R_{B_j}(f)\right\| = \inf_{\alpha>0} \Big\{\frac{1}{\alpha} :\ I(R_{B_j}(f), \alpha) \le 1\Big\} \le K\|f\|.$$

The sufficiency has been proved.

In order to prove the necessity of condition (3.1), we need some estimates. For any positive number l, we define two regions $I(l)$ and $J(l)$ in $\mathbb{R}^+_n$:

$$I(l) = \big\{x :\ |x| \le l,\ x_j > 0\ j = 1, 2, \cdots, n\big\},$$
$$J(l) = \big\{x :\ |x| > 2l,\ 0 < \theta_j < \frac{\pi}{4}\big\},$$

where θ_j is the angle between $\vec{ox}$ and the jth axis. We define the characteristic function $f_l(x)$ by

$$f_l(x) = \begin{cases} 1, & x \in I(l), \\ 0, & \text{otherwise} . \end{cases}$$

We wish to prove that

$$\left\|R_{B_j}(f_l)(x)\right\|_{p,\nu} \ge B(p)\,\|f_l(x)\|_{p,\nu} \qquad \text{for } 1 < p < 2, \tag{3.2}$$

where $B(p)$ is independent of l and $B(p) \to \infty$ as $p \to 1^+$, and

$$\left\|R_{B_j}(f_l)(x)\right\|_{p,\nu} \ge C(p)\,\|f_l(x)\|_{p,\nu} \qquad \text{for } 2 \le p < \infty, \tag{3.3}$$

where $C(p)$ is independent of l and $C(p) \to \infty$ as $p \to \infty$. If we use Lemma 2.1, then for any $y \in I(l)$ and $x \in J(l)$, we have

$$\begin{aligned} R_{B_j}\ (f_l)(x) &= c_\nu\ p.v.\int_{\mathbb{R}^+_n} \frac{y_j}{|y|^{n+1+2\nu}}\,[T^y_x\ (f_l)(x)]\, y_n^{2\nu}\ dy \\ &= c_\nu \int_{|\tilde{x}-\tilde{y}|>\epsilon} \frac{x_j - y_j}{|\tilde{x}-\tilde{y}|^{n+1+2\nu}} f_l\Big(y', \sqrt{y_n^2 + y_{n+1}^2}\Big) y_{n+1}^{2\nu-1}\ d\tilde{y}. \end{aligned}$$

Since $x_j - y_j + l \ge x_1$ and $\frac{x_j}{|x|} = \cos\theta_j \ge \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$,

$$x_j - y_j \ge \frac{\sqrt{2}}{2}|x| - l \ge \frac{\sqrt{2}}{2}|x| - \frac{|x|}{2} = \Big(\frac{\sqrt{2}}{2} - \frac{1}{2}\Big)|x|.$$

In addition, as $|x| > 2l$, $|\tilde{y}| \leq l$ and, $|\tilde{x}| = x$, it follows that

$$|\tilde{x} - \tilde{y}| \leq |\tilde{x}| + |\tilde{y}| \leq |x| + l \leq \frac{3}{2}|x|.$$

Thus, for any $x \in J(l)$,

$$|R_{B_j}\ (f_l)(x)| \geq \left[c_\nu(\sqrt{2}-1)\frac{3^{n+1+2\nu}}{2^{n+2+2\nu}}\right] \int_{\mathbb{R}^+_{n+1}(\tilde{y})} \frac{1}{|x|^{n+2\nu}} \left|f_l\Big(y', \sqrt{y_n^2 + y_{n+1}^2}\Big)\right| |y_{n+1}|^{2\nu-1}\ d\tilde{y}.$$

By Lemma 2.2, we have

$$\begin{aligned}
|R_{B_j}\ (f_l)(x)| &\geq \left[c_\nu(\sqrt{2}-1)\frac{3^{n+1+2\nu}}{2^{n+2+2\nu}}\right] \frac{1}{|x|^{n+2\nu}} \int_{\mathbb{R}^+_n(s)} |f_l(s)||s_n|^{2\nu}\ ds \\
&= \left[c_\nu(\sqrt{2}-1)\frac{3^{n+1+2\nu}}{2^{n+2+2\nu}}\right] \frac{1}{|x|^{n+2\nu}} \int_{I(l)} |s_n|^{2\nu}\ ds \\
&= \left[c_\nu(\sqrt{2}-1)\frac{3^{n+2\nu}}{2^{n+1+2\nu}} c \frac{l^{n+2\nu}}{n+2\nu}\right] |x|^{-n-2\nu},
\end{aligned}$$

where the constant c is defined by

$$\int_{s\in I(l)} |s_n|^{2\nu}\ ds = c\int_0^l r^{n+2\nu-1} dr = c\frac{l^{n+2\nu}}{n+2\nu}. \tag{3.4}$$

Let $c' = \dfrac{c_\nu(\sqrt{2}-1)3^{n+2\nu}}{(n+2\nu)2^{n+1+2\nu}}$. A simple computation shows that

$$\begin{aligned}
\left\|R_{B_j} f_l(x)\right\|_{p,\nu} &\geq \Big(\int_{x\in J(l)} |R_{B_j} f_l(x)|^p x_n^{2\nu}\ dx\Big)^{\frac{1}{p}} \\
&\geq c' l^{n+2\nu} \Big(\int_{x\in J(l)} |x|^{-(n+2\nu)} x_n^{2\nu} dx\Big)^{\frac{1}{p}} \\
&= c' l^{n+2\nu} \Big(c'' \int_{r=2l}^{\infty} r^{-(n+2\nu)p+n+2\nu-1} dr\Big)^{\frac{1}{p}} \\
&= c'(c'')^{\frac{1}{p}} l^{n+2\nu} \left[\frac{(2l)^{(n+2\nu)-(n+2\nu)p}}{(n+2\nu)p-(n+2\nu)}\right]^{\frac{1}{p}} \\
&= c'(c'')^{\frac{1}{p}} \left\{\frac{2^{\frac{(n+2\nu)-(n+2\nu)p}{p}}}{[(n+2\nu)p-(n+2\nu)]^{\frac{1}{p}}}\right\} l^{\frac{n+2\nu}{p}} \\
&= c'(c'')^{\frac{1}{p}} \left\{\frac{2^{\frac{(n+2\nu)-(n+2\nu)p}{p}} (n+2\nu)^{\frac{1}{p}}}{c^{\frac{1}{p}}[(n+2\nu)p-(n+2\nu)]^{\frac{1}{p}}}\right\} \|f_l\|_{L_{p,\nu}},
\end{aligned}$$

where we have used $\|f_l\|_p = \Big(\dfrac{cl^{n+2\nu}}{n+2\nu}\Big)^{\frac{1}{p}}$ and c'' is a constant independent of l and p. If we let

$$B(p) = c'(c'')^{\frac{1}{p}} \left\{\frac{2^{\frac{(n+2\nu)-(n+2\nu)p}{p}} (n+2\nu)^{\frac{1}{p}}}{c^{\frac{1}{p}}[(n+2\nu)p-(n+2\nu)]^{\frac{1}{p}}}\right\},$$

then $B(p)$ is independent of l and tends to infinity as $p \to 1^+$. Hence, (3.2) is proved.

Next, we show that the estimate of the form (3.3) is also true. In fact, if it were not the case, then there would exist some constant k such that

$$\left\|R_{B_j} f_l(x)\right\|_{p_m,\nu} \leq k \left\|f_l(x)\right\|_{p_m,\nu}, \qquad m = 1, 2, \cdots, \text{ and } p_m \to \infty \text{ as } m \to \infty,$$

where k is independent of p_m and l. In particular,

$$\Big(\int_{I(l)} |R_{B_j} f_l(x)|^{p_m} x_n^{2\nu}\ dx\Big)^{\frac{1}{p_m}} < k \left\| f_l(x) \right\|_{p_m,\nu}.$$

Now letting $m \to \infty$, we have (note $\left\| f_l(x) \right\|_{\infty,\nu} = 1$) $|R_{B_j} f_l(x)| \le k$ almost everywhere in $I(l)$, or

$$\begin{aligned}
&\Big|\int_{I(l)} \frac{y_j}{|y|^{n+1+2\nu}} \left[T_x^y f_l(x)\right] y_n^{2\nu}\ dy\Big| \\
&= \Big|\int_{|\tilde{x}-\tilde{y}|>\epsilon} \frac{x_j - y_j}{|\tilde{x}-\tilde{y}|^{n+1+2\nu}} f_l\Big(y', \sqrt{y_n^2 + y_{n+1}^2}\Big) y_{n+1}^{2\nu-1}\ d\tilde{y}\Big| \\
&= \Big|\int_{|\tilde{x}-\tilde{y}|>\epsilon} \frac{x_j - y_j}{|\tilde{x}-\tilde{y}|^{n+1+2\nu}}\ y_{n+1}^{2\nu-1}\ d\tilde{y}\Big| \\
&\le k.
\end{aligned}$$

Let $x \to 0$, we have

$$\int_{|\tilde{y}|>\epsilon} \frac{y_j}{|\tilde{y}|^{n+1+2\nu}}\ y_{n+1}^{2\nu-1}\ d\tilde{y} \le k,$$

which is a contradiction because the integration is divergent.

Now we can prove the necessity of condition (3.1). Let us suppose that the Riesz-Bessel transformations $R_{B_j},\ j = 1, 2, \cdots, n-1$, are bounded operators on Ba spaces. Then, for all $f \in Ba$ there exists a constant A, independent of f, such that

$$\left\| R_{B_j} f \right\| \le A \left\| f \right\|. \tag{3.5}$$

By using (3.2) and (3.5), it follows from the definition of the Ba-norm that

$$\begin{aligned}
&\sum_{m=1}^{\infty} \Big\{a_m \big[B(p_m) \left\| f_l(x) \right\|_{p_m,\nu}\big]^m / (A \left\| f_l \right\|)^m\Big\} \\
&\le \sum_{m=1}^{\infty} \Big\{a_m \big(\left\| R_{B_j} f_l(x) \right\|_{p_m,\nu}\big)^m / \left\| R_{B_j} f_l \right\|^m\Big\} = 1.
\end{aligned}$$

In particular,

$$a_m^{\frac{1}{m}} B(p_m) \left\| f_l \right\|_{p_m,\nu} / \big(A \left\| f_l \right\|\big) \le 1,$$

or

$$a_m^{\frac{1}{m}} \left\| f_l \right\|_{p_m,\nu} / \left\| f_l \right\|) \le A / B(p_m). \tag{3.6}$$

Note that as $p_m \to 1^+,\ B(p_m) \to \infty$, so, if α in (3.1) does not exist, we may find a $p_{m'} > 1$ such that

$$a_m^{\frac{1}{m}} \left\| f_l \right\|_{p_m,\nu} / \left\| f_l \right\|) < \frac{1}{2} \qquad \text{for } p_m \in (1, p_{m'}] \text{ and } l \in (0, \infty). \tag{3.7}$$

Without loss of generality, we assume that there exists an $a_{m''}$ such that $a_{m''} \ne 0,\ p_{m''} < p_{m'}$ and

$$0 < a_{m''}^{\frac{1}{m''}} \left\| f_l \right\|_{p_{m''},\nu} / \left\| f_l \right\|) < \frac{1}{2}. \tag{3.8}$$

Now choose l_0 large enough so that $c\frac{l_0^{n+2\nu}}{n+2\nu} > 1$ and

$$M\Big(c\frac{l_0^{n+2\nu}}{n+2\nu}\Big)^{\frac{1}{p_{m'}}} < a_{m''}^{\frac{1}{m''}} \Big(c\frac{l_0^{n+2\nu}}{n+2\nu}\Big)^{\frac{1}{p_{m''}}},$$

where $M = \sup(a_m^{\frac{1}{m}},\ m = 1, 2, \cdots) < \infty$ and c is defined by (3.4). We can see for any $p_m > p_{m'}$,

$$a_m^{\frac{1}{m}}\Big(c\frac{l_0^{n+2\nu}}{n+2\nu}\Big)^{\frac{1}{p_m}} \le M\Big(c\frac{l_0^{n+2\nu}}{n+2\nu}\Big)^{\frac{1}{p_{m'}}} \le a_{m''}^{\frac{1}{m''}}\Big(c\frac{l_0^{n+2\nu}}{n+2\nu}\Big)^{\frac{1}{p_{m''}}},$$

so, by using (3.8) and the fact

$$\|f_{l_0}\|_{p,\nu} = (c\frac{l_0^{n+2\nu}}{n+2\nu}),$$

we have for any $p_m > p_{m'}$,

$$a_m^{\frac{1}{m}} \|f_{l_0}\|_{p_m,\nu} / \|f_{l_0}\| \le a_{m''}^{\frac{1}{m''}} \|f_{l_0}\|_{p_{m''},\nu} / \|f_{l_0}\| < \frac{1}{2}. \tag{3.9}$$

(3.7) and (3.9) together give

$$\sum_{m=1}^{\infty} a_m \|f_{l_0}\|_{p_m,\nu}^m / \|f_{l_0}\|^m < \sum_{m=1}^{\infty} (\frac{1}{2})^m = 1,$$

which is a contradiction to the definition of the Ba-norm, since it can be easily checked that $I(f_{l_o}, 1/\|f_{l_o}\|) = 1$.

Next we prove the existence of the β in the theorem. Using (3.3) and (3.5), we obtain (by using similar estimates)

$$\sum_{m=1}^{\infty} \left\{a_m[C(p_m)\|f_l\|_{p_m,\nu}]^m/(A\|f_l\|)^m\right\} \le 1.$$

Thus

$$a_m^{\frac{1}{m}} \|f_l\|_{p_m,\nu} / \|f_l\| \le A/C(p_m).$$

Note that as $p_m \to \infty,\ C(p_m) \to \infty$; so, if β does not exist, we can find $p_{m'}$ large enough such that

$$a_m^{\frac{1}{m}} \|f_l\|_{p_m,\nu} / \|f_l\| < \frac{1}{2}, \qquad \text{for } p_m \in [p_{m'}, \infty) \text{ and } l \in (0, \infty). \tag{3.10}$$

Similarly, we may assume that there exists a positive integer m'' such that $p_{m''} > p_{m'}$ and

$$0 < a_{m''}^{\frac{1}{m''}} \|f_l\|_{p_m,\nu} / \|f_l\| < \frac{1}{2}, \qquad \text{for } l \in (0, \infty). \tag{3.11}$$

Choose l_1 small enough such that $c\frac{l_1^{n+2\nu}}{n+2\nu} < 1$ and

$$M\Big(c\frac{l_1^n}{n}\Big)\frac{1}{p_{m'}} < a_{m''}^{\frac{1}{m''}}\Big(c\frac{l_1^{n+2\nu}}{n+2\nu}\Big)\frac{1}{p_{m''}}.$$

Then for any $p_m < p_{m'}$,

$$a_m^{\frac{1}{m}}\Big(c\frac{l_1^{n+2\nu}}{n+2\nu}\Big)\frac{1}{p_m} \le M\Big(c\frac{l_1^n}{n}\Big)\frac{1}{p_{m'}} < a_{m''}^{\frac{1}{m''}}\Big(c\frac{l_1^{n+2\nu}}{n+2\nu}\Big)\frac{1}{p_{m''}}.$$

Thus, from (3.11) we have, for any $p_m < p_{m'}$,

$$a_m^{\frac{1}{m}} \|f_l\|_{p_m,\nu} / \|f_l\| \le a_{m''}^{\frac{1}{m''}} \|f_l\|_{p_{m''},\nu} / \|f_l\| < \frac{1}{2}. \tag{3.12}$$

(3.10) and (3.12) together give

$$\sum_{m=1}^{\infty} a_m \|f_{l_0}\|_{p_m,\nu}^m / \|f_{l_0}\|^m < 1,$$

which is again a contradiction to the definition of the Ba-norm, and the theorem is finally proved.

Acknowledgment. We feel great pleasure to have the opportunity to appreciate Professor A.D. Gadzhiev for his exceptional advises and comments on this paper.

References

[1] Aliev, I. A., On Riesz transformations generated by a generalized shift operators, *Izvestiya Acad. of Sciences of Azerbaydian,* **1**(1987), 7–13.

[2] Aliev, I. A. & Gadzhiev, A. D., Weighted estimates for multidimensional singular integrals generated by the generalized shift operator, *Matematika Sbornik Rossiyskaya Akad. Nauk.,* **183:**9(1992), 45–66. English translation: *Russian Acad. Sci. Sb. Math.,* **77:**1(1994) 37–55.

[3] Chang, W. D., Boundedness of the Calderon-Zygmund singular integral operators on Ba spaces, *Proc. Amer. Math. Soc.,* **109:**2(1990), 403–408.

[4] Ding, X. X. & Luo, P., Ba spaces and some estimates of Laplace operators, *J. Systems Sci. Math. Sci.,* **1**(1981), 9–33.

[5] Ding, X. X., On a new class of function spaces, *Kexue Tongbao,* **26**(1981), 973–976.

[6] Ekincioğlu, İ., On high order Riesz transformations generated by a generalized shift operator, *Tr. J. Math.,* **21**(1997), 51–60.

[7] Ekincioğlu, İ. & Şerbetçi, A., On the singular integral operators generated by the generalized shift operator, *Int. J. App. Math.,* **1:**1(1999), 29-38.

[8] Levitan, B. M., *Uspehi Mat. Nauk.,* **6**(1967), 2(42), 102,143–163.

[9] Stein, E. M., Singular integrals and differentiability properties of functions, Princeton Univ. Press, Princeton, N.J., 1970.

[10] Şerbetçi, A. & Ekincioğlu, I., Boundedness of Riesz Potential generated by the generalized shift operator on Ba spaces, *Czec. Math. J.,* (2003)(to appear).