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MINIMAL SURFACES IN 3-DIMENSIONAL SOLVABLE LIE GROUPS**

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(*Dedicated to Professor Su Bu-chin on the Occasion of His 100th Birthday*)

Abstract

The author studies minimal surfaces in 3-dimensional solvable Lie groups with left invariant Riemannian metrics. A Weierstraß type integral representation formula for minimal surfaces is obtained.

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§1. Introduction

Minimal surfaces in (or more generally harmonic maps into) Riemannian space forms have been studied extensively. As is well known, minimal surfaces in Euclidean n-space $\mathbf{E}^n$ are represented by the Weierestraß-Enneper integral formula.

Several kinds of generalisations of Weierstraß-Enneper formula are obtained.

J. Dorfmeister, F. Pedit and H. Wu obtained a scheme to construct constant mean curvature surfaces in $\mathbf{E}^3$ from holomorphic data in terms of loop groups[9]. R. Bryant[5] established a representation formula for constant mean curvature 1 surfaces (CMC-1 surfaces) in hyperbolic 3-space H^3 of constant curvature -1.

The hyperbolic 3-space H^3 is represented by $\mathrm{SL}_2\mathbf{C}/\mathrm{SU}(2)$ as a Riemannian symmetric space. Bryant showed that any CMC-1 surface in H^3 is given by a "holomorphic data"—$\mathfrak{sl}_2\mathbf{C}$-valued holomorphic 1-forms which satisfy certain condition.

These two generalisations emphasize the aspect "holomorphic construction" of the Weieratraß-Enneper formula.

On the other hand, the Weierstraß-Enneper formula has another aspect—"integral representation formula".

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K. Kenmotsu[15] obtained an integral representation formula for surfaces with prescribed mean curvature in $\mathbf{E}^3$. The data of Kenmotsu's representation formula are smooth maps into the unit 2-sphere which satisfy second order elliptic partial differential equation. In particular, in case of constant mean curvature surfaces, the partial differentail equation coincides with the harmonic map equation into S^2.

C. C. Góes and P. A. Q. Simões [10] obtained an integral representation formula for minimal surfaces in hyperbolic 3-space and hyperbolic 4-space. The data of the formula due to Góes and Simões are smooth maps into S^2 which are solutions to certain second order elliptic partial differential equation.

Independently, M. Kokubu[16] obtained an integral representation for minimal surfaces in H^n. Kokubu's formula coincides with the formula due to Góes and Simões for $n = 3, 4$. Thus Kokubu's formula is a generalisation of Góes-Simões' formula to general dimension.

To obtain such a representation formula, Kokubu introduced a matrix group model of hyperbolic space. By using the matrix group model, Kokubu clarified the differential geometric meaning of the data for the formula due to Góes-Simões-Kokubu. More precisely every minimal surface in H^3 is constructed by a harmonic map into S^2 with certain singular metric (Kokubu metric). Moreover Kokubu showed that the harmonic map becomes the normal Gauß map of the minimal surface. Note that Kokubu's formula is valid for simlpy connected Riemannian space forms of nonpositive curvature.

The formulae due to Kenmotsu, Góes–Simões and Kokubu are generalisations of the Weieratraß-Enneper formula from the viewpoint of integral representation formula with data derived from solutions to second order elliptic PDE's.

The underlying Riemannian manifold of Kokubu's model is a warped product representation of H^n. It is easy to see that the matrix group model used by Kokubu is solvable. For instance in 3-dimension, according to the polar decomposition (Iwasawa decomposition) of $\mathrm{SL}_2\mathbf{C}$, the hyperbolic 3-space $H^3 = \mathrm{SL}_2\mathbf{C}/\mathrm{SU}(2)$ is identified with the solvable part of $\mathrm{SL}_2\mathbf{C}$. It seems to be interesting to generalise Kokubu's formula to 3-dimensional solvable Lie groups with more general left invariant metrics.

The Minkowski motion group $E(1,1)$ is a typical example of 3-dimensional solvable Lie group. The Minkowski motion group equipped with natural left invariant metric has been studied in some contexts. For example, O. Kowalski[17] proved that simply connected proper 3-dimensional generalised Riemannian symmetric space is isometric to $E(1,1)$. Tsunero Takahashi[21] proved that this is the only (simply connected) 3-dimensional homogeneous Riemannian manifold isometrically immersed in hyperbolic 4-space with type number 2.

The Euclidean 3-space, hyperbolic 3-space and Minkowski motion group are included in the following 3-parameter family of Riemannian homogeneous spaces $(\mathbf{R}^3, g[\mu_1, \mu_2, \mu_3])$,

$$g[\mu_1, \mu_2, \mu_3] = \mathrm{e}^{-2\mu_1 t} dx^2 + \mathrm{e}^{-2\mu_2 t} dy^2 + \mu_3^2 dt^2.$$

Here μ_1, μ_2 are real constants and μ_3 is a positive constant. Every homogeneous Riemannian manifold in this family can be represented as a solvable matrix group with left invariant metric.

In this paper, we shall generalise Kokubu's formula to minimal surfaces in $(\mathbf{R}^3, g[\mu_1, \mu_2, \mu_3])$.

According to the classification of all 3-dimensional unimodular Lie groups, there exist

3-dimensional solvable Lie groups of another type (see [6,18]). The model space of such a class is Euclidean motion group $E(2)$. This paper will end with some remarks on surfaces in $E(2)$ furnished with the standard left invariant flat metric.

§2. Solvable Lie Groups

Let $(\mathbf{R}(t),+)$ be a real line. For any real numbers μ_1 and μ_2 we define a representation $\rho=\rho_{\mu_1,\mu_2}$ of $\mathbf{R}(t)$ on $(\mathbf{R}^2(x,y),+)$ by

$$\rho=\rho_{\mu_1,\mu_2}:\mathbf{R}(t)\to \mathrm{GL}_2\mathbf{R};\quad \rho(t)=\begin{pmatrix} \mathrm{e}^{\mu_1 t} & 0\\ 0 & \mathrm{e}^{\mu_2 t}\end{pmatrix}. \tag{2.1}$$

Then the semi-direct product group G of $\mathbf{R}(t)$ and $\mathbf{R}^2(x,y)$ via ρ is $\mathbf{R}^3(x,y,t)$ with multiplication:

$$(x,y,t)\cdot(x',y',t')=(x+\mathrm{e}^{\mu_1 t}x',y+\mathrm{e}^{\mu_2 t}y',t+t'). \tag{2.2}$$

The unit element of G is $\mathbf{0}=(0,0,0)$. The inverse element of (x,y,t) is given by $(-\mathrm{e}^{-\mu_1 t}x, -\mathrm{e}^{-\mu_2 t}y,-t)$. In particular if $\mu_1=\mu_2=0$ then G is $(\mathbf{R}^3(x,y,t),+)$.

The Lie algebra $\mathfrak{g}$ of G is $\mathbf{R}^3$ with commutation relations:

$$[E_1,E_2]=0,\ [E_2,H]=-\mu_2 E_2,\ [H,E_1]=\mu_1 E_1 \tag{2.3}$$

with respect to the basis $E_1=(1,0,0),\ E_2=(0,1,0),\ H=(0,0,1)$.

These formulae imply that G is solvable. In fact the derived series $\{\mathfrak{D}^i\}_{i\in\mathbf{N}}$ of $\mathfrak{g}$ is given by

$$\mathfrak{D}^1:=[\mathfrak{g},\mathfrak{g}]=\begin{cases}\mathbf{R}E_1\oplus\mathbf{R}E_2, & \mu_1\neq 0,\ \mu_2\neq 0,\\ \mathbf{R}E_1, & \mu_1\neq 0,\ \mu_2=0,\\ \mathbf{R}E_2, & \mu_1=0,\ \mu_2\neq 0,\\ \{\mathbf{0}\}, & \mu_1=\mu_2=0,\end{cases}\qquad \mathfrak{D}^2:=[\mathfrak{D}^1,\mathfrak{D}^1]=\{0\}.$$

The left translated vector fields of $E_1,\ E_2,\ H$ are

$$e_1=\mathrm{e}^{\mu_1 t}\,\frac{\partial}{\partial x},\quad e_2=\mathrm{e}^{\mu_2 t}\,\frac{\partial}{\partial y},\quad h=\frac{\partial}{\partial t}. \tag{2.4}$$

We shall equip an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ so that $\{E_1,\ E_2,\ E_3=H/\mu_3\}$ is an orthonormal basis of $\mathfrak{g}$. Here μ_3 is a positive constant. Then the left translated Riemannian metric $g=g[\mu_1,\mu_2,\mu_3]$ is

$$g[\mu_1,\mu_2,\mu_3]=\mathrm{e}^{-2\mu_1 t}dx^2+\mathrm{e}^{-2\mu_2 t}dy^2+\mu_3^2dt^2. \tag{2.5}$$

Hereafter we shall denote $G=G(\mu_1,\mu_2,\mu_3)$ the Lie group G with left invariant metric $g[\mu_1,\mu_2,\mu_3]$.

§3. Matrix Group Model of $G(\mu_1,\mu_2,u_3)$

The Lie group $G(\mu_1,\mu_2,\,u_3)$ can be realised as a closed subgroup of affine transformation group $\mathrm{GL}_3\mathbf{R}\ltimes\mathbf{R}^3$ of $\mathbf{R}^3$. In fact G is imbedded in $\mathrm{GL}_3\mathbf{R}\ltimes\mathbf{R}^3$ by $\iota: G\to \mathrm{GL}_3\mathbf{R}\ltimes\mathbf{R}^3$;

$$\iota(x,y,t)=\begin{pmatrix}1 & 0 & 0 & t\\ 0 & \mathrm{e}^{\mu_1 t} & 0 & x\\ 0 & 0 & \mathrm{e}^{\mu_2 t} & y\\ 0 & 0 & 0 & 1\end{pmatrix}. \tag{3.1}$$

It is obvious that ι is an injective Lie group homomorphism into $\mathrm{GL}_3\mathbf{R} \ltimes \mathbf{R}^3$. Hence G is identified with the following closed subgroup of $\mathrm{GL}_4\mathbf{R}$:

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & t \\ 0 & \mathrm{e}^{\mu_1 t} & 0 & x \\ 0 & 0 & \mathrm{e}^{\mu_2 t} & y \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x, y, t \in \mathbf{R} \right\}. \tag{3.2}$$

The Lie algebra $\mathfrak{g}$ corresponds to

$$\left\{ \begin{pmatrix} 0 & 0 & 0 & w \\ 0 & \mu_1 w & 0 & u \\ 0 & 0 & \mu_2 w & v \\ 0 & 0 & 0 & 0 \end{pmatrix} \;\middle|\; u, v, w \in \mathbf{R} \right\}. \tag{3.3}$$

The orthonormal basis $\{E_i\}_{i=1}^3$ is identified with

$$E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad E_3 = \frac{1}{\mu_3} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & \mu_1 & 0 & 0 \\ 0 & 0 & \mu_2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The Levi-Civita connection ∇ of $G(\mu_1, \mu_2, \mu_3)$ is given by the following formulae:

$$\begin{aligned} &\nabla_{E_1} E_1 = \frac{\mu_1}{\mu_3} E_3, \quad \nabla_{E_1} E_2 = 0, \quad \nabla_{E_1} E_3 = -\frac{\mu_1}{\mu_3} E_1, \\ &\nabla_{E_2} E_1 = 0, \quad \nabla_{E_2} E_2 = \frac{\mu_2}{\mu_3} E_3, \quad \nabla_{E_2} E_3 = -\frac{\mu_2}{\mu_3} E_2, \\ &\nabla_{E_3} E_1 = \nabla_{E_3} E_2 = \nabla_{E_3} E_3 = 0. \end{aligned} \tag{3.4}$$

The Riemannian curvature tensor of G is described as follows:

$$R^1_{212} = -\frac{\mu_1 \mu_2}{\mu_3^2}, \; R^1_{313} = -\frac{\mu_1^2}{\mu_3^2}, \; R^2_{323} = -\frac{\mu_2^2}{\mu_3^2}. \tag{3.5}$$

The Ricci tensor field Ric is given by

$$R_{11} = -\frac{\mu_1(\mu_1 + \mu_2)}{\mu_3^2}, \; R_{22} = -\frac{\mu_2(\mu_1 + \mu_2)}{\mu_3^2}, \; R_{33} = -\frac{\mu_1^2 + \mu_2^2}{\mu_3^2}. \tag{3.6}$$

The scalar curvature s of G is

$$s = -\frac{2}{\mu_3^2}(\mu_1^2 + \mu_2^2 + \mu_1 \mu_2).$$

The natural-reducibility obstruction U defined by

$$2g(U(X, Y), Z) = g(X, [Z, Y]) + g(Y, [Z, X]), \quad X, Y, Z \in \mathfrak{g} \tag{3.7}$$

is given by

$$\begin{aligned} &U(E_1, E_1) = \frac{\mu_1}{\mu_3} E_3, \quad U(E_2, E_2) = \frac{\mu_2}{\mu_3} E_3, \\ &U(E_1, E_3) = -\frac{\mu_1}{2\mu_3} E_1, \quad U(E_2, E_3) = -\frac{\mu_2}{2\mu_3} E_2. \end{aligned} \tag{3.8}$$

Note that U measures the non-right-invariance of the metric. In fact $U = 0$ if and only if g is right invariant (and hence biinvariant). The formulae (3.8) imply that g is biinvariant if and only if $\mu_1 = \mu_2 = 0$.

For later use we shall introduce a left invariant almost contact structure on G. Define a linear endomorphism $F_{\mathbf{0}}$ on $\mathfrak{g}$ by

$$F_{\mathbf{0}}(E_1) = E_2, \quad F_{\mathbf{0}}(E_2) = -E_1, \quad F_{\mathbf{0}}(E_3) = 0.$$

We denote the left translated endomorphism field of $F_{\mathbf{0}}$ by F. Then the triple (F,ξ,η) defined by $\xi = e_3,\ \eta = \mu_3 dt$ gives a left invariant almost contact structure compatible to g (see [2]). Since η is exact, this almost contact structure is not contact. The formulae (3.4) imply that the vector field ξ is Killing if and only if $\mu_1 = \mu_2 = 0$.

Example 3.1 (Euclidean 3-Space). If we choose $\mu_1 = \mu_2 = 0,\ \mu_3 = 1$, then G is isomorphic and isometric to Euclidean 3-space $(\mathbf{E}^3,+)$. In fact G is a translation group of $\mathbf{R}^3$.

Example 3.2 (Warped Product Model of the Hyperbolic 3-Space).

Let $\mu_1 = \mu_2 = c,\ \mu_3 = 1,\ c > 0$. Then G is a warped product model of hyperbolic 3-space of curvature $-c^2$:

$$H^3(-c^2) = \left(\mathbf{R}^3(x,y,t),\ \mathrm{e}^{-2ct}(dx^2+dy^2)+dt^2\right). \tag{3.9}$$

Namely $H^3(-c^2)$ is represeted as a warped product $\mathbf{E}^1(t) \times_{\mathrm{e}^{-ct}} \mathbf{E}^2(x,y)$ with base $\mathbf{E}^1$ and fibre $\mathbf{E}^2$. Kokubu used this matrix group model $G(c,c,1)$ to obtain his Weierstraß type representation formula for minimal surfaces[16]. In this case, the left invariant almost contact structure satisfies

$$(\nabla_X F)Y = -c\{\eta(Y)FX + g(X,FY)\xi\},\ \nabla_X\xi = -c\{X - \eta(X)\xi\} \tag{3.10}$$

for all $X, Y \in \mathfrak{X}(G)$. These formulae imply that $H^3(-1) = (G(1,1,1);F,\xi,\eta)$ is a Kenmotsu manifold[13,14].

Example 3.3 (Riemannian 4-Symmetric Space). Let $\mu_1 \neq 0,\ \mu_2 \neq 0,\ \mu_3 \neq 0$. Then G is isomorphic to the following closed Lie subgroup of $\mathrm{GL}_2\mathbf{R} \ltimes \mathbf{R}^2$:

$$\left\{ \begin{pmatrix} \mathrm{e}^{\mu_1 t} & 0 & x \\ 0 & \mathrm{e}^{\mu_2 t} & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x,y,t \in \mathbf{R} \right\}. \tag{3.11}$$

In particular if $\mu_1 = -\mu_2 = 1$, then G is isomorphic to the identity component of the isometry group of the Minkowski plane $\mathbf{E}^2_1 = (\mathbf{R}^2(x,y), dxdy)$. The Riemannian homogeneous 3-space $G(1,-1,\mu_3)$ is a Riemannian 4-symmetric space of dimension 3 which is not a Riemannian symmetric space. The 4-symmetric space representation of G is $G = G/\{\mathbf{0}\}$ with fourth order automorphism τ:

$$\tau(x,y,t) = (-y,x,-t). \tag{3.12}$$

O. Kowalski proved that $G(1,-1,\mu_3)$ is the only proper simply connected generalised Riemannian symmetric space of dimension 3 (see [17, Theorem VI.2]). In particular $G(1,-1,\mu_3)$ is irreducible. Furthermore Tsunero Takahashi showed that $G(1,-1,1)$ can be isometrically imbedded in hyperbolic 4-space $H^4(-1)$. He proved that simply connected Riemannian homogeneous 3-space (M^3, g_M) can be isometrically immmersed in H^4 with type number 2 if and only if M^3 is isometric to $G(1,-1,1)$ (see [21]). $G(1,-1,1)$ has been called Takahashi's B-manifold. Note that $G(1,-1,1)$ is the model space Sol^3 of 3-dimensional solvegeometry in the sense of W. M. Thurston[22].

Remark 3.1. (1) The hyperbolic 3-space $H^3(-c^2) = G(c,c,1)$ does not admit any left invariant contact Riemannian structure. In addition $H^3(-c^2)$ does not admit any other Lie group structure (see [3, p. 413]).

On the other hand, the Minkowski motion group $E(1,1)$ admits left invariant contact Riemannian structure whose Webster scalar curvature W and torsion invariant τ satisfy

$4\sqrt{2}W = -\|\tau\|$ (see [20, Theorem 3.1]).

(2) Both $H^3 = G(1,1,1)$ and $E(1,1) = G(1,1,-1)$ can be realised as hypersurfaces in hyperbolic 4-space H^4 (see [21]).

§4. Harmonic Map Equation

Let $\mathfrak{D}$ be a Riemann surface and K a Lie group with left invariant metric. We denote by $\mathfrak{k}$ the Lie algebra of K. For a smooth map $\varphi : \mathfrak{D} \to K$, we denote by α the pulled-back one form of the Maurer-Cartan form of K. If K is a matrix group, α is given by

$$\alpha = \varphi^{-1} d\varphi. \tag{4.1}$$

The $\mathfrak{k}$-valued 1-form α satisfies the following equation:

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0. \tag{4.2}$$

The equation (4.2) is a complete integrability condition of the existence for a map φ satisfying $\varphi^{-1}d\varphi = \alpha$ (see [11, p. 54, Proposition]).

Let z be a local complex coordinate of $\mathfrak{D}$. We decompose α with respect to the conformal structure of $\mathfrak{D}$;

$$\alpha = \alpha' dz + \alpha'' d\bar{z}.$$

Then (4.2) is rewritten as

$$\frac{\partial}{\partial z}\alpha'' - \frac{\partial}{\partial \bar{z}}\alpha' + [\alpha', \alpha''] = 0. \tag{4.2$'$}$$

For a smooth map $\varphi : \mathfrak{D} \to K$, the energy of φ is

$$E(\varphi) = \int_{\mathfrak{D}} \frac{1}{2}|d\varphi|^2 dV.$$

A smooth map φ is a harmonic map provided that φ is a critical point of the energy.

Now we recall the harmonic map equation for φ, i.e., the Euler-Lagrange equation of $E(\varphi)$ in terms of α.

It is known that φ is harmonic if and only if

$$\frac{\partial}{\partial \bar{z}}\alpha' + \frac{\partial}{\partial z}\alpha'' + 2U(\alpha', \alpha'') = 0. \tag{4.3}$$

Here U is the natural-reducibility obstruction defined by (3.7) (see [8, Corollary 2.4], [16, Lemma 3.1]).

The integrability condition (4.2$'$) together with the harmonicity (4.3) are equivalent to

$$\frac{\partial}{\partial \bar{z}}\alpha' + U(\alpha', \alpha'') = \frac{1}{2}[\alpha', \alpha''], \quad \alpha'' = \overline{\alpha'}. \tag{4.4}$$

Here we recall the following fundamental fact (see [15, Section 3]).

Lemma 4.1. *Let $\varphi : \mathfrak{D} \to K$ be a harmonic map. Then $\alpha = \varphi^{-1}d\varphi$ satisfies* (4.4). *Conversely assume that $\mathfrak{D}$ is simply connected. Let α be a $\mathfrak{k}$-valued* 1*-form satisfying* (4.4) *on $\mathfrak{D}$. Then there exists a harmonic map φ into K such that $\varphi^{-1}d\varphi = \alpha$.*

Now we shall apply these observations for harmonic maps into $G(\mu_1, \mu_2, \mu_3)$.

We shall write a smooth map φ into $G(\mu_1, \mu_2, \mu_3)$ by

$$\varphi(z, \bar{z}) = (x^1(z, \bar{z}), x^2(z, \bar{z}), t(z, \bar{z})). \tag{4.5}$$

Here we used the convension $x^1 = x,\ x^2 = y$. The $(1,0)$-part α' of α is given by

$$\alpha' = \sum_{i=1}^{2} (\mathrm{e}^{-\mu_i t} x^i_z) e_i + \mu_3 t_z e_3. \tag{4.6}$$

The integrability together with harmonicity (4.4) for $\varphi : \mathfrak{D} \to G(\mu_1, \mu_2, \mu_3)$ is given by

$$\begin{cases} x^i_{z\bar{z}} - \mu_i (t_z x^i_{\bar{z}} + t_{\bar{z}} x^i_z) = 0, \\ t_{z\bar{z}} + \frac{1}{\mu_3^2} \sum \mu_i \mathrm{e}^{-2\mu_i t} x^i_z x^i_{\bar{z}} = 0. \end{cases} \tag{4.7}$$

If μ_1, μ_2 are nonnegative, we get the following

Proposition 4.1. *Let $\varphi : \mathfrak{D} \to G(\mu_1, \mu_2, \mu_3)$ be a harmonic map from a compact Riemann surface into G. If $\mu_1, \mu_2 \geq 0$ then φ is constant.*

Proof. Let φ be a harmonic map. Then the second equation of (4.7) implies t is a subharmonic function. Hence t is constant by the maximum principle for subharmonic functions. Furthermore the first equation of (4.7) implies that x^i are harmonic functions and hence constants.

Define three $(1,0)$-forms χ, ω^1 and ω^2 on $\mathfrak{D}$ by

$$\chi = \mu_3 t_z dz,\ \omega^i = \mathrm{e}^{-\mu_i t} x^i_z dz,\ i = 1, 2. \tag{4.8}$$

Then

$$\alpha' dz = \omega^1 E_1 + \omega^2 E_2 + \chi E_3.$$

Further (4.7) can be rewritten as

$$\begin{cases} \bar{\partial}\chi = \frac{1}{\mu_3} \sum \mu_i \omega^i \wedge \overline{\omega^i}, \\ \bar{\partial}\omega^i = \frac{1}{\mu_3} \mu_i \overline{\omega^i} \wedge \chi. \end{cases} \tag{4.9}$$

Thus we get the following formula:

Theorem 4.1. *Let χ, ω^1 and ω^2 be $(1,0)$-forms on a simply connected Riemann surface $\mathfrak{D}$ satisfying (4.9). Then the mapping $\varphi : \mathfrak{D} \to G$ defined by*

$$\varphi(z, \bar{z}) = 2 \int_{z_0}^{z} \mathrm{Re}\ \left(\mathrm{e}^{\ \mu_1 t(z,\bar{z})} \cdot \omega^1, \mathrm{e}^{\ \mu_2 t(z,\bar{z})} \cdot \omega^2, \frac{1}{\mu_3} \chi \right) \tag{4.10}$$

is a harmonic map into $G(\mu_1, \mu_2, \mu_3)$.

Conversely any harmonic map of $\mathfrak{D}$ into $G(\mu_1, \mu_2, \mu_3)$ can be represented in this form.

This integral representation formula is a generalisation of the formula due to Góes-Simões[10] and Kokubu[16]. In case $G(c, c, 1) = H^3(-c^2)$, (4.10) coincides with their formula.

Arguments similar to [16, Lemma 4.5] yields the following

Corollary 4.1. *Let $\chi,\ \omega^1$ and ω^2 be $(1,0)$-forms on $\mathfrak{D}$ satisfying*

$$\begin{cases} \bar{\partial}\omega^i = \mu_3^{-1} \mu_i \overline{\omega^i} \wedge \chi, \\ \chi \otimes \chi + \sum \omega^i \otimes \omega^i = 0. \end{cases} \tag{4.11}$$

Then φ defined by (4.10) is a weakly conformal harmonic map. Moreover φ is a minimal immersion if

$$\chi \otimes \bar{\chi} + \sum \omega^i \otimes \overline{\omega^i} \neq 0.$$

Example 4.1. Let us take the following three 1-forms:

$$\omega^1 = \frac{\sqrt{-1}\sqrt{\mu_3}}{z + \bar{z}} dz,\quad \omega^2 = 0,\quad \chi = \frac{\mu_1}{z + \bar{z}} dz,\ \mu_1 \neq 0.$$

Then one can check that $\{\omega^1, \omega^2, \chi\}$ satisfies the first equation of (4.9). And the data satisfy the second equation of (4.9) if and only if $\mu_3 = \mu_1^2$. The data satisfy the weak conformality (the second equation of (4.11)) if and only if $\mu_3 = \mu_1^2$. Note that

$$\chi \otimes \bar{\chi} + \sum \omega^i \otimes \overline{\omega^i} > 0 \quad \text{for all} \quad \mu_1, \mu_2 \in \mathbf{R}, \ \mu_3 > 0.$$

Now we choose $\mu_3 = \mu_1^2$. Then

$$t = \frac{1}{\mu_1} \log |u|, \quad x = -u^{\frac{\mu_1^2 - 1}{\mu_1^2}} v, \quad y = \text{constant}, \ z = u + \sqrt{-1}v.$$

Thus the minimal surface (x, y, t) is the plane determined by y =constant.

Remark 4.1. (1) Geodesics of $G(1, -1, 1)$ are explicitly described by F. Borghero and R. Caddeo[4]. We can see geodesics in (the warped product model of) hyperbolic 3-space and Minkowski motion group $G(1, -1, 1)$ are given by similar formulae (see Cas B, Cas C and Remarque finale in [4]).

(2) A representation formula for minimal surfaces in Heisenberg group is obtained in [12].

(3) In [1], V. Balan and J. Dorfmeister generalised the "loop group theoretic Weierstraß type representation" for harmonic maps from Riemann surfaces into compact Lie groups (furnished with biinvariant Riemannian metrics)[9] to general Lie groups with biinvariant semi-Riemannian metrics. Moreover they established a Weierstraß-type representation for maps from Riemann surfaces into arbitrary Lie groups which satisfy

$$\frac{\partial}{\partial \bar{z}} \alpha' + \frac{\partial}{\partial z} \alpha'' = 0. \tag{4.12}$$

In case of $\varphi : \mathfrak{D} \to G(\mu_1, \mu_2, \mu_3)$, the equation (4.12) becomes

$$t_{z\bar{z}} = 0, \quad x^i_{z\bar{z}} - \frac{1}{2} \mu_i (x^i_z t_{\bar{z}} + x^i_{\bar{z}} t_z) = 0, \quad i = 1, 2. \tag{4.12$'$}$$

The integrability condition (4.2$'$) toghether with (4.12$'$) is equivalent to

$$x^1_{z\bar{z}} = x^2_{z\bar{z}} = t_{z\bar{z}} = 0.$$

Namley $\varphi = (x^1, x^2, t)$ is a vector-valued harmonic function.

§5. Primitive Maps into Minkowski Motion Group

In this section we study harmonic maps into $G(1, 1, -1)$ which are solutions to certain first order PDE. More precisely, we classify primitive maps into Minkowski motion group $G(1, 1, -1)$. As we will see below, every primitive map is harmonic.

First of all we shall recall the notion of primitive map into generalised symmetric spaces. For general theory of primitive maps we refer to [7].

Let $N = G/H$ be a Riemannian k-symmetric space with corresponding automorphism τ of G and $k > 2$. We denote the induced automorphism on $\mathfrak{g}$ by the same letter. The eigenvalues of τ on $\mathfrak{g}$ are $\{\omega^i \mid i \in \mathbf{Z}_k\}$. Here we denote the primitive k-th root of 1 by ω. We have an eigenspace decomposition of the complexification $\mathfrak{g}^{\mathbf{C}}$ of $\mathfrak{g}$:

$$\mathfrak{g}^{\mathbf{C}} = \sum_{i \in \mathbf{Z}_k} \mathfrak{g}[i],$$

where $\mathfrak{g}[i]$ is the eigenspace of τ corresponding to the eigenvalue ω^i. Clearly,

$$\mathfrak{g}_{[0]} = \mathfrak{h}^{\mathbf{C}} \quad \text{and} \quad \overline{\mathfrak{g}[i]} = \mathfrak{g}[-i].$$

Furthermore one can check that

$$[\ \mathfrak{g}[i],\ \mathfrak{g}[j]\ ] = \mathfrak{g}[i+j] \pmod{k}.$$

Note that $\mathfrak{g}[1] \cap \mathfrak{g}[-1] = \{\mathbf{0}\}$ since $k > 2$.

For a smooth map $\varphi : M \to G/H$ from a Riemann surface M is said to be a primitive map if $\varphi^*\beta'$ takes value in $[\mathfrak{g}[-1]\ ]$. Here β' is the $(1,0)$-part of the Maurer-Cartan form β of G/H (see [7, p. 240] and [11, p. 137]). The vector bundle $[\mathfrak{g}[-1]\ ]$ is the subbundle of $G/H \times \mathfrak{g}^{\mathbf{C}}$ defined by

$$[\mathfrak{g}[-1]]_{gH} := \mathrm{Ad}(g)\ \mathfrak{g}[-1].$$

Now we shall study primitive maps into the Minkowski motion group $G(1,1,-1) = G/\{\mathbf{0}\}$ with automorphism τ defined by (3.12). Since the primitive 4-th root of unity is $\sqrt{-1}$, the eigenvalues of τ are $-1,\ \pm\sqrt{-1}$. One can give eigenspaces of $\mathfrak{g}^{\mathbf{C}}$ corresponding to ω^i explicitly.

$$\mathfrak{g}[0] = \{\mathbf{0}\},\ \mathfrak{g}[1] = \mathbf{C}\begin{bmatrix} 1 \\ \sqrt{-1} \\ 0 \end{bmatrix},\quad \mathfrak{g}[2] = \mathbf{C}h = \mathbf{C}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\quad \mathfrak{g}[3] = \mathbf{C}\begin{bmatrix} 1 \\ -\sqrt{-1} \\ 0 \end{bmatrix}.$$

Define $S_{\mathbf{0}} := \mathfrak{g}_{[1]}$. Then

$$\mathfrak{g}^{\mathbf{C}} = S_{\mathbf{0}} \oplus \bar{S}_{\mathbf{0}} \oplus \mathbf{C}h.$$

Furthermore $S_{\mathbf{0}}$ defines a left invariant CR-structure S on $G(1,-1,1)$.

$$S_g := L_{g*}S_{\mathbf{0}},\ g \in G(1,-1,1).$$

One can check that the almost contact structure (F,ξ,η) on $G(1,-1,1)$ defined in Section 3 is compatible to this CR-structure. Namely CR-structure S is given by

$$S = \{X - \sqrt{-1}FX \mid X \in D\}.$$

Here D denotes the real expression of S defined by

$$D_{\mathbf{0}} = \{X \in \mathfrak{g} |\ \eta(X) = 0\}.$$

Since $G(1,1,-1)$ itself is a Lie group, the primitive map equation takes more simpler form. In fact a smooth map $\varphi : \mathfrak{D} \to G$ is primitive if and only if

$$\alpha\Big(\frac{\partial}{\partial z}\Big) \in \mathfrak{g}_{[-1]}.$$

Direct computatios show that $\varphi : \mathfrak{D} \to G$ is primitive if and only if

$$t_z = 0,\ -\sqrt{-1}\mathrm{e}^{-t}x_z = \mathrm{e}^t y_z.$$

Proposition 5.1. *Let $\varphi : \mathfrak{D} \to G$ be a smooth map into 3-dimensional 4-symmetric space. Then φ is a primitive map if and only if $t =$ constant and x and $\mathrm{e}^{2t}y$ are conjugate harmonic functions.*

Corollary 5.1. (1) *Every primitive map $\varphi : \mathfrak{D} \to G(1,1,-1)$ is a harmonic map.*
(2) *Let ψ be a holomorphic function on $\mathfrak{D}$ and put $f = \psi + \bar{\psi}$ and $\hat{f} = \psi - \bar{\psi}$. Then the data*

$$\omega^1 = \mathrm{e}^{-c}f_z dz,\ \omega^2 = \mathrm{e}^{-c}\hat{f}_z dz,\ \chi = 0,\ c \in \mathbf{R}$$

define a primitive map into $G(1,1,-1)$.

By using the almost contact structure of $G(1,-1,1)$, one can check that the primitivity is equivalent to the anti F-holomorphicity. Recall that a smooth map $\varphi : \mathfrak{D} \to (M; F, \xi, \eta)$

into an almost contact manifold M is said to be an F-holomorphic curve if φ satisfies

$$d\varphi \circ J = F \circ d\varphi.$$

And φ is said to be an anti F-holomorphic curve if

$$d\varphi \circ J = -F \circ d\varphi.$$

Here J denotes the complex structure of $\mathfrak{D}$.

Proposition 5.2. *Let $\varphi : \mathfrak{D} \to G(1,-1,1)$ be a smooth map. Then φ is F-holomorphic if and only if $\sqrt{-1}\mathrm{e}^{-t}x_z = \mathrm{e}^t y_z$ and t is constant. Equivalently t is constant and x and $-\mathrm{e}^{2t}y$ are conjugate harmonic functions.*

§6. Surfaces in Euclidean Motion Group $E(2)$

We start with recalling the definition of Euclidean motion group $E(2)$. Let $\mathbf{E}^2$ be a Euclidean 2-space. Then the rigid motion group of $\mathbf{E}^2$ is the semi-direct product of rotation group SO(2) and translation group $(\mathbf{R}^2,+)$. The semi-direct product structure of $\mathrm{SO}(2)\ltimes\mathbf{R}^2$ is

$$(A,\mathbf{p})\cdot(B,\mathbf{q}) := (AB, \mathbf{p}+A\mathbf{q}), \quad A,B\in \mathrm{SO}(2),\ \mathbf{p},\mathbf{q}\in\mathbf{R}^2. \tag{5.1}$$

The semi-direct product $\mathrm{SO}(2)\ltimes\mathbf{R}^2$ is isomorphic to the following closed subgroup of $\mathrm{GL}_3\mathbf{R}$:

$$E(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; x,y\in\mathbf{R},\ \theta\in S^1 \right\}. \tag{5.2}$$

We may regard (x,y,θ) as a global coordinate system of $E(2)$. Thus $E(2)$ is $\mathbf{R}^2(x,y)\times S^1$ with multipication rule:

$$(x,y,\theta)*(x',y',\theta') = (x+\cos\theta\ x' - \sin\theta\ y', y+\sin\theta\ x'+\cos\theta\ y', \theta+\theta'). \tag{5.3}$$

The Lie algebra $\mathfrak{e}(2)$ corresponds to

$$\mathfrak{e}(2) = \left\{ \begin{pmatrix} 0 & -w & u \\ w & 0 & v \\ 0 & 0 & 0 \end{pmatrix} \;\middle|\; u,v,w\in\mathbf{R} \right\}. \tag{5.4}$$

We take a basis $\{E_i\}$ of $\mathfrak{e}(2)$:

$$E_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the left translated vector fields of $E_1,\ E_2,\ H$ are

$$e_1 = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \quad e_2 = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}, \quad h = \frac{\partial}{\partial\theta}.$$

We equip an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{e}(2)$ so that $\{E_1,\ E_2,\ E_3 = H/\mu\}$ is an orthonormal basis of $\mathfrak{e}(2)$. Here μ is a positive constant (cf. [8]). Then the left translated Riemannian metric g_μ is

$$g_\mu = dx^2 + dy^2 + \mu^2 d\theta^2.$$

Namely, as a Riemannian manifold, $E(2)$ is a warped product with base $\mathbf{E}^2(x,y)$, fibre S^1 and constant warping function μ. Obviously, $(E(2), g_\mu)$ is flat.

The Levi-Civita connection ∇ of $E(2)$ is described as follows:

$$\begin{aligned}
&\nabla_{e_1}e_1 = 0, \quad \nabla_{e_1}e_2 = 0, \quad \nabla_{e_1}e_3 = 0,\\
&\nabla_{e_2}e_1 = 0, \quad \nabla_{e_2}e_2 = 0, \quad \nabla_{e_2}e_3 = 0,\\
&\nabla_{e_3}e_1 = \mu^{-1}e_2, \quad \nabla_{e_3}e_2 = -\mu^{-1}e_1, \quad \nabla_{e_3}e_3 = 0.\\
&[e_1, e_2] = 0, \quad [e_2, e_3] = \mu^{-1}e_1, \quad [e_3, e_1] = \mu^{-1}e_2.
\end{aligned}$$

Let $\mathbf{E}^3(x, y, z)$ be a Euclidean 3-space with natural Riemannian metric $dx^2 + dy^2 + dz^2$. For any vector $\mathbf{v} \in \mathbf{R}^3$, we denote by $\Gamma(\mathbf{v})$ the discrete subgroup of $(\mathbf{R}^3, +)$ generated by $\mathbf{v}$:

$$\Gamma(\mathbf{v}) = \{\ 2\pi m\mathbf{v}\ |m \in \mathbf{Z}\ \}.$$

In particular, if we choose $\mathbf{v} = \mathbf{e}_3 = (0, 0, 1)$ then the factor space $\mathbf{E}^3/\Gamma(\mathbf{e}_3)$ is the underlying flat Riemannian manifold of $(E(2), g_1)$. This fundamental observation suggests us to use group structure of $E(2)$ for the study of surfaces in Euclidean 3-space which are invariant under $\Gamma(\mathbf{e}_3)$-action. Denote by $\widetilde{G}$ the universal covering group $G = E(2)$. Then the left translation by $(\alpha, \beta, \gamma) \in \widetilde{G}$ is written in the following formula:

$$L_{(\alpha,\beta,\gamma)}\begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0\\ \sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix} + \begin{bmatrix} \alpha\\ \beta\\ \gamma\end{bmatrix}.$$

In particular the one parameter subgroup $\{(0, 0, \gamma) \mid \gamma \in \mathbf{R}\}$ of $\widetilde{G}$ acts on $\mathbf{R}^3$ as helicoidal motion group (or screw motion group) of pitch 1:

$$L_{(0,0,\gamma)}\begin{bmatrix} x\\ y\\ z\end{bmatrix} = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0\\ \sin\gamma & \cos\gamma & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix} x\\ y\\ z\end{bmatrix} + \begin{bmatrix} 0\\ 0\\ \gamma\end{bmatrix}.$$

The action of $\{(0, 0, \gamma) \mid \gamma \in \mathbf{R}\ \}$ decends to SO(2)-action on $G = E(2)$.

As is well known, helicoids are helicoidal minimal surfaces, i.e., minimal surfaces which are orbits of spatial curves under a helicoidal motion. For instance a helicoid $z = \tan^{-1}(y/x)$ is an orbit of x-axis under the action of $\{(0, 0, t) \mid t \in \mathbf{R}\}$.

In fact, the helicoid $z = \tan^{-1}(y/x)$ is given by the immersion: $\widetilde{\varphi} : \mathbf{R}^2(u, v) \to \mathbf{E}^3$:

$$\widetilde{\varphi}(u, v) = (u \cos v,\ u \sin v,\ v).$$

By using the multiplication $*$, the helicoid $\widetilde{\varphi}$ is written in the following form:

$$\widetilde{\varphi}(u, v) = L_{(0,0,v)}(u, 0, 0).$$

Thus $\widetilde{\varphi}$ is an orbit of x-axis under the one-parameter subgroup

$$\{(0, 0, v) \mid v \in \mathbf{R}\ \} \cong (\mathbf{R}(v), +).$$

The helicoid $\tilde{\varphi}$ induces a minimal surface φ in $\mathbf{E}^3/\Gamma(\mathbf{e}_3)$. In $\mathbf{E}^3/\Gamma(\mathbf{e}_3)$, the helicoid φ is regarded as S^1-orbit of the x-axis. Namely, with respect to the group structure $*$, the helicoid is regarded as a "rotational surface" in $\mathbf{E}^3/\Gamma(\mathbf{e}_3)$.

Note that since $*$ is noncommutative,

$$(0, 0, v) * (u, 0, 0) \neq (u, 0, 0) * (0, 0, v).$$

In fact,

$$(u, 0, 0) * (0, 0, v) = (u, 0, v).$$

The image of this immersion is xz-plane. This is also obviously $\Gamma(\mathbf{e}_3)$-invariant.

For more detailed study on minimal surfaces in $\mathbf{E}^3/\Gamma(\mathbf{v})$, we refer to [19]. In particular Moriya studied moduli space of minimal surfaces in $\mathbf{E}^3/\Gamma(\mathbf{v})$.

References

[1] Balan, V. & Dorfmeister, J., A Weierstrass-type representation for harmonic maps from Riemann surfaces to general Lie groups, *Balkan J. Geom. Appl.,* **5**(2000), 7–37 (Available at http://www.emis.de/journals/BJGA/5.1/2.html).

[2] Blair, D. E., Contact manifolds in Riemannian geometry, Lecture Notes in Math. **509** (1976), Springer-Verlag, Berlin.

[3] Boeckx, E., Bueken, P. & Vanhecke, L., φ-symmetric contact metric spaces, *Glasgow Math. J.,* **41** (1999), 409–416.

[4] Borghero, F. & Caddeo, R., Une structure de Séparabilité et Géodésiques dans les Huit Géométries Tridimensionalles de Thurston , *Rend. Mat. Appl.,* (7) **9**(1989), 607–624.

[5] Bryant, R., Surfaces of mean curvature one in hyperbolic space , *Astérisque,* **154–155**(1987), 321–347.

[6] Bryant, R., An introduction to Lie groups and symplectic geometry, Geometry and Quantum Field Theory (D. S. Freed and K. K. Uhlenbeck, eds.), IAS/Park City Math. Series, vol. 1, Amer. Math. Soc., 1991, pp. 5–181.

[7] Burstall, F. E., & Pedit, F., Harmonic maps via Adler-Kostant-Symes theory, Harmonic Maps and Integrable Systems, Aspects Math., vol. E23, Vieweg, 1994, pp. 221–272.

[8] Dai, Y.-J., M. Shoji, M. & Urakawa, H., Harmonic maps into Lie groups and homogeneous spaces, *Differential Geom. Appl.,* **7**(1997), 143–160.

[9] Dorfmeister, J., Pedit, F. & Wu, H., Weierstrass type representation of harmonic maps into symmetric spaces, *Comm. Anal. Geom.,* **7**(1998), 633–668.

[10] Góes, C. C. & Simões, P. A. Q., The generalized Gauss map of minimal surfaces in H^3 and H^4, *Bol. Soc. Brasil. Mat.,* **18**(1987), 35–47.

[11] Guest, M. A., Harmonic maps, loop groups and integrable systems, London Math. Soc. Student Texts 38, Cambridge Univ. Press, 1997.

[12] Inoguchi, J., Kumamoto, T., Ohsugi, N. & Suyama, Y., Differential geometry of curves and surfaces in 3-dimensional homogeneous spaces II, *Fukuoka Univ. Sci. Report,* **30**(2000), 17–47.

[13] Janssens, D. & Vanhecke, L., Almost contact structures and curvature tensors, *Kōdai Math. J.,* **4**(1981), 1–27.

[14] Kenmotsu, K., A class of almost contact Riemannian manifolds, *Tôhoku Math. J.,* **24**(1972), 93–103.

[15] Kenmotsu, K., Weierstrass formula for surfaces of prescribed mean curvature, *Math. Ann.,* **245**(1979), 89–99.

[16] Kokubu, M., Weierstrass representation for minimal surfaces in hyperbolic space, *Tôhoku Math. J.,* **49** (1997), 367–377.

[17] Kowalski, O., Generalized symmetric spaces, Lecture Notes in Math., **805** (1980), Springer Verlag, Berlin.

[18] Milnor, J., Curvatures of left invariant metrics on Lie groups, *Advances in Math.,* **21**(1976), 293–329.

[19] Moriya, K., On a variety of minimal surfaces invariant under a screw motion, *J. Math. Sci. Univ. Tokyo,* **7**(2000), 311–323.

[20] Perrone, D., Homogeneous contact Riemannian three-manifolds, *Illinois J. Math.,* **42**(1998), 243–256.

[21] Takahashi, T., An isometric immersion of a homogeneous Riemannian manifold of dimension 3 in the hyperbolic space, *J. Math. Soc. Japan,* **23**(1971), 649–661.

[22] Thurston, W. M., Three-dimensional geometry and topology I, Princeton Math. Series,, vol. 35(S. Levy, ed.), 1997.