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BIFURCATIONS OF ROUGH 3-POINT-LOOP WITH HIGHER DIMENSIONS***

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Abstract

The authors study the bifurcation problems of rough heteroclinic loop connecting three saddle points for the case $\beta_1 > 1$, $\beta_2 > 1$, $\beta_3 < 1$ and $\beta_1\beta_2\beta_3 < 1$. The existence, number, coexistence and incoexistence of 2-point-loop, 1-homoclinic orbit and 1-periodic orbit are studied. Meanwhile, the bifurcation surfaces and existence regions are given.

Keywords Local coordinates, Poincaré map, 1-homoclinic orbit, 1-periodic orbit, Bifurcation surface

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§1. Introduction

In recent years, the bifurcation problems of heteroclinic and homoclinic orbits in higher dimensional space were studied and many results were obtained. For example, [7, 8, 9, 10] discussed the problems of the homoclinic loop bifurcations. In [11], Zhu and Xia studied the bifurcations of heteroclinic loop with two saddle points (abbr. 2-point-loop). And in [12], Tian and Zhu considered the bifurcation problems of fine 2-point-loop. In [13], the authors studied the bifurcations of rough 2-point-loop for the case $\beta_1 > 1$, $\beta_2 < 1$, $\beta_1\beta_2 < 1$, where $\beta_i = \rho_i/\lambda_i$, $-\rho_i^1$ and λ_i^1 are the principal eigenvalues of unperturbed system at saddle point p_i, $i = 1, 2$. In this paper, we consider the following C^r system

$$\dot{z} = f(z) + g(z, \mu), \tag{1.1}$$

and its unperturbed system

$$\dot{z} = f(z), \tag{1.2}$$

where $r \geq 4$, $z \in \mathbf{R}^{m+n}$, $\mu \in \mathbf{R}^l$, $l \geq 3$, $0 \leq |\mu| \ll 1$, $g(z, 0) = 0$. For $i = 1, 2, 3$, we assume $f(p_i) = 0$, $g(p_i, \mu) = 0$ and

(H1) $z = p_i$ is a hyperbolic critical point of (1.2). The stable manifold W_i^s and the unstable manifold W_i^u of $z = p_i$ are m-dimensional and n-dimensional, respectively. Moreover,

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$-\rho_i^1$, λ_i^1 are the simple eigenvalues of $D_z f(p_i)$ such that the other eigenvalues of $D_z f(p_i)$, $-\rho_i^j$, λ_i^k, satisfy

$$-\mathrm{Re}\,\rho_i^j < -\rho_i^0 < -\rho_i^1 < 0 < \lambda_i^1 < \lambda_i^0 < \mathrm{Re}\,\lambda_i^k, \tag{1.3}$$

where $1 < j \le m$, $1 < k \le n$ and ρ_i^0 and λ_i^0 are some positive constants.

(H2) System (1.2) has a heteroclinic loop $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where

$$\Gamma_i = \{z = r_i(t) : t \in \mathbf{R}\}, \quad r_i(+\infty) = r_{i+1}(-\infty) = p_{i+1}, \quad r_4(t) = r_1(t), \quad p_4 = p_1.$$

For any point $P_i \in \Gamma_i$, $\dim(T_{P_i} W_i^u \cap T_{P_i} W_{i+1}^s) = 1$, $W_4^s = W_1^s$.

(H3) Define $e_i^{\pm} = \lim\limits_{t \to \mp\infty} \dot r_i(t)/|\dot r_i(t)|$. Then, $e_i^+ \in T_{p_i} W_i^u$ and $e_i^- \in T_{p_{i+1}} W_{i+1}^s$ are unit eigenvectors corresponding to λ_i^1 and $-\rho_{i+1}^1$, respectively.

(H4) $\mathrm{span}(T_{r_i(t)} W_i^u, T_{r_i(t)} W_{i+1}^s, e_{i+1}^+) = \mathbf{R}^{m+n}, \ \ t \gg 1,$
$\mathrm{span}(T_{r_i(t)} W_i^u, T_{r_i(t)} W_{i+1}^s, e_{i-1}^-) = \mathbf{R}^{m+n}, \ \ t \ll -1,$
where $e_4^+ = e_1^+$, $e_0^- = e_3^-$.

Under the hypotheses **(H1)**–**(H4)**, [14] studied the bifurcations of rough heteroclinic loop with three hyperbolic saddle points (abbr. 3-point-loop) for the case $\beta_i = \rho_i^1/\lambda_i^1 > 1$, $i = 1, 2, 3$. In this paper, we study the problems of bifurcation of rough 3-point-loop Γ for the case $\beta_1 > 1$, $\beta_2 > 1$, $\beta_3 < 1$, $\beta_1\beta_2\beta_3 < 1$. Under some transversal conditions and the nontwisted condition, we discuss the existence, uniqueness, incoexistence and the related bifurcation surfaces of 1-heteroclinic 3-point-loop, 2-point-loop, and 1-homoclinic orbit near Γ . Meanwhile, we also discuss the existence, number and existence regions of 1-periodic orbits and 2-fold 1-periodic orbits near Γ, and the coexistence, incoexistence of 3-point-loop, 2-point-loop, 1-homoclinic orbits, 1-periodic orbits and 2-fold 1-periodic orbits near Γ. Our results show that the bifurcation pattern studied here is much more complicated than that studied by [14].

Remark 1.1. Under the hypotheses **(H1)** and **(H2)**, the hypotheses **(H3)** and **(H4)** are generic (refer to [11]).

§2. Local Coordinates and Bifurcation Equations

In this section, we use the linear independent solutions of the linear variational equation along Γ_i as the demanded local coordinates to construct the Poincaré map. The method was suggested and used by [9, 12], which is similar to and easier than that in [8, 11].

Suppose that U_i is a sufficiently small neighborhood of p_i and **(H1)**–**(H4)** hold. Then, for $|\mu|$ small enough, there always exists a C^r transformation such that system (1.1) has the following form in U_i:

$$\begin{cases} \dot x = [\lambda_i^1(\mu) + \cdots]\, x + O(u)[O(y) + O(v)], \\ \dot y = [-\rho_i^1(\mu) + \cdots]\, y + O(v)[O(x) + O(u)], \\ \dot u = [B_i^1(\mu) + \cdots]\, u + O(x)[O(x) + O(y) + O(v)], \\ \dot v = [-B_i^2(\mu) + \cdots]\, v + O(y)[O(y) + O(x) + O(u)], \end{cases} \tag{2.1}$$

where $\lambda_i^1(0) = \lambda_i^1$, $\rho_i^1(0) = \rho_i^1$, $\mathrm{Re}\,\sigma(B_i^1(0)) > \lambda_i^0$, $\mathrm{Re}\,\sigma(-B_i^2(0)) < -\rho_i^0$, $z = (x, y, u^*, v^*)^*$, $x \in R^1$, $y \in R^1$, $u \in R^{n-1}$, $v \in R^{m-1}$, and (2.1) is C^{r-1}. Thus, in U_i, we have

$$\begin{aligned} &\Gamma \cap W_i^u = \{z : u = u(x), y = 0, v = 0\}, && \Gamma \cap W_i^s = \{z : x = 0, u = 0, v = v(y)\}, \\ &W_i^{uu} = \{z : x = 0, y = 0, v = 0\}, && W_i^{ss} = \{z : x = 0, y = 0, u = 0\}, \\ &W_i^u = \{z : y = 0, v = 0\}, && W_i^s = \{z : x = 0, u = 0\}, \end{aligned}$$

where $u(0) = \dot{u}(0) = 0$, $v(0) = \dot{v}(0) = 0$, W_i^{uu} and W_i^{ss} are the strong unstable and stable manifold of p_i, respectively.

The sign $*$ means transposition. Denote

$$r_i(t) = (r_i^x(t), r_i^y(t), (r_i^u(t))^*, (r_i^v(t))^*)^*.$$

$-T_i^0$ and T_i^1 are the moments such that

$$r_i(-T_i^0) = (\delta, 0, \delta_u^*, 0^*)^*, \qquad r_i(T_i^1) = (0, \delta, 0^*, \delta_v^*)^*,$$

where δ is small enough so that

$$\{(x, y, u^*, v^*)^* : |x|, |y|, |u|, |v| < 2\delta\} \subset U_i.$$

Consider the linear system

$$\dot{z} = Df(r_i(t))z \tag{2.2}$$

and its adjoint system

$$\dot{\phi} = -(Df(r_i(t)))^*\phi. \tag{2.3}$$

Due to [11, 14], system (2.2) has a fundamental solution matrix $Z_i(t) = (z_i^1(t), z_i^2(t), z_i^3(t), z_i^4(t))$ satisfying

$$z_i^1(t) \in (T_{r_i(t)}W_i^u)^c \cap (T_{r_i(t)}W_{i+1}^s)^c, \quad z_i^2(t) = -\dot{r}_i(t)/|\dot{r}_i^y(T_i^1)|,$$
$$z_i^3(t) \in (T_{r_i(t)}W_i^u) \cap (T_{r_i(t)}W_{i+1}^s)^c, \quad z_i^4(t) \in (T_{r_i(t)}W_i^u)^c \cap (T_{r_i(t)}W_{i+1}^s),$$

$$Z_i(-T_i^0) = \begin{pmatrix} w_i^{11} & w_i^{21} & 0 & w_i^{41} \\ w_i^{12} & 0 & 0 & w_i^{42} \\ w_i^{13} & w_i^{23} & I & w_i^{43} \\ 0 & 0 & 0 & w_i^{44} \end{pmatrix}, \quad Z_i(T_i^1) = \begin{pmatrix} 1 & 0 & w_i^{31} & 0 \\ 0 & 1 & w_i^{32} & 0 \\ 0 & 0 & w_i^{33} & 0 \\ w_i^{14} & w_i^{24} & w_i^{34} & I \end{pmatrix},$$

where $W_4^s = W_1^s$, $w_i^{21} < 0$, $w_i^{12} \neq 0$, $\det w_i^{33} \neq 0$, $\det w_i^{44} \neq 0$. Moreover, for δ small enough,

$$||w_i^{1j}(w_i^{12})^{-1}|| \ll 1 \quad \text{for} \quad j \neq 2,$$
$$||w_i^{2j}(w_i^{21})^{-1}|| \ll 1 \quad \text{for} \quad j = 3, 4,$$
$$||w_i^{3j}(w_i^{33})^{-1}|| \ll 1 \quad \text{for} \quad j \neq 3,$$
$$||w_i^{4j}(w_i^{44})^{-1}|| \ll 1 \quad \text{for} \quad j \neq 4.$$

Thus, we select $z_i^1(t)$, $z_i^2(t)$, $z_i^3(t)$, $z_i^4(t)$ as a local coordinate system in the small tube neighborhood of Γ_i. Denote

$$\Phi_i(t) = (\phi_i^1(t), \phi_i^2(t), \phi_i^3(t), \phi_i^4(t)) = (Z_i^{-1}(t))^*.$$

Obviously, $\Phi_i(t)$ is a fundamental solution matrix of (2.3).

Let $w_i^{12} = \Delta_i|w_i^{12}|$. We say that Γ is nontwisted as $\Delta = \Delta_1\Delta_2\Delta_3 = 1$, and twisted as $\Delta = -1$. In this paper, we only consider the case $\Delta = 1$.

Make a transformation as following

$$z(t) = h_i(t) = r_i(t) + Z_i(t)N_i,$$

where $N_i = (n_i^1, 0, (n_i^3)^*, (n_i^4)^*)^*$. Denote by $S_i^0 = \{z = h_i(-T_i^0) : |x|, |y|, |u|, |v| < 2\delta\}$, $S_i^1 = \{z = h_i(T_i^1) : |x|, |y|, |u|, |v| < 2\delta\}$ the cross sections of Γ_i at $t = -T_i^0$ and $t = T_i^1$, respectively, where δ is small enough so that $S_i^0 \subset U_i$, $S_i^1 \subset U_{i+1}$, $U_4 = U_1$.

Now, we construct the Poincaré map $F_i = F_i^1 \circ F_i^0$: $S_{i-1}^1 \mapsto S_i^1$, where

$$F_i^0 : q_{i-1}^1 \in S_{i-1}^1 \mapsto q_i^0 \in S_i^0, \quad F_i^1 : q_i^0 \in S_i^0 \mapsto q_i^1 \in S_i^1.$$

Denote

$$\begin{aligned}
q_i^0 &= (x_i^0, y_i^0, (u_i^0)^*, (v_i^0)^*)^* = r_i(-T_i^0) + Z_i(-T_i^0)N_i^0,\\
N_i^0 &= (n_i^{0,1}, 0, (n_i^{0,3})^*, (n_i^{0,4})^*)^*,\\
q_i^1 &= (x_i^1, y_i^1, (u_i^1)^*, (v_i^1)^*)^* = r_i(T_i^1) + Z_i(T_i^1)N_i^1,\\
N_i^1 &= (n_i^{1,1}, 0, (n_i^{1,3})^*, (n_i^{1,4})^*)^*, \quad i = 1, 2, 3,
\end{aligned}$$

and

$$r_0(T_0^1) = r_3(T_3^1), \quad Z_0(T_0^1) = Z_3(T_3^1), \quad N_0^1 = N_3^1.$$

Using the expressions of $Z_i(-T_i^0)$ and $Z_i(T_i^1)$, we have $x_i^0 \approx \delta$, $y_i^1 \approx \delta$,

$$\begin{aligned}
n_i^{0,1} &= (w_i^{12})^{-1}(y_i^0 - w_i^{42}(w_i^{44})^{-1}v_i^0),\\
n_i^{0,3} &= u_i^0 - \delta_u - w_i^{13}(w_i^{12})^{-1}y_i^0 + [w_i^{13}(w_i^{12})^{-1}w_i^{42} - w_i^{43}](w_i^{44})^{-1}v_i^0,\\
n_i^{0,4} &= (w_i^{44})^{-1}v_i^0,
\end{aligned} \tag{2.4}$$

$$\begin{aligned}
n_i^{1,1} &= x_i^1 - w_i^{31}(w_i^{33})^{-1}u_i^1,\\
n_i^{1,3} &= (w_i^{33})^{-1}u_i^1,\\
n_i^{1,4} &= -w_i^{14}x_i^1 + (w_i^{14}w_i^{31} - w_i^{34})(w_i^{33})^{-1}u_i^1 + v_i^1 - \delta_v.
\end{aligned} \tag{2.5}$$

For simplicity, we may as well assume $\rho_i^1 \geq \lambda_i^1$. Let τ_i be the flying time from q_{i-1}^1 to q_i^0, and $s_i = e^{-\lambda_i^1(\mu)\tau_i}$, which is called the Silnikov time. Then by [11, 14], we obtain the map F_i^1 which is given by

$$n_i^{1,j} = n_i^{0,j} + M_i^j\mu + \text{h.o.t.}, \quad j = 1, 3, 4 \tag{2.6}$$

and the map $F_i^0 : S_{i-1}^1 \mapsto S_i^0$, $q_{i-1}^1 \mapsto q_i^0$, $S_0^1 = S_3^1$ defined by

$$\begin{aligned}
x_{i-1}^1 &\approx s_i\delta, \quad y_i^0 \approx s_i^{\rho_i^1(\mu)/\lambda_i^1(\mu)}\delta,\\
u_{i-1}^1 &\approx s_i^{B_i^1(\mu)/\lambda_i^1(\mu)}u_i^0, \quad v_i^0 \approx s_i^{B_i^2(\mu)/\lambda_i^1(\mu)}v_{i-1}^1,
\end{aligned} \tag{2.7}$$

if we neglect the higher order terms.

We call

$$M_i^j = \int_{-\infty}^{+\infty} \phi_i^{j^*}(t)g_\mu(r_i(t), 0)dt, \quad i = 1, 2, 3, \quad j = 1, 3, 4$$

Melnikov vectors, and (s_i, u_i^0, v_{i-1}^1), $i = 1, 2, 3$ Silnikov coordinates.

Thus, by (2.4)$-$(2.7), we get the expression of the sucessive function $G_i(q_{i-1}^1) = F_i(q_{i-1}^1) - q_i^1$ as following

$$\begin{aligned}
G_i^1 &= \delta[(w_i^{12})^{-1}s_i^{\rho_i^1(\mu)/\lambda_i^1(\mu)} - s_{i+1}] + M_i^1\mu + \text{h.o.t.},\\
G_i^3 &= u_i^0 - \delta_u - w_i^{13}(w_i^{12})^{-1}\delta s_i^{\rho_i^1(\mu)/\lambda_i^1(\mu)} - (w_i^{33})^{-1}s_{i+1}^{B_{i+1}^1(\mu)/\lambda_{i+1}^1(\mu)}u_{i+1}^0\\
&\quad + M_i^3\mu + \text{h.o.t.},\\
G_i^4 &= -v_i^1 + \delta_v + w_i^{14}\delta s_{i+1} + (w_i^{44})^{-1}s_i^{B_i^2(\mu)/\lambda_i^1(\mu)}v_{i-1}^1 + M_i^4\mu + \text{h.o.t.}.
\end{aligned} \tag{2.8}$$

Remark 2.1. If $\rho_i^1 < \lambda_i^1$ and $\rho_j^1 > \lambda_j^1$ for $j \neq i$, $1 \leq i, j \leq 3$, then we take $s_i = e^{-\rho_i^1(\mu)\tau_i}$. In this case, (2.7) becomes

$$\begin{aligned} x_{i-1}^1 &\approx s_i^{\lambda_i^1(\mu)/\rho_i^1(\mu)}\delta, \quad y_i^0 \approx s_i\delta, \\ u_{i-1}^1 &\approx s_i^{B_i^1(\mu)/\rho_i^1(\mu)}u_i^0, \quad v_i^0 \approx s_i^{B_i^2(\mu)/\rho_i^1(\mu)}v_{i-1}^1, \end{aligned} \tag{2.7'}$$

and (2.8) becomes

$$\begin{aligned} G_i^1 &= \delta[(w_i^{12})^{-1}s_i - s_{i+1}] + M_i^1\mu + \text{h.o.t.}, \\ G_i^3 &= u_i^0 - \delta_u - w_i^{13}(w_i^{12})^{-1}\delta s_i - (w_i^{33})^{-1}s_{i+1}^{B_{i+1}^1(\mu)/\lambda_{i+1}^1(\mu)}u_{i+1}^0 \\ &\quad + M_i^3\mu + \text{h.o.t.}, \\ G_i^4 &= -v_i^1 + \delta_v + w_i^{14}\delta s_{i+1} + (w_i^{44})^{-1}s_i^{B_i^2(\mu)/\rho_i^1(\mu)}v_{i-1}^1 + M_i^4\mu + \text{h.o.t.}, \end{aligned} \tag{2.8'}$$

$$\begin{aligned} G_{i-1}^1 &= \delta[(w_{i-1}^{12})^{-1}s_{i-1}^{\rho_{i-1}^1(\mu)/\lambda_{i-1}^1(\mu)} - s_i^{\lambda_i^1(\mu)/\rho_i^1(\mu)}] + M_{i-1}^1\mu + \text{h.o.t.}, \\ G_{i-1}^3 &= u_{i-1}^0 - \delta_u - w_{i-1}^{13}(w_{i-1}^{12})^{-1}\delta s_{i-1}^{\rho_{i-1}^1(\mu)/\lambda_{i-1}^1(\mu)} - (w_{i-1}^{33})^{-1}s_i^{B_i^1(\mu)/\rho_i^1(\mu)}u_i^0 \\ &\quad + M_{i-1}^3\mu + \text{h.o.t.}, \\ G_{i-1}^4 &= -v_{i-1}^1 + \delta_v + w_{i-1}^{14}\delta s_i^{\lambda_i^1(\mu)/\rho_i^1(\mu)} + (w_{i-1}^{44})^{-1}s_{i-1}^{B_{i-1}^2(\mu)/\lambda_{i-1}^1(\mu)}v_{i-2}^1 \\ &\quad + M_{i-1}^4\mu + \text{h.o.t.}. \end{aligned} \tag{2.8''}$$

Remark 2.2. If $\rho_i^1 > \lambda_i^1$ and $\rho_j^1 < \lambda_j^1$ for $j \neq i$, $1 \leq i, j \leq 3$, then we only need to take $t \mapsto -t$; the others are similar.

Remark 2.3. There is a $1-1$ correspondence between the 3-point-loop, 2-point-loop, 1-homoclinic loop and 1-periodic orbit of (1.1) and the solution $Q = (s_1, s_2, s_3, u_1^0, u_2^0, u_3^0, v_1^1, v_2^1, v_3^1)$ of the following equation with $s_1 \geq 0$, $s_2 \geq 0$, $s_3 \geq 0$:

$$(G_1^1, G_2^1, G_3^1, G_1^3, G_2^3, G_3^3, G_1^4, G_2^4, G_3^4) = 0. \tag{2.9}$$

(2.9) is called the bifurcation equation.

Remark 2.4. G_i is C^{r-2} with respect to Q in the region $s_1 > 0$, $s_2 > 0$, $s_3 > 0$ and at least C^1 at $s_1 = s_2 = s_3 = 0$.

§3. Bifurcations of 2-Point-Loop and 1-Homoclinic Loop from Γ

(AI) $\beta_i = \rho_i^1/\lambda_i^1 > 1$, $i = 1, 2$, $\beta_3 = \rho_3^1/\lambda_3^1 < 1$, $\beta_1\beta_2\beta_3 < 1$.

Denote

$$\begin{aligned} R_{12}^3 &= \{\mu : M_2^1\mu > 0, \Delta_3 M_3^1\mu < 0, |\mu| \ll 1\}, \\ R_{23}^1 &= \{\mu : M_3^1\mu > 0, \Delta_1 M_1^1\mu < 0, |\mu| \ll 1\}, \\ R_{31}^2 &= \{\mu : M_1^1\mu > 0, \Delta_2 M_2^1\mu < 0, |\mu| \ll 1\}, \\ R_1^{23} &= \{\mu : M_1^1\mu > 0, \Delta_3 M_3^1\mu < 0, |\mu| \ll 1\}, \\ R_2^{31} &= \{\mu : M_2^1\mu > 0, \Delta_1 M_1^1\mu < 0, |\mu| \ll 1\}, \\ R_3^{12} &= \{\mu : M_3^1\mu > 0, \Delta_2 M_2^1\mu < 0, |\mu| \ll 1\}. \end{aligned}$$

Suppose that hypotheses **(H1)**–**(H4)** and **(AI)** hold. Then the following theorem are true.

Theorem 3.1. (1) *If $M_i \neq 0$, then there exists a unique surface L_i with codimension* 1 *and normal vector M_i^1 at $\mu = 0$, such that* (1.1) *has a heteroclinic orbit connecting p_i and p_{i+1} near Γ_i if and only if $\mu \in L_i$ and $|\mu| \ll 1$.*

If rank$(M_i^1,\ M_j^1) = 2$, *$i, j = 1, 2, 3$, $i \neq j$, then $L_{ij} = L_i \cap L_j$ is an $(l-2)$-dimensional surface and* $0 \in L_{ij}$ *such that* (1.1) *has two heteroclinic orbits near $\Gamma_i \cup \Gamma_j$ as $\mu \in L_{ij}$ and* $|\mu| \ll 1$.

If rank$(M_1^1,\ M_2^1, M_3) = 3$, *then $L = L_1 \cap L_2 \cap L_3$ is an $(l-3)$-dimensional surface and* $0 \in L$ *such that* (1.1) *has a* 3*-point-loop near Γ as $\mu \in L$ and $|\mu| \ll 1$, that is, Γ is persistent.*

(2) *If $M_2^1 \neq 0$, $M_3^1 \neq 0$, then there exists an $(l-1)$-dimensional surface $L_{12}^3 \subset R_{12}^3$ tangent to L_2 at $\mu = 0$ such that* (1.1) *has a unique heteroclinic orbit Γ_{12}^3 near $\Gamma_2 \cup \Gamma_3$ if and only if $\mu \in L_{12}^3$. Moreover, if M_1^1 and M_2^1 are linearly independent, then* (1.1) *has a unique* 2*-point-loop Γ_{12} connecting p_1 and p_2 if and only if $\mu \in L_1 \cap L_{12}^3$.*

If $M_3^1 \neq 0$, $M_1^1 \neq 0$, then there exists an $(l-1)$-dimensional surface $L_{23}^1 \subset R_{23}^1$ tangent to L_1 at $\mu = 0$ such that (1.1) *has a unique heteroclinic orbit Γ_{23}^1 near $\Gamma_3 \cup \Gamma_1$ if and only if $\mu \in L_{23}^1$. Moreover, if M_2^1 and M_1^1 are linearly independent, then* (1.1) *has a unique* 2*-point-loop Γ_{23} connecting p_2 and p_3 if and only if $\mu \in L_2 \cap L_{23}^1$.*

If $M_1^1 \neq 0$, $M_2^1 \neq 0$, then there exists an $(l-1)$-dimensional surface $L_{31}^2 \subset R_{31}^2$ tangent to L_2 at $\mu = 0$ such that (1.1) *has a unique heteroclinic orbit Γ_{31}^2 near $\Gamma_1 \cup \Gamma_2$ if and only if $\mu \in L_{31}^2$. Moreover, if M_3^1 and M_2^1 are linearly independent, then* (1.1) *has a unique* 2*-point-loop Γ_{31} connecting p_3 and p_1 if and only if $\mu \in L_3 \cap L_{31}^2$.*

(3) *If* rank$(M_1^1, M_2^1, M_3^1) = 3$, *then there exist surfaces $L_1^{23} \subset R_1^{23}$, $L_2^{31} \subset R_2^{31}$ and $L_3^{12} \subset R_3^{12}$ all with codimension 1 and normal vector M_2^1 at $\mu = 0$, such that* (1.1) *has a unique homoclinic loop connecting p_1, p_2 and p_3 for $\mu \in L_1^{23}$, L_2^{31} and L_3^{12}, respectively.*

(4) *The* 3*-point-loop,* 2*-point-loop and* 1*-homoclinic orbit cannot coexist.*

Proof. For the study of the bifurcations of (1.1) near Γ, we only need to consider the solutions of equation (2.9). It is not difficult to see that the equation $(G_1^3, G_2^3, G_3^3, G_1^4, G_2^4, G_3^4) = 0$ always has a solution $u_i^0 = u_i^0(s_1, s_2, s_3, \mu)$, $v_i^1 = v_i^1(s_1, s_2, s_3, \mu)$ $i = 1, 2, 3$ for δ, $|\mu|$, s_1, s_2, s_3 sufficiently small. Substituting it into $(G_1^1, G_2^1, G_3^1) = 0$, we get

$$\begin{aligned} s_2 &= (w_1^{12})^{-1} s_1^{\beta_1} + \delta^{-1} M_1^1 \mu + \text{h.o.t.}, \\ s_3^{1/\beta_3} &= (w_2^{12})^{-1} s_2^{\beta_2} + \delta^{-1} M_2^1 \mu + \text{h.o.t.}, \\ s_1 &= (w_3^{12})^{-1} s_3 + \delta^{-1} M_3^1 \mu + \text{h.o.t.}. \end{aligned} \tag{3.1}$$

(1) Suppose that (3.1) has zero solution $s_1 = s_2 = s_3 = 0$, then (3.1) reads as

$$M_i^1 \mu + \text{h.o.t.} = 0, \quad i = 1, 2, 3. \tag{3.2}$$

If $M_i^1 \neq 0$, then, by the implicit function theorem, (3.2) defines a surface L_i with codimension 1 and normal vector M_i^1 at $\mu = 0$ such that the ith equation of (3.1) has a solution $s_i = s_{i+1} = 0$ as $\mu \in L_i$ and $|\mu| \ll 1$, that is to say, Γ_i is persistent.

Moreover, if rank$(M_i^1, M_j^1) = 2$, $i \neq j$, then $L_{ij} = L_i \cap L_j$ is an $(l-2)$-dimensional surface (refer to [1]) such that the ith and jth equations of (3.1) have a solution $s_1 = s_2 = s_3 = 0$ for $\mu \in L_{ij}$ and $|\mu| \ll 1$, that is, Γ_i and Γ_j are both persistent. Particularly, if rank$(M_1^1, M_2^1, M_3^1) = 3$, then $L = L_1 \cap L_2 \cap L_3$ is an $(l-3)$-dimensional surface such that (3.1) has a solution $s_1 = s_2 = s_3 = 0$ as $\mu \in L$ and $|\mu| \ll 1$, that is, Γ is persistent.

(2) Suppose that $s_1 = s_2 = 0$, $s_3 > 0$ is a solution of (3.1). Then (3.1) becomes

$$M_1^1\mu + \text{h.o.t.} = 0, \tag{3.3}$$

$$s_3 = -\delta^{-1}w_3^{12}M_3^1\mu + \text{h.o.t.}, \tag{3.4}$$

$$(-\delta^{-1}w_3^{12}M_3^1\mu + \text{h.o.t.})^{1/\beta_3} = \delta^{-1}M_2^1\mu + \text{h.o.t.}. \tag{3.5}$$

By the implicit function theorem, (3.5) defines an $(l-1)$-dimensional surface L_{12}^3 in the region $M_2^1\mu > 0$, $\Delta_3M_3^1\mu < 0$ with a normal vector M_2^1 at $\mu = 0$, which means L_{12}^3 is tangent to L_2 at $\mu = 0$. Thus, for $\mu \in L_{12}^3$ and $|\mu| \ll 1$, the second and third equations of (3.1) have solution $s_1 = s_2 = 0$, $s_3 > 0$, which means that (1.1) has an orbit Γ_{12}^3 heteroclinic to p_1 and p_2 and situated in the neighborhood of $\Gamma_2 \cup \Gamma_3$. Moreover, if M_1^1 and M_2^1 are linearly independent, then (3.1) has a unique solution $s_1 = s_2 = 0$, $s_3 > 0$ as $\mu \in L_1 \cap L_{12}^3$ and $|\mu| \ll 1$, which means that system (1.1) has a unique 2-point-loop Γ_{12} near Γ connecting p_1 and p_2 for $\mu \in L_1 \cap L_{12}^3$ and $|\mu| \ll 1$, where $L_1 \cap L_{12}^3$ is an $(l-2)$-dimensional surface.

In the same way, we can discuss the case $s_2 = s_3 = 0$, $s_1 > 0$ (resp. $s_3 = s_1 = 0$, $s_2 > 0$) and obtain the surface L_{23}^1 (resp. L_{31}^2) in the region $M_3^1\mu > 0$, $\Delta_1M_1^1\mu < 0$ (resp. $M_1^1\mu > 0$, $\Delta_2M_2^1\mu < 0$) tangent to L_1 (resp. L_2) at $\mu = 0$, such that system (1.1) has an orbit Γ_{23}^1 (resp. Γ_{31}^2) heteroclinic to p_2 and p_3 (resp. p_3 and p_1) and situated in the neighborhood of $\Gamma_3 \cup \Gamma_1$ (resp. $\Gamma_1 \cup \Gamma_2$) as $\mu \in L_{23}^1$ (resp. $\mu \in L_{31}^2$) and $|\mu| \ll 1$. Moreover, if M_2^1 and M_1^1 (resp. M_3^1 and M_2^1) are linearly independent, then system (1.1) has a unique 2-point-loop Γ_{23} (resp. Γ_{31}) near Γ connecting p_2 and p_3 (resp. p_3 and p_1) for $\mu \in L_2 \cap L_{23}^1$ (resp. $\mu \in L_3 \cap L_{31}^2$) and $|\mu| \ll 1$.

(3) Suppose that (3.1) has solution $s_1 = 0$, $s_2 > 0$, $s_3 > 0$. Then (3.1) becomes the following form:

$$s_2 = \delta^{-1}M_1^1\mu + \text{h.o.t.}, \tag{3.6}$$

$$s_3 = -\delta^{-1}w_3^{12}M_3^1\mu + \text{h.o.t.}, \tag{3.7}$$

$$(\delta^{-1}M_1^1\mu)^{\beta_2} = w_2^{12}(-\delta^{-1}w_3^{12}M_3^1\mu)^{1/\beta_3} - \delta^{-1}w_2^{12}M_2^1\mu + \text{h.o.t.}. \tag{3.8}$$

In the region $\{\mu\colon M_1^1\mu > 0, \Delta_3M_3^1\mu < 0, |\mu| \ll 1\}$, (3.8) defines an $(l-1)$-dimensional surface L_1^{23} which is tangent to L_2 at $\mu = 0$. Clearly, for $\mu \in L_1^{23}$, system (1.1) has a 1-homoclinic orbit Γ_1^{23} homoclinic to p_1 in the neighborhood of Γ.

Similarly, we can discuss the case $s_2 = 0$, $s_3 > 0$, $s_1 > 0$ (resp. $s_3 = 0$, $s_1 > 0$, $s_2 > 0$) and get the surface L_2^{31} (resp. L_3^{12}) situated in the region $\{\mu\colon M_2^1\mu > 0, \Delta_1M_1^1\mu < 0, |\mu| \ll 1\}$ (resp. $\{\mu\colon M_3^1\mu > 0, \Delta_2M_2^1\mu < 0, |\mu| \ll 1\}$). L_2^{31} and L_3^{12} are both tangent to L_2, and system (1.1) has a 1-homoclinic orbit Γ_2^{31} (resp. Γ_3^{21}) homoclinic to p_2 (resp. p_3) as $\mu \in L_2^{31}$ (resp. $\mu \in L_3^{12}$).

(4) By the above discussion and the existence regions defined above, it is easy to see that 3-point-loop, 2-point-loop and 1-homoclinic orbit cannot coexist.

The proof is complete.

§4. Bifurcations of 1-Periodic Orbits from Γ

At first, we give three lemmas which are on the bifurcation results of rough 2-point-loop (for the detail of proofs, see [13]).

Suppose that system (1.2) has two saddle points, $\beta_1 > 1$, $\beta_2 < 1$, $\beta_1\beta_2 < 1$, $\Delta_1 = \Delta_2 = 1$, $\mathrm{rank}(M_1^1,\ M_2^1) = 2$ and all hypotheses of Section one hold. Denote $R_1^2 = \{\mu : M_1^1\mu > 0,\ M_2^1\mu < 0,\ |\mu| \ll 1\}$, $R_2^1 = \{\mu : M_1^1\mu < 0,\ M_2^1\mu > 0,\ |\mu| \ll 1\}$. Then, we have the following three lemmas.

Lemma 4.1. (1) *There exists an* $(l-1)$*-dimensional surface* L_i *with normal vector* M_i^1 *at* $\mu = 0$*, such that* (1.1) *has a heteroclinic orbit joining* p_1 *and* p_2 *near* Γ_i *if and only if* $\mu \in L_i$ *and* $|\mu| \ll 1$*,* $i = 1, 2$*. Moreover,* (1.1) *has a heteroclinic loop near* Γ *if and only if* $|\mu| \ll 1$ *and* $\mu \in L_{12} = L_1 \cap L_2$ *which is an* $(l-2)$*-dimensional surface.*

(2) *There exists an* $(l-1)$*-dimensional surface* $L_1^2 \subset R_1^2$ *(resp.* $L_2^1 \subset R_2^1$*) tangent to* L_1 *at* $\mu = 0$ *such that* (1.1) *has a unique homoclinic loop* Γ_1^2 *(resp.* Γ_2^1*) connecting* p_1 *(resp.* p_2*) for* $\mu \in L_1^2$ *(resp.* $\mu \in L_2^1$*).*

Lemma 4.2. *In* R_1^2*, there are an* $(l-1)$*-dimensional surface* $\tilde{L}_1^2$ *near* $\mu = 0$ *tangent to* L_1 *at* $\mu = 0$*, and three open regions* $(R_1^2)_1$ *with boundaries* L_1 *and* L_1^2*,* $(R_1^2)_2$ *with boundaries* L_1^2 *and* $\tilde{L}_1^2$*, and* $(R_1^2)_0$ *with boundaries* $\tilde{L}_1^2$ *and* L_2*, such that*

(1) *System* (1.1) *has exactly one simple* 1*-periodic orbit near* Γ *as* $\mu \in (R_1^2)_1$*.*

(2) *System* (1.1) *has exactly one simple* 1*-periodic orbit and one* 1*-homoclinic orbit homoclinic to* p_1 *near* Γ *as* $\mu \in L_1^2$*.*

(3) *System* (1.1) *has exactly two simple* 1*-periodic orbits near* Γ *as* $\mu \in (R_1^2)_2$*.*

(4) *System* (1.1) *has a unique two-fold* 1*-periodic orbit near* Γ *as* $\mu \in \tilde{L}_1^2$*.*

(5) *System* (1.1) *has not any* 1*-periodic and* 1*-homoclinic orbit near* Γ *as* $\mu \in (R_1^2)_0$*.*

Lemma 4.3. *In the region* R_2^1*, there are two open regions* $(R_2^1)_0$ *and* $(R_2^1)_1$ *with boundaries* L_1*,* L_2^1 *and* L_2^1*,* L_2*, respectively, such that*

(1) (1.1) *has not any* 1*-periodic orbit and* 1*-homoclinic orbit near* Γ *as* $\mu \in (R_2^1)_0$*.*

(2) (1.1) *has exactly one* 1*-homoclinic orbit homoclinic to* p_2 *near* Γ *as* $\mu \in L_2^1$*.*

(3) (1.1) *has exactly one simple* 1*-periodic orbits near* Γ *as* $\mu \in (R_2^1)_1$*.*

Now, we consider the periodic orbits bifurcated from the heteroclinic loop, that is, consider the solutions of (3.1) satisfying $s_1 > 0$, $s_2 > 0$, $s_3 > 0$. For simplicity, we assume

(AII) $\Delta_1 = \Delta_2 = \Delta_3 = 1$.

Now, we have

$$\begin{aligned}
R_{12}^3 &= \{\mu : M_2^1\mu > 0, M_3^1\mu < 0, |\mu| \ll 1\},\\
R_{23}^1 &= \{\mu : M_3^1\mu > 0, M_1^1\mu < 0, |\mu| \ll 1\},\\
R_{31}^2 &= \{\mu : M_1^1\mu > 0, M_2^1\mu < 0, |\mu| \ll 1\},\\
R_1^{23} &= \{\mu : M_1^1\mu > 0, M_3^1\mu < 0, |\mu| \ll 1\},\\
R_2^{31} &= \{\mu : M_2^1\mu > 0, M_1^1\mu < 0, |\mu| \ll 1\},\\
R_3^{12} &= \{\mu : M_3^1\mu > 0, M_2^1\mu < 0, |\mu| \ll 1\}.
\end{aligned}$$

Theorem 4.1. *Suppose that hypotheses* **(H1)**–**(H4)**, (AI) *and* (AII) *hold. Then*

(1) *System* (1.1) *has exactly one* 2*-point-loop* Γ_{12} *and one simple* 1*-periodic orbit near* Γ *as* $\mu \in L_1 \cap L_{12}^3$ *and* $|\mu| \ll 1$*. Moreover, the* 1*-periodic orbit is persistent for* μ *changes near* $L_1 \cap L_{12}^3$*.*

(2) *Near* $L_1 \cap L_{12}^3$*, there exists an open region* S_{12}*, such that* (1.1) *has exactly two* 1*-periodic orbits near* Γ *as* $\mu \in S_{12}$*. Meanwhile,* L_1^{23} *and* L_2^{31} *are the boundaries of* S_{12}*.*

Proof. By (3.1), we get

$$D_1^{1/\beta_3}(s_1-\delta^{-1}M_3^1\mu)^{1/\beta_3}=(s_1^{\beta_1}+\delta^{-1}w_1^{12}M_1^1\mu)^{\beta_2}+\delta^{-1}w_2^{12}(w_1^{12})^{\beta_2}M_2^1\mu+\text{h.o.t.},\quad (4.1)$$

where $D_1=(w_3^{12})(w_2^{12})^{\beta_3}(w_1^{12})^{\beta_2\beta_3}$. Let $V_1(s_1)$ and $N_1(s_1)$ be the left and right hand of (4.1), respectively.

If $\mu\in L_1\cap L_{12}^3$, that is, (3.1) has solution $s_1=s_2=0$, $s_3>0$, then, by (3.1), (3.3) and (3.4), we have

$$\delta^{-1}M_1^1\mu+\text{h.o.t.}=0,\ \ s_2=(w_1^{12})^{-1}s_1^{\beta_1}+\text{h.o.t.},\quad (4.2)$$

and $M_2^1\mu>0$, $M_3^1\mu<0$. Let $\bar{s}_1=-\delta^{-1}M_3^1\mu$. By (3.1), (3.5), (4.1), (4.2) and some simplicity calculation, we can easily get $V_1(0)=N_1(0)$ and

$$\dot{V}_1(s_1)=\frac{1}{\beta_3}D_1^{1/\beta_3}(s_1+\bar{s}_1)^{1/\beta_3-1}+\text{h.o.t.},\ \ \dot{N}_1(s_1)=\beta_1\beta_2s_1^{\beta_1\beta_2-1}+\text{h.o.t.}.$$

Obviously, $0=\dot{N}_1(0)<\dot{V}_1(0)=\frac{1}{\beta_3}D_1^{1/\beta_3}(\bar{s}_1)^{1/\beta_3-1}+\text{h.o.t.}$ Therefore, there exists a positive number $\hat{s}_1$, $0<\hat{s}_1\ll\bar{s}_1$, such that

$$V_1(s_1)>N_1(s_1)\ \text{ for }\ 0<s_1<\hat{s}_1\ll\bar{s}_1.\quad (4.3)$$

On the other hand, it is easy to see that

$$V_1(\bar{s}_1)=D_1^{1/\beta_3}(2\bar{s}_1)^{1/\beta_3}<N_1(\bar{s}_1)=(\bar{s}_1)^{\beta_1\beta_2}+\delta^{-1}w_2^{12}(w_1^{12})^{\beta_2}M_2^1\mu+\text{h.o.t.}\quad (4.4)$$

as $0<\bar{s}_1,|\mu|\ll 1$ and $\beta_1\beta_2<1/\beta_3$.

Combining (4.3) with (4.4), we see that $V_1(s_1)=N_1(s_1)$ has at least one solution $s_1=s_1^*$ satisfying $0<s_1^*<\bar{s}_1$.

Now, we prove $N_1(s_1)>V_1(s_1)$ as $\bar{s}_1<s_1\ll 1$.

It is not difficult to see that

$$\begin{aligned}\dot{V}_1(s_1)&=\frac{1}{\beta_3}D_1^{1/\beta_3}(s_1+\bar{s}_1)^{1/\beta_3-1}+\text{h.o.t.}<\frac{1}{\beta_3}D_1^{1/\beta_3}(2s_1)^{1/\beta_3-1}+\text{h.o.t.}\\&=\frac{1}{2\beta_3}(2D_1)^{1/\beta_3}(s_1)^{1/\beta_3-1}+\text{h.o.t.}<\beta_1\beta_2s_1^{\beta_1\beta_2-1}+\text{h.o.t.}=\dot{N}_1(s_1),\end{aligned}\quad (4.5)$$

as $0<\bar{s}_1\le s_1\ll 1$, $0<|\mu|\ll 1$ and $\beta_1\beta_2<1/\beta_3$.

By (4.4) and (4.5) we obtain $N_1(s_1)>V_1(s_1)$ as $\bar{s}_1<s_1\ll 1$.

Next, we prove the uniqueness of the positive solution.

Based on the fact that $V_1(s_1)-N_1(s_1)=0$ has solutions $s_1=0$ and $s_1=s_1^*$, one can see that $\dot{V}_1(s_1)-\dot{N}_1(s_1)=0$ has at least one solution $s_1=\tilde{s}_1$ in $(0,\bar{s}_1)$. That is,

$$\dot{V}_1(\tilde{s}_1)=\frac{1}{\beta_3}D_1^{1/\beta_3}(\tilde{s}_1+\bar{s}_1)^{1/\beta_3-1}=\dot{N}_1(\tilde{s}_1)=\beta_1\beta_2(\tilde{s}_1)^{\beta_1\beta_2-1}+\text{h.o.t.},\quad (4.6)$$

$$\frac{\tilde{s}_1}{\tilde{s}_1+\bar{s}_1}=(\beta_1\beta_2\beta_3)^{\frac{1}{1-\beta_1\beta_2}}D_1^{\frac{1}{\beta_3(\beta_1\beta_2-1)}}(\tilde{s}_1+\bar{s}_1)^{\frac{1-\beta_1\beta_2\beta_3}{\beta_3(\beta_1\beta_2-1)}}+\text{h.o.t.}.\quad (4.7)$$

It follows from $\tilde{s}_1/(\tilde{s}_1+\bar{s}_1) \ll 1$ as $0<|\mu| \ll 1$ that we have

$$\begin{aligned}
&d^2[V_1(\tilde{s}_1)-N_1(\tilde{s}_1)]/d{s_1}^2\\
&=\frac{1-\beta_3}{\beta_3}\frac{1}{\beta_3}D_1^{\frac{1}{\beta_3}}(\tilde{s}_1+\bar{s}_1)^{\frac{1}{\beta_3}-2}-(\beta_1\beta_2-1)\beta_1\beta_2(\tilde{s}_1)^{\beta_1\beta_1-2}+\text{h.o.t.}\\
&=\frac{1-\beta_3}{\beta_3}(\tilde{s}_1+\bar{s}_1)^{-1}\dot{V}_1(\tilde{s}_1)-(\beta_1\beta_2-1)(\tilde{s}_1)^{-1}\dot{N}_1(\tilde{s}_1)\\
&=(\beta_1\beta_1-1)(\tilde{s}_1)^{-1}\dot{N}_1(\tilde{s}_1)\Big[\frac{1-\beta_3}{\beta_3(\beta_1\beta_2-1)}\cdot\frac{\tilde{s}_1}{\tilde{s}_1+\bar{s}_1}-1\Big]<0
\end{aligned}$$

as $0<|\mu| \ll 1$. Therefore, $s_1=s_1^*$ is the unique sufficiently small positive solution of equation $V_1(s_1)=N_1(s_1)$. Moreover, it turns out that (3.1) has a unique sufficiently small positive solution $s_2=s_2^*$, $s_3=s_3^*$ corresponding to $s_1=s_1^*$. That is, in addition to a 2-point-loop Γ_{12}, system (1.1) has a unique simple 1-periodic orbit near Γ for $\mu \in L_1 \cap L_{12}^3$. Clearly, s_1^* is a simple zero of $V_1(s_1)=N_1(s_1)$ which is persistent under small perturbation of μ. So, since μ changes in a sufficiently small neighborhood of $L_1 \cap L_{12}^3$, the simple 1-periodic orbit mentioned above can not vanish.

Thus, we have shown the first conclusion of the theorem.

By $\beta_1>1$, $\beta_2>1$, the bifurcations of 2-point rough heteroclinic loop (cf. [11]), it is not difficult to check the conclusion (2) of the theorem, we omit the detail.

Theorem 4.2. *Suppose that hypotheses* **(H1)**–**(H4)**, (AI) *and* (AII) *are valid. Then, the following conclusions are valid.*

(1) *For $\mu \in L_2 \cap L_{23}^1$, system* (1.1) *has not any* 1*-periodic orbit except the* 2*-point-loop Γ_{23} near Γ.*

(2) *In region $R_2^{31} \cap R_{23}^1$ there is an $(l-1)$-dimensional surface $\tilde{L}_2^{31}$ such that system* (1.1) *has exactly one* 2*-fold* 1*-periodic orbit for $\mu \in \tilde{L}_2^{31}$, where, $\tilde{L}_2^{31}$ is situated in an open region bounded by L_{23}^1 and L_2^{31}. Moreover, there exist three open regions $(R_2^{31})_i \subset R_2^{31} \cap R_{23}^1$, $i=0,1,2$, with boundaries L_{23}^1 and $\tilde{L}_2^{31}$, L_2^{31} and L_2, and $\tilde{L}_2^{31}$ and L_2^{31}, respectively, such that*

(i) (1.1) *has no* 1*-periodic orbit for $\mu \in (R_2^{31})_0$.*

(ii) (1.1) *has exactly two simple* 1*-periodic orbits for $\mu \in (R_2^{31})_2$.*

(iii) (1.1) *has exactly one* 1*-homoclinic orbit and one simple* 1*-periodic orbit for $\mu \in L_2^{31}$.*

(iv) (1.1) *has exactly one simple* 1*-periodic orbit for $\mu \in (R_2^{31})_1$.*

(3) *In region $R_{23}^1 \cap R_3^{12}$, there are two open regions $(R_3^{12})_0$ and $(R_3^{12})_1$ with boundaries L_2, L_3^{12} and L_3^{12}, L_{23}^1, respectively, such that system* (1.1) *has not any* 1*-periodic orbit for $\mu \in (R_3^{12})_0$, exactly one* 1*-homoclinic orbit for $\mu \in L_3^{12}$ and exactly one simple* 1*-periodic orbit for $\mu \in (R_3^{12})_1$, respectively.*

Proof. Due to (3.1), we have

$$V_2(s_2)=N_2(s_2), \tag{4.8}$$

where

$$\begin{aligned}
V_2(s_2)&=D_2^{1/\beta_1}(s_2-\delta^{-1}M_1^1\mu+\text{h.o.t.})^{1/\beta_1},\\
D_2&=w_1^{12}(w_3^{12})^{\beta_1}(w_2^{12})^{\beta_1\beta_3},\\
N_2(s_2)&=(s_2^{\beta_2}+\delta^{-1}w_2^{12}M_2^1\mu+\text{h.o.t.})^{\beta_3}+\delta^{-1}w_3^{12}(w_2^{12})^{\beta_3}M_3^1\mu+\text{h.o.t.}
\end{aligned}$$

If $\mu \in L_2 \cap L_{23}^1$, i.e. (3.1) has solution $s_2 = s_3 = 0$, $s_1 > 0$, then $V_2(0) = N_2(0)$. By (3.1), we can easily get $M_3^1\mu > 0$, $M_1^1\mu < 0$ and

$$\delta^{-1}M_2^1\mu + \text{h.o.t.} = 0, \quad s_3 = (w_2^{12})^{-\beta_3}s_2^{\beta_2\beta_3} + \text{h.o.t.}, \tag{4.9}$$

$$(\delta^{-1}M_3^1\mu + \text{h.o.t.})^{\beta_1} + \delta^{-1}w_1^{12}M_1^1\mu + \text{h.o.t.} = 0. \tag{4.10}$$

Let $\bar{s}_2 = -\delta^{-1}M_1^1\mu$. Substituting it into (4.8) and using (4.9) and (4.10), we have

$$\begin{aligned} V_2(s_2) &= D_2^{1/\beta_1}(s_2 + \bar{s}_2 + \text{h.o.t.})^{1/\beta_1}, \\ N_2(s_2) &= s_2^{\beta_2\beta_3} + D_2^{1/\beta_1}(\bar{s}_2 + \text{h.o.t.})^{1/\beta_1}, \end{aligned} \tag{4.11}$$

$$\begin{aligned} \dot{V}_2(s_2) &= \frac{1}{\beta_1}D_2^{1/\beta_1}(s_2 + \bar{s}_2 + \text{h.o.t.})^{1/\beta_1 - 1}, \\ \dot{N}_2(s_2) &= \beta_2\beta_3 s_2^{\beta_2\beta_3 - 1} + \text{h.o.t.}. \end{aligned} \tag{4.12}$$

Notice that $\beta_2\beta_3 < 1/\beta_1 < 1$ means

$$1 \ll \dot{V}_2(0) = \frac{1}{\beta_1}D_2^{1/\beta_1}(\bar{s}_2 + \text{h.o.t.})^{1/\beta_1 - 1} < +\infty \quad \text{and} \quad \dot{N}_2(0^+) = +\infty,$$

for $0 < |\mu| \ll 1$. So, $\dot{V}_2(0^+) < \dot{N}_2(0^+)$. Thus we have

$$V_2(s_2) < N_2(s_2) \quad \text{as} \ \ 0 < |\mu|, s_2 \ll 1. \tag{4.13}$$

On the other hand, it is easy to see that, for $\beta_2\beta_3 < 1/\beta_1 < 1$,

$$\dot{V}_2(s_2) < \frac{1}{\beta_1}D_2^{1/\beta_1}(s_2 + \text{h.o.t.})^{1/\beta_1 - 1} < \beta_2\beta_3 s_2^{\beta_2\beta_3 - 1} + \text{h.o.t.} = \dot{N}_2(s_2) \tag{4.14}$$

as $0 < |\mu|, s_2 \ll 1$.

Combining (4.13) with (4.14), we have $V_2(s_2) < N_2(s_2)$ for $0 < s_2 = O(|\mu|)$. That is to say, for $\mu \in L_2 \cap L_{23}^1$, (3.1) has not any solution satisfing $0 < s_2 \ll 1$.

Thus, we have shown the first conclusion of the theorem.

The conclusions (2) and (3) can be obtained by the bifurcation results of 2-point-loop for the case $\beta_2 > 1$, $\beta_3 < 1$ and $\beta_2\beta_3 < 1$ (see Lemmas 4.1, 4.2 and 4.3 or [13]).

Theorem 4.3. *Suppose that hypotheses* **(H1)**–**(H4)** *hold, and* (AI), (AII) *are valid. Then, the following conclusions are valid.*

(1) *For $\mu \in L_3 \cap L_{31}^2$, system* (1.1) *has not any* 1*-periodic orbit except the* 2*-point-loop* Γ_{31} *near* Γ.

(2) *In $R_1^{23} \cap R_{31}^2$, there is an $(l-1)$-dimensional surface $\tilde{L}_1^{23}$ such that system* (1.1) *has exactly one* 2*-fold* 1*-periodic orbit for $\mu \in \tilde{L}_1^{23}$, where, $\tilde{L}_1^{23}$ is situated in an open region bounded by L_3 and L_1^{23}. Moreover, there exist three open regions $(R_1^{23})_i \subset R_1^{23} \cap R_{31}^2$, $i = 0, 1, 2$, such that* (1.1) *has not any* 1*-periodic orbit for $\mu \in (R_1^{23})_0$, exactly two simple* 1*-periodic orbits for $\mu \in (R_1^{23})_2$, exactly one* 1*-homoclinic orbit and one simple* 1*-periodic orbit for $\mu \in L_1^{23}$, exactly one simple* 1*-periodic orbit for $\mu \in (R_1^{23})_1$, respectively. Here $(R_1^{23})_0$, $(R_1^{23})_1$ and $(R_1^{23})_2$ have boundaries L_3 and $\tilde{L}_1^{23}$, L_1^{23} and L_{31}^2, and $\tilde{L}_1^{23}$ and L_1^{23}, respectively.*

(3) *In $R_{31}^2 \cap R_3^{12}$, there exist two open regions $(R_3^{12})_0^*$ and $(R_3^{12})_1^*$ with boundaries L_{31}^2, L_3^{12} and L_3^{12}, L_3, respectively, such that system* (1.1) *has no* 1*-periodic orbit for $\mu \in (R_3^{12})_0^*$, exactly one* 1*-homoclinic orbit for $\mu \in L_3^{12}$ and exactly one simple* 1*-periodic orbit for $\mu \in (R_3^{12})_1^*$, respectively.*

The Proof is similar to that of Theorem 4.2.

Remark 4.1. For the case $\Delta_i = \Delta_j = -1$, $i \neq j$, $\Delta_1\Delta_2\Delta_3 = 1$, we can discuss in a similar way.

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