Chin. Ann. Math.
24B:1(2003),97-110.

EXPLICIT CONSTRUCTION FOR LOCAL ISOMETRIC IMMERSIONS OF SPACE FORMS***

HE QUN* SHEN YIBING**

Abstract

By using Darboux transformations, the authors give the explicit construction for local isometric immersions of space forms $M^n(c)$ into space forms $M^{2n-1}(c+\varepsilon^2)$ via purely algebraic algorithm.

Keywords Space form, Isometric immersion, Darboux transformation
2000 MR Subject Classification 53C42, 35Q58
Chinese Library Classification O86.16 **Document Code** A
Article ID 0252-9599(2003)01-0097-14

§0. Introduction

Isometric immersions of space forms $M^n(c)$ of curvature c into space forms $M^N(\tilde{c})$ of curvature $\tilde{c}$ have been studied by many geometers. Due to the complicated structure of the integrability condition for isometric immersions, i.e., Gauss-Codazzi-Ricci equations, over the past decades one focused mainly on the study of the nonexistence rather than the explicit construction (see, e.g., [2, 6, 9, 15], etc.). Recently, it has been found that these equations admit "Lax pairs", i.e., they can be written as the condition for a family of connections to be flat. This enable us to use the soliton theory to study some problems on isometric immersions of space forms. In [11] the local isometric immersions from $M^n(c)$ into $M^{2n}(c)$ with flat normal bundle and linearly independent curvature normals were discussed. The Darboux transformation for the explicit expressions of such isometric immersions was given in [16]. A general soliton theory on isometric immersions of space forms $M^n(c)$ into $M^N(\tilde{c})$ with flat normal bundle and $0 \neq c \neq \tilde{c} \neq 0$ was proposed in [3]. When $c > \tilde{c}$, there exists a standard isometric, totally umbilical embedding $i_0 : M^n(c) \to M^{n+1}(\tilde{c})$ (e.g., see [8]). When $c < \tilde{c}$, it is proved by E.Cartan[1] that $M^n(c)$ cannot be locally, isometrically, immersed into $M^{2n-2}(\tilde{c})$, but can be into $M^{2n-1}(\tilde{c})$. Moreover, by the work of J.D.Moore[7]

Manuscript received July 16, 2001.
*Department of Applied Mathematics, Tongji University, Shanghai 200092, China.
E-mail: qun_he@163.com
**Department of Mathematics, West Brook, Zhejiang University, Hangzhou 310028, China.
E-mail: ybsh@mail.hz.zj.cn
***Project supported by the National Natural Science Foundation of China (No.10271106), the Science Foundation of the Ministry of Education of China, and the Natural Science Foundation of Zhejiang Province, China.

and E.Cartan[1], the isometric immersions $M^n(c) \to M^{2n-1}(\tilde{c})$ with $c < \tilde{c}$ must have flat normal bundle and linearly independent curvature normals.

The purpose of this paper is to give the explicit expressions of the local isometric immersions of the space form $M^n(c)$ into the space form $M^{2n-1}(c+\varepsilon^2)$ via the Darboux transformation.

For this problem, the Lax pair has a twisted $so(n)$ reduction described as in [16]. Here we use the dressing action by two simple rational elements to express explicitly the Darboux transformation instead of the singular Darboux transformation (a Darboux transformation by a limit process[16]). A more general method to construct the Darboux matrix for the twisted $so(p, n-p)$-hierarchy is proposed (Theorem 1.1).

Section 2 gives the Darboux transformatin for the local isometric immersion from $M^n(c)$ into $M^{2n-1}(c+\varepsilon^2)$, the Lax set of which was shown in [11]. We present the general explicit expression of the transformation for the position vector of $M^n(c)$ into R_c^{2n+1} (Theorem 2.1), where R_c^{2n+1} denotes R^{2n+1} or $R^{2n,1}$.

In Section 3, we give the concrete explicit expression for local isometric immersions of $M^n(c)$ into $M^{2n-1}(c+\varepsilon^2)$ derived from trivial solutions. Some interesting examples are given, including those corresponding to the well-known sine-Gordon equation and the wave equation (see [4, 5, 10, 12]).

§1. Bäcklund and Darboux Transformations for the Twisted $so(p, N-p)$-hierarchy

Let

$$J = \begin{pmatrix} -I_p & 0 \\ 0 & I_{N-p} \end{pmatrix} \qquad (p \in \mathbb{N}; 0 \le p \le N-1), \tag{1.1}$$

where I_p and I_{N-p} are identity matrices of orders p and $N-p$, respectively. We endow $\mathbb{C}^N$ the following J-Hermitian metric $\langle , \rangle_J$:

$$\langle w, z \rangle_J = \langle w, Jz \rangle, \qquad \forall w, z \in \mathbb{C}^N,$$

where $\langle , \rangle$ stands for the canonical Hermitian metric of $\mathbb{C}^N$. The isometric group $U(p, N-p)$ of $(\mathbb{C}^N, \langle , \rangle_J)$ and its Lie algebra $u(p, N-p)$ are respectively

$$U(p, N-p) = \{y \in GL(N, \mathbb{C}) \mid yJy^* = J\},$$
$$u(p, N-p) = \{X \in gl(N, \mathbb{C}) \mid XJ + JX^* = 0\}.$$

Consider the group $SU(p, N-p) = \{y \in U(p, N-p) \mid \det y = 1\}$ and its Lie algebra $su(p, N-p) = \{X \in u(p, N-p) \mid \operatorname{tr} X = 0\}$. Clearly, as real forms of $SU(p, N-p)$ and $su(p, N-p)$, $SO(p, N-p)$ and $so(p, N-p)$ can be expressed respectively as

$$\begin{aligned} G = SO(p, N-p) &= \{y \in SU(p, N-p) \mid \overline{y} = y\}, \qquad G^{\mathbb{C}} = SU(p, N-p), \\ g = so(p, N-p) &= \{X \in su(p, N-p) \mid \overline{X} = X\}, \qquad g^{\mathbb{C}} = su(p, N-p). \end{aligned} \tag{1.2}$$

Let σ $(\neq I_N, J)$ be a diagonal matrix such that $\sigma^2 = I_N$, which induces an involution on g, $X \mapsto \sigma X \sigma$. Thus, there is the Cartan decomposition $g = \mathcal{K} \oplus \mathcal{P}$ where $\mathcal{K}$ and $\mathcal{P}$ are the $+1$ and -1 eigenspaces, respectively, satisfying

$$[\mathcal{K}, \mathcal{K}] \subset \mathcal{K}, \quad [\mathcal{K}, \mathcal{P}] \subset \mathcal{P}, \quad [\mathcal{P}, \mathcal{P}] \subset \mathcal{K}.$$

Let K be the subgroup corresdonding to $\mathcal{K}$. Then G/K is a symmetric space. An $A(\lambda)$ $(\in SL(N, \mathbb{C})$ for $\lambda \in \mathbb{C})$ is said to satisfy the G/K-reality condition if

$$A(\lambda) J A(\bar{\lambda})^* = J, \qquad \overline{A(\bar{\lambda})} = A(\lambda), \qquad \sigma A(\lambda) \sigma = A(-\lambda). \tag{1.3}$$

For a fixed $a \in \mathcal{P}$, let

$$g_a = \{X \in g \mid [X,a]=0\}, \quad g_a^\perp = \{X \in g \mid \mathrm{tr}(XY)=0,\ \forall Y \in g_a\},$$
$$\wedge_\sigma g = \Big\{A(\lambda) = \sum_k A_k \lambda^k \mid A_k \in g,\ \sigma A(\lambda)\sigma = A(-\lambda)\Big\}.$$

Clearly, $A(\lambda) \in \wedge_\sigma g$ if and only if A_k is in $\mathcal{K}$ when k is even, and is in $\mathcal{P}$ when k is odd.

Consider the following linear system:

$$\begin{cases} d\Phi_\lambda = \Phi_\lambda(\lambda a + [a,v]), \\ \Phi_\lambda(0) = I_N, \end{cases} \tag{1.4}$$

where a is a $\mathcal{P}$-valued 1-form, $v : \mathbb{R}^n \to g_a^\perp \cap \mathcal{P}$. Thus, $[a,v]$ is a $(g_a^\perp \cap \mathcal{K})$-valued 1-form, and $(\lambda a + [a,v])$ is a $\wedge_\sigma g$-valued 1-form. Suppose that $\Phi(x,\lambda) = \Phi_\lambda(x)$, a solution to (1.4), is holomorphic with respect to $\lambda \in \mathbb{C}$. By the uniqueness of the solution to (1.4), we see that Φ_λ satisfies the G/K-reality condition (1.3).

Let O_∞ be an open subset near ∞ in $S^2 = \mathbb{C} \cup \{\infty\}$, and let

$$G_+ = \{f : \mathbb{C} \to GL(N,\mathbb{C}) \mid\ f \text{ is holomorphic, } f(\lambda)Jf(\bar\lambda)^* = J\},$$
$$G_- = \{f : O_\infty \to GL(N,\mathbb{C}) \mid\ f \text{ is holomorphic, } f(\lambda)Jf(\bar\lambda)^* = J,\ f(\infty) = I_N\},$$
$$G_-^m = \{f(\lambda) \in G_- \mid f(\lambda) \text{ is a rational faction}\},$$
$$(G_-^m)_\sigma = \{f(\lambda) \in G_-^m \mid f(\lambda) \text{ satisfies } G/K\text{-reality condition (1.3)}\}.$$

Suppose that π is a J-Hermitian projection in $\mathbb{C}^N$, i.e., π satisfies $\pi^2 = \pi,\ \pi^* = J\pi J$. Let $\pi^\perp = I - \pi$ be the complementary J-orthogonal projection (with respect to the J-Hermitian metric $\langle , \rangle_J$). Then a simple element of G_-^m is

$$\xi_{\alpha,\pi}(\lambda) = \pi + \frac{\lambda-\alpha}{\lambda-\bar\alpha}\pi^\perp = I - \frac{\alpha-\bar\alpha}{\lambda-\bar\alpha}\pi^\perp \tag{1.5}$$

for a parameter $\alpha \in \mathbb{C}$. Obviously, we have $\xi_{\alpha,\pi}^{-1} = \xi_{\bar\alpha,\pi},\ \xi_{-\alpha,\pi}(\lambda) = \xi_{\alpha,\pi}(-\lambda)$. By using the method of the proof of Theorem 5.4 in [14], one can prove that G_-^m is generated by simple elements formed as (1.5). Thus, we need only consider the dressing actions of simple elements.[13] Let $\tilde\pi$ be a J-Hermitian projection in the trivial bundle $\mathbb{R}^n \times \mathbb{C}^N$, and let $\underline{\tilde\pi} = \mathbb{R}^n \times (\mathrm{Im}\tilde\pi)$.

Lemma 1.1. *Let $\Phi_\lambda : \mathbb{R}^n \to G_+$ be a solution to* (1.4), *and $\xi_{\bar\alpha,\pi}$ a simple element. Suppose that π is a J-Hermitian projection in $\mathbb{C}^N$. Set*

$$\underline{\tilde\pi} = \Phi_\alpha^{-1}\underline{\pi} = J\Phi_{\bar\alpha}^* J\underline{\pi}, \qquad \tilde\Phi_\lambda = \xi_{\bar\alpha,\pi}\Phi_\lambda\xi_{\alpha,\tilde\pi}.$$

Then there is an open neighborhood U near the origin 0 *in $\mathbb{R}^n$ such that $\tilde\Phi_\lambda : U \to G_+$ satisfies the following system*

$$\begin{cases} d\tilde\Phi_\lambda = \tilde\Phi_\lambda(\lambda a + [a, v + (\bar\alpha - \alpha)\tilde\pi^\perp]), \\ \tilde\Phi_\lambda(0) = I_N. \end{cases} \tag{1.6}$$

Proof. By Proposition 4.2 and the proof of Theorem 4.3 in [13], we see that there is an open neighborhood U near the origin of $\mathbb{R}^n$ such that on U the J-orthogonal complementary subbundle of $\underline{\tilde\pi}$ is

$$\underline{\tilde\pi}^\perp = \Phi_{\bar\alpha}^{-1}\underline{\pi}^\perp = J\Phi_\alpha^* J\underline{\pi}^\perp.$$

Thus, $\tilde\Phi_\lambda : U \to G_+$ and $\tilde\Phi_\lambda^{-1} d\tilde\Phi_\lambda$ is holomorphic with respect to $\lambda \in \mathbb{C}$. The asymptotic expansion of $\xi_{\alpha,\tilde\pi}$ at ∞ is

$$\xi_{\alpha,\tilde\pi} \sim I + (\bar\alpha - \alpha)\lambda^{-1}\tilde\pi^\perp + O(\lambda^{-2}).$$

We then have

$$\tilde\Phi_\lambda^{-1}d\tilde\Phi_\lambda = \xi_{\alpha,\tilde\pi}^{-1}(\Phi_\lambda^{-1}d\Phi_\lambda)\xi_{\alpha,\tilde\pi} + \xi_{\alpha,\tilde\pi}^{-1}d(\xi_{\alpha,\tilde\pi})$$
$$= \lambda a + [a, v + (\bar\alpha - \alpha)\tilde\pi^\perp] + O(\lambda^{-1}).$$

Since $\tilde\Phi_\lambda^{-1}d\tilde\Phi_\lambda$ is holomorphic in $\mathbb{C}$, then $O(\lambda^{-1}) = 0$, which implies $(1.6)_1$. The condition $(1.6)_2$ follows directly from the fact that $\tilde\pi(0) = \pi$.

As in Lemma 1.1, $\tilde\Phi_\lambda = \xi_{\bar\alpha,\pi}\Phi_\lambda\xi_{\alpha,\tilde\pi}$ is called the dressing action by the simple element $\xi_{\alpha,\pi}$ in [13].

By Lemma 1.1 and the uniqueness of the Birkhoff factorization, we have immediatly the following Bäcklund transformation.[13]

Proposition 1.1. *Let Φ_λ be a solution to (1.4), and $f(\lambda) \in (G_-^m)_\sigma$. Then there are an open neighborhood U near the origin 0 in $\mathbb{R}^n$ and a unique smooth map $D : U \to (G_-^m)_\sigma$ such that $\tilde\Phi_\lambda = f(\lambda)\Phi_\lambda D_\lambda$ satisfies the following system:*

$$\begin{cases} d\tilde\Phi_\lambda = \tilde\Phi_\lambda(\lambda a + [a, \tilde v]), \\ \tilde\Phi_\lambda(0) = I_N, \end{cases} \tag{1.7}$$

where $\tilde v = v + (d_1)_{g_a^\perp} \in g_a^\perp \cap \mathcal{P}$, d_1 is the coefficient of the term λ^{-1} in the asymptotic expansion of D_λ at ∞.

In order to express explicitly the Bäcklund transformation in Proposition 1.2, we consider the dressing action by two simple elements of $(G_-^m)_\sigma$ because there is no non-trivial simple element in $(G_-^m)_\sigma$. Let τ be a diagonal complex matrix such that $\tau^2 = \sigma$. Note that $\tau^{-1} = \tau^* = \bar\tau = \tau^3$.

Lemma 1.2. *Let π_0 be a real J-Hermitian projection in $\mathbb{C}^N$, i.e., $\bar\pi_0 = \pi_0$, such that $\sigma\pi_0\sigma\pi_0 = \pi_0\sigma\pi_0\sigma$. Set $\pi = \tau^{-1}\pi_0\tau$. Then, $f(\lambda) = \xi_{\alpha,\pi}\xi_{-\alpha,\sigma\pi\sigma}$ is in $(G_-^m)_\sigma$ for $\alpha \in i\mathbb{R}$.*

Proof. It is clear that $\sigma\pi\sigma\pi = \pi\sigma\pi\sigma$ and $\bar\pi = \sigma\pi\sigma$. Then we have

$$\sigma f(\lambda)\sigma = \xi_{\alpha,\sigma\pi\sigma}(\lambda)\xi_{-\alpha,\pi}(\lambda) = \xi_{\alpha,\pi}(-\lambda)\xi_{-\alpha,\sigma\pi\sigma}(-\lambda) = f(-\lambda),$$
$$\overline{f(\bar\lambda)} = \xi_{\bar\alpha,\bar\pi}(\lambda)\xi_{-\bar\alpha,\sigma\bar\pi\sigma}(\lambda) = \xi_{-\alpha,\sigma\pi\sigma}(\lambda)\xi_{\alpha,\pi}(\lambda) = f(\lambda).$$

In fact, if $\sigma\pi\sigma\pi = \pi\sigma\pi\sigma$, then $(\sigma\pi^\perp\sigma)\pi^\perp = \pi^\perp(\sigma\pi^\perp\sigma)$, which yields that $\underline{\pi}^\perp$ can be decomposed as $\underline{\pi_1}^\perp \oplus \underline{\pi_2}^\perp$ such that $\pi_1^\perp\pi_2^\perp = 0$, $\pi_1^\perp\sigma\pi_1^\perp = 0$ and $\sigma\pi_2^\perp\sigma = \pi_2^\perp$. By using a direct computation, we can see that $\xi_{\alpha,\pi}\xi_{-\alpha,\sigma\pi\sigma} = \xi_{\alpha,\pi_1}\xi_{-\alpha,\sigma\pi_1\sigma}$. Hence, without loss of generality, we need only consider the case that π satisfies $\pi^\perp\sigma\pi^\perp = 0$.

Let Q be a real constant $s \times N$ matrix satisfying that $QJ\sigma Q^T = 0$, $\det(QJQ^T) \neq 0$. On putting

$$\pi_0^\perp = JQ^T(QJQ^T)^{-1}Q, \qquad \pi^\perp = \tau^{-1}\pi_0^\perp\tau, \tag{1.8}$$

we see easily that $\pi^\perp\sigma\pi^\perp = 0$. Set

$$h = Q\tau\Phi_\alpha, \qquad \tilde\pi_1^\perp = Jh^*(hJh^*)^{-1}h,$$
$$\hat\Phi_\lambda = \xi_{\bar\alpha,\pi}\Phi_\lambda\xi_{\alpha,\tilde\pi_1}, \qquad \tilde h = Q\tau\sigma\hat\Phi_{-\alpha}, \qquad \tilde\pi_2^\perp = J\tilde h^*(\tilde hJ\tilde h^*)^{-1}\tilde h.$$

For $\alpha \in i\mathbb{R}$, we have

$$\hat\Phi_{-\alpha} = \hat\Phi_{\bar\alpha} = \pi\Phi_{\bar\alpha} + \pi^\perp\Phi_{\bar\alpha}\tilde\pi_1^\perp - 2\alpha\pi\dot\Phi_{\bar\alpha}\tilde\pi_1^\perp,$$
$$\tilde h = Q\tau\sigma\Phi_{\bar\alpha} - 2\alpha Q\tau\sigma\dot\Phi_{\bar\alpha}\tilde\pi_1^\perp = h\sigma + i\alpha\rho\Delta^{-1}h,$$

where $\dot\Phi_{\bar\alpha} = (d\Phi_\lambda/d\lambda)|_{\lambda=\bar\alpha}$, $\rho = iQ\tau\sigma\dot\Phi_{\bar\alpha}Jh^*$, $\Delta = \frac{1}{2}hJh^*$. Since

$$\dot\Phi_{\bar\alpha} = -\sigma\dot\Phi_\alpha\sigma, \qquad \dot\Phi_\alpha J\Phi_{\bar\alpha}^* = -\Phi_\alpha J(\dot\Phi_{\bar\alpha})^*,$$

noting that $\tau\Phi_\alpha\tau$ and $i\tau\dot{\Phi}_\alpha\tau$ are real we have

$$\rho = iQ\tau\sigma\dot{\Phi}_{\bar{\alpha}}J\Phi^*_\alpha\tau^*Q^T = iQ(\tau\Phi_\alpha\tau)J(\tau\dot{\Phi}_{\bar{\alpha}}\tau)^*\sigma Q^T = -\rho^* = -\rho^T,$$

which means that ρ is a real skew-symmetric $s\times s$ matrix.

Let $\tilde{\Delta} = \frac{1}{2}\tilde{h}\ J\tilde{h}^*$. Then it is easy to see that $\tilde{\Delta} = \Delta + \alpha^2\rho\Delta^{-1}\rho$. It follows that

$$\begin{aligned} D_\lambda &:= \xi_{\alpha,\tilde{\pi}_1}\xi_{-\alpha,\tilde{\pi}_2} = \Big(I - \frac{2\alpha}{\lambda+\alpha}\tilde{\pi}_1^\perp\Big)\Big(I + \frac{2\alpha}{\lambda-\alpha}\tilde{\pi}_2^\perp\Big) \\ &= I - \frac{\alpha}{\lambda+\alpha}Jh^*\Delta^{-1}h \\ &\quad + \frac{\alpha}{\lambda-\alpha}J(\sigma h^* - i\alpha h^*\Delta^{-1}\rho)(\Delta + \alpha^2\rho\Delta^{-1}\rho)^{-1}(h\sigma + i\alpha\rho\Delta^{-1}h) \\ &\quad + \frac{2i\alpha^3}{\lambda^2-\alpha^2}Jh^*\Delta^{-1}\rho(\Delta+\alpha^2\rho\Delta^{-1}\rho)^{-1}(h\sigma + i\alpha\rho\Delta^{-1}h). \end{aligned} \tag{1.9}$$

$$\begin{aligned} d_1 &= \frac{dD_{\lambda^{-1}}}{d\lambda}\Big|_{\lambda=0} = 2\alpha(\tilde{\pi}_2^\perp - \tilde{\pi}_1^\perp) \\ &= \alpha J\{(\sigma h^* - i\alpha h^*\Delta^{-1}\rho)(\Delta+\alpha^2\rho\Delta^{-1}\rho)^{-1}(h\sigma + i\alpha\rho\Delta^{-1}h) - h^*\Delta^{-1}h\}. \end{aligned} \tag{1.10}$$

If $s = 1$, i.e., Q is a nonzero row vector in $\mathbb{R}^n$, then we have $\rho = 0$, $\tilde{h} = h\sigma$, i.e., $\tilde{\pi}_2^\perp = \sigma\tilde{\pi}_1^\perp\sigma$. In such a case, (1.9) and (1.10) are reduced to

$$D_\lambda = I - \frac{\alpha}{\Delta}J\left\{\frac{h^*h}{\lambda+\alpha} - \frac{\sigma h^*h\sigma}{\lambda-\alpha}\right\},\quad d_1 = \frac{\alpha}{\Delta}J\sigma[h^*h,\sigma],$$

which have been shown in [16] in a different way.

Summing up, we have proved the following

Theorem 1.1. *Let Φ_λ be a solution to* (1.4), *and Q a real constant $s\times N$ matrix satisfying that $QJ\sigma Q^T = 0$ and $det(QJQ^T)\neq 0$. Set $h = Q\tau\Phi_\alpha$ for $\alpha\in i\mathbb{R}$ and $\alpha\neq 0$. Then $\tilde{\Phi}_\lambda = D_\lambda(0)^{-1}\Phi_\lambda D_\lambda$ is a solution to* (1.7), *where D_λ and d_1 are given respactively by* (1.9) *and* (1.10), *and $\tilde{v} = v + (d_1)_{g_a^\perp}$.*

Remark 1.1. If we take $\tilde{\Phi}_\lambda = \Phi_\lambda D_\lambda$ in Theorem 1.1, then $\tilde{\Phi}_\lambda$ satisfies the equation $(1.7)_1$. Hence, D_λ defined by (1.9) is a Darboux matrix of order two. Such $\tilde{\Phi}_\lambda$ without the normarized condition $(1.7)_2$ may have polar points.

§2. Local Isometric Immersions of Space Forms into Space Forms

Consider local isometric immersions from an n-dimensional space form $M^n(c)$ of constant curvature c into a $(2n-1)$-dimensional space form $M^{2n-1}(c+\varepsilon^2)$ of constant curvature $c+\varepsilon^2$ with $\varepsilon\in\mathbb{R}\setminus\{0\}$. Without loss of generality, we can assume that $c = 0, \pm 1$. Let $U\subset M^n(c)$ be a simply connected open subset of $M^n(c)$, and $\varphi : U\to M^{2n-1}(c+\varepsilon^2)$ a local isometric immersion. By the work of J.D.Moore[7] and E. Cartan[1], it is known that the normal bundle of φ is flat, and there exist a line of curvature coordinates $(x_1,\cdots,x_n)$ on U such that the first and second fundamental forms of the immersion φ are given by

$$\mathrm{I} = \sum_i b_i^2 dx_i^2,\quad \mathrm{II} = \sum_{\alpha,i}\varepsilon a_{\alpha i}b_i dx_i^2\mathbf{n}_\alpha, \tag{2.1}$$

where $\{\mathbf{n}_1,\cdots,\mathbf{n}_{n-1}\}$ is a parallel normal frame field. Here and from now on, we use the following convention on ranges of indices unless otherwise stated:

$$1\le i,j,k,\cdots\le n;\qquad 1\le\alpha,\beta,\cdots\le n-1.$$

It is known from [11] and [10] that

$$A=(a_{ij}):U\to O(n)\qquad \text{with } a_{nj}=b_j \tag{2.2}$$

is a smooth map.

Let $i_0 : M^{2n-1}(c+\varepsilon^2)\to M^{2n}(c)$ be the standard isometric, totally umbilical embedding (see [8]), and $\mathbf{r}_c : M^{2n}(c)\to R_c^{2n+1}$ the standard isometric embedding given by

$$\begin{aligned}
M^{2n}(0)&=\{(x_0,x_1,\cdots,x_{2n})\in\mathbb{R}^{2n+1}\mid x_0=0\},\\
M^{2n}(1)&=\{(x_0,x_1,\cdots,x_{2n})\in\mathbb{R}^{2n+1}\mid x_1^2+\cdots+x_{2n}^2+x_0^2=1\},\\
M^{2n}(-1)&=\{(x_0,x_1,\cdots,x_{2n})\in\mathbb{R}^{2n,1}\mid x_1^2+\cdots+x_{2n}^2-x_0^2=-1\}.
\end{aligned}$$

Then the compositon map $\mathbf{r}=\mathbf{r}_c\circ i_0\circ\varphi : U\to R_c^{2n+1}$, i.e.,

$$\mathbf{r}:U\subset M^n(c)\xrightarrow{\varphi}M^{2n-1}(c+\varepsilon^2)\xrightarrow{i_0}M^{2n}(c)\xrightarrow{\mathbf{r}_c}R_c^{2n+1} \tag{2.3}$$

is a local isometric immersion into R_c^{2n+1} with flat normal bundle.

Set $J_c=\begin{pmatrix}c&0\\0&I_{2n}\end{pmatrix}$. Then we have $\mathbf{r}J_c\mathbf{r}^T=c$ for $c=\pm1$, and $\mathbf{r}\in\mathbb{R}^{2n}$ for $c=0$. On putting $\mathbf{r}_i=\partial_i\mathbf{r}$ where $\partial_i=\partial/\partial x_i$, we see the structure equations of immersions (2.3) are

$$\begin{aligned}
\partial_j\mathbf{r}_i&=\frac{\partial_j b_i}{b_i}\mathbf{r}_i+\frac{\partial_i b_j}{b_j}\mathbf{r}_j\qquad(i\neq j)\\
\partial_i\mathbf{r}_i&=-\sum_{k\neq i}\frac{b_i\partial_k b_i}{b_k^2}\mathbf{r}_k+\frac{\partial_i b_i}{b_i}\mathbf{r}_i+\varepsilon b_i a_{\alpha i}\mathbf{n}_\alpha+\varepsilon b_i^2\mathbf{n}_n-cb_i^2\mathbf{r},\\
\partial_i\mathbf{n}_\alpha&=-\frac{\varepsilon}{b_i}a_{\alpha i}\mathbf{r}_i,\quad \partial_i\mathbf{n}_n=-\varepsilon\mathbf{r}_i,
\end{aligned}\tag{2.4}$$

where $\mathbf{n}_n$ is the normal frame field of the immersion $i_0 : M^{2n-1}(c+\varepsilon^2)\to M^{2n}(c)$.

Let

$$e_0=\mathbf{r},\qquad e_i=b_i^{-1}\mathbf{r}_i,\qquad e_{n+i}=\sum_j a_{ji}\mathbf{n}_j, \tag{2.5}$$

and set

$$\begin{aligned}
&b=(b_1,\cdots,b_n)(\in S^{n-1}),\qquad \delta=\mathrm{diag}(dx_1,\cdots,dx_n),\\
&\Xi=(e_0,e_1,\cdots,e_{2n})\qquad\text{with}\qquad \Xi(0)=J_c^2,\\
&F=(f_{ij})\in gl(n)_*=\{Y=(y_{ij})\in gl(n)\mid y_{ii}=0\},\quad\text{where } f_{ij}=\frac{\partial_j b_i}{b_j}\ (i\neq j).
\end{aligned}\tag{2.6}$$

For simplisity, we write an $m\times(2n+1)$ matrix Ψ as a row matrix

$$\begin{matrix}\ \ 1 & n & n\end{matrix}$$
$$\Psi=(\ \Psi^{(1)}\quad \Psi^{(2)}\quad \Psi^{(3)}\ ).$$

In particular, we write a $(2n+1)\times(2n+1)$ matrix Ψ as a block matrix

$$\begin{matrix} & 1 & n & n & \\ & & & & \end{matrix}$$
$$\begin{pmatrix}\Psi^{(1,1)}&\Psi^{(1,2)}&\Psi^{(1,3)}\\ \Psi^{(2,1)}&\Psi^{(2,2)}&\Psi^{(2,3)}\\ \Psi^{(3,1)}&\Psi^{(3,2)}&\Psi^{(3,3)}\end{pmatrix}\begin{matrix}1\\ n\\ n\end{matrix}$$

If we take

$$a=\begin{pmatrix}0&0&0\\0&0&\delta\\0&-\delta&0\end{pmatrix},\quad v=\begin{pmatrix}0&0&-cb\\0&0&-F^T\\b^T&F&0\end{pmatrix},\quad [a,v]=\begin{pmatrix}0&-cb\delta&0\\ \delta b^T&\omega&0\\0&0&\theta\end{pmatrix}, \tag{2.7}$$

where $\theta = \delta F^T - F\delta,\ \omega = \delta F - F^T\delta$, then the system (2.4) can be written as

$$d\Xi = \Xi(-\varepsilon a + [a, v]). \tag{2.8}$$

Clearly, $\mathbf{r} = \Xi^{(1)},\ b^T = v^{(3,1)},\ F = v^{(3,2)}$. Then the Gauss-Codazzi-Ricci equation of the immersion, i.e., the integrability condition of (2.8), is a system for (A, F) :

$$\begin{cases} dA = A\theta, \\ d\omega + \omega\wedge\omega - c\delta b^T \wedge b\delta = 0, \\ b = E_n A, \end{cases} \tag{2.9}$$

where

$$E_i = \mathrm{diag}(\underbrace{0, \cdots, 0}_{(i-1)}, 1, 0, \cdots, 0). \tag{2.10}$$

Consider the following Lax set for (2.8):

$$\begin{cases} d\Phi_\lambda = \Phi_\lambda\Theta_\lambda, \qquad \text{where } \Theta_\lambda = a\lambda + [a, v], \\ \Phi_\lambda(0) = J_c^2. \end{cases} \tag{2.11}$$

Clearly, $\Xi = \Phi_{-\varepsilon},\ A = A(0)\Phi_0^{(3,3)}$.

Let $so_{ex}(p, q, r) = \{X \in sl(p + q + r, \mathbb{R}) \mid X\tilde{J} + \tilde{J}X^T = 0\}$, where

$$\tilde{J} = \mathrm{diag}(\underbrace{0, \cdots, 0}_{p}, \underbrace{-1, \cdots, -1}_{q}, \underbrace{1, \cdots, 1}_{r}).$$

Obviously, $so_{ex}(0, 0, r) = so(r)$ and $so_{ex}(0, 1, r) = so(1, r)$. Let

$$g = \begin{cases} so_{ex}(1, 0, 2n) & \text{for } c = 0, \\ so(2n + 1) & \text{for } c = 1, \\ so(1, 2n) & \text{for } c = -1. \end{cases}$$

Set

$$\sigma = \begin{pmatrix} -I_{n+1} & 0 \\ 0 & I_n \end{pmatrix}, \qquad \mathcal{K} = \left\{ \begin{pmatrix} 0 & -c\xi & 0 \\ \xi^T & Y & 0 \\ 0 & 0 & X \end{pmatrix} \Bigg| X, Y \in so(n),\ \xi \in \mathbb{R}^n \right\},$$

$$\mathcal{P} = \left\{ \begin{pmatrix} 0 & 0 & -c\xi \\ 0 & 0 & -X \\ \xi^T & X^T & 0 \end{pmatrix} \Bigg| X \in gl(n, \mathbb{R}),\ \xi \in \mathbb{R}^n \right\}.$$

Clearly, a is a $\mathcal{P}$-valued 1-form, and $v(x) \in \mathcal{P}\cap g_a^\perp$, i.e., Θ_λ is a $\wedge_\sigma g$-valued 1-form. When $c \neq 0$, (2.9) holds if and only if the system (2.11) has a unique solution Φ_λ satisfying that $\Phi_\lambda(x) \in \wedge_\sigma G$, $b(x) \in S^{n-1}$. When $c = 0$, the (2.9) holds if and only if the system (2.11) has a unique solution Φ_λ such that Φ_λ satisfies (1.3) and $b(x) \in S^{n-1}$. Hence, in the following, we give a unified treatment for the cases that $c \neq 0$ and $c = 0$. The following result can be found in [11] and [10].

Proposition 2.1.[11,10] *Let $U \subset M^n(c)$ be a simply connected open neighborhood at $x_0 = 0$, and $\varphi : U \to M^{2n-1}(c+\varepsilon^2)$ a local isometric immersion. Then there exists a smooth map $(F, b) : U \to gl(n)_* \times S^{n-1}$ such that Θ_λ defind in (2.11) is a flat connection, i.e., there exists a unique solution Φ_λ to (2.11) such that $\Phi_{-\varepsilon}^{(1)} = \mathbf{r}_c \circ i_0 \circ \varphi$. Conversely, if Φ_λ for some $(F, b) : \mathbb{R}^n \to gl(n)_* \times S^{n-1}$ is a unique solution to (2.11), then there exists a smooth map $A = (a_{ij}) : \mathbb{R}^n \to O(n)$ such that $b = E_n A$. Moreover, if $U = \{x \in \mathbb{R}^n \mid b_i(x) \neq 0 \quad \text{for all } i\}$*

is not empty, then there exists a local isometric immersion $\varphi : U \to M^{2n-1}(c+\varepsilon^2)$ *such that the first and second fundamental forms for* φ *are given by* (2.1), *and* $\mathbf{r}_c \circ i_0 \circ \varphi = \Phi^{(1)}_{-\varepsilon}$.

We now consider Darboux transformations preserving $b(x) \in S^{n-1}$ for solutions to (2.11).

Lemma 2.1. *Let* Φ_λ *be a solution of* (2.11) *with* $b(x) = E_n A \in S^{n-1}$, *and* Q *a complex constant* $s\times(2n+1)$ *matrix. Set* $\lambda_0 \in \mathbb{C}$, $h = Q\Phi_{\lambda_0} = (\xi, \eta, \zeta)$. *Then we have* $d(\zeta b^T - \lambda_0\xi) = 0$.

Proof. From (2.11) it follows that $dh = h\Theta_{\lambda_0}$, i.e.,

$$d\xi = \eta\delta b^T, \quad d\eta = -c\xi b\delta + \eta\omega - \lambda_0\zeta\delta, \quad d\zeta = \lambda_0\eta\delta + \zeta\theta.$$

On the other hand, we see from (2.9) that $db = b\theta$. Hence, we have

$$\begin{aligned} d(\zeta b^T - \lambda_0\xi) &= (d\zeta)b^T + \zeta db^T - \lambda_0 d\xi \\ &= \zeta\theta b^T + \lambda_0\eta\delta b^T + \zeta\theta^T b^T - \lambda_0\eta\delta b^T = 0. \end{aligned}$$

Let $\mu \in \mathbb{R}\setminus\{0\}$, and Q be a real constant $s\times(2n+1)$ matrix. Set

$$h = Q\tau\Phi_{i\mu} \qquad h' = \frac{dh}{d\mu} = iQ\tau\dot{\Phi}_{i\mu}.$$

Since $\tau\Phi_{i\mu}\tau$ is a real matrix, both $h\tau$ and $h'\tau$ are real matrices. If we write $h\tau = (-\xi, -\eta, \zeta)$, then $h = (i\xi, i\eta, \zeta)$, $h' = (i\xi', i\eta', \zeta')$, where ξ, η, ζ satisfy

$$\begin{cases} d\xi = \eta\delta b^T, \\ d\eta = -c\xi b\delta + \eta\omega - \mu\zeta\delta, \\ d\zeta = -\mu\eta\delta + \zeta\theta. \end{cases} \tag{2.12}$$

By Lemma 2.1 and Theorem 1.1, if we choose Q such that

$$QJ_c\sigma Q^T = 0, \qquad \det(QJ_cQ^T) \neq 0, \qquad Q^{(3)}b(0)^T + c^2\mu Q^{(1)} = 0, \tag{2.13}$$

then there exists an open neighborhood U at $x = 0$ such that on U we have

$$\begin{cases} hJ_c\sigma h^* = \zeta\zeta^T - \eta\eta^T - c\xi\xi^T = 0, \\ \det(hJ_ch^*) \neq 0, \\ \zeta b^T + \mu\xi = 0. \end{cases} \tag{2.14}$$

Moreover, D_λ and d_1 defined in (1.9) and (1.10) can be expressed explicitly as

$$D_\lambda = I - \frac{2\mu}{\lambda^2+\mu^2}\begin{pmatrix} \mu c\xi^T(\Delta-\mu\rho)^{-1}\xi & \mu c\xi^T(\Delta-\mu\rho)^{-1}\eta & \lambda c\xi^T(\Delta+\mu\rho)^{-1}\zeta \\ \mu\eta^T(\Delta-\mu\rho)^{-1}\xi & \mu\eta^T(\Delta-\mu\rho)^{-1}\eta & \lambda\eta^T(\Delta+\mu\rho)^{-1}\zeta \\ -\lambda\zeta^T(\Delta-\mu\rho)^{-1}\xi & -\lambda\zeta^T(\Delta-\mu\rho)^{-1}\eta & \mu\zeta^T(\Delta+\mu\rho)^{-1}\zeta \end{pmatrix}, \tag{2.15}$$

$$d_1 = 2\mu\begin{pmatrix} 0 & 0 & -c\xi^T(\Delta+\mu\rho)^{-1}\zeta \\ 0 & 0 & -\eta^T(\Delta+\mu\rho)^{-1}\zeta \\ \zeta^T(\Delta-\mu\rho)^{-1}\xi & \zeta^T(\Delta-\mu\rho)^{-1}\eta & 0 \end{pmatrix},$$

where

$$\begin{aligned} \Delta &= \frac{1}{2}hJ_ch^* = \zeta\zeta^T = \eta\eta^T + c\xi\xi^T, \\ \rho &= -h'\sigma J_ch^* = c\xi'\xi^T + \eta'\eta^T - \zeta'\zeta^T = -(c\xi\xi'^T + \eta\eta'^T - \zeta\zeta'^T). \end{aligned} \tag{2.16}$$

Let $\tilde{\Phi}_\lambda = \Phi_\lambda D_\lambda$. Then, by Theorem 1.1, $\tilde{\Phi}_\lambda$ satisfies

$$d\tilde{\Phi}_\lambda = \tilde{\Phi}_\lambda(\lambda a + [a, \tilde{v}]),$$

where $\tilde{v} = v + (d_1)_{g_a^\perp}$ and $(\lambda a + [a, \tilde{v}])$ is a $\wedge_\sigma g$-valued 1-form. Then we have

$$\begin{aligned} \tilde{F} &= \tilde{v}^{(3,2)} = F + 2\mu(\zeta^T(\Delta - \mu\rho)^{-1}\eta)_{\rm off}, \\ \tilde{b} &= (\tilde{v}^{(3,1)})^T = b + 2\mu\xi^T(\Delta + \mu\rho)^{-1}\zeta, \\ \tilde{A} &= A(0)\tilde{\Phi}_0^{(3,3)} = AD_0^{(3,3)} = A - 2A\zeta^T(\Delta + \mu\rho)^{-1}\zeta, \end{aligned} \tag{2.17}$$

where $(\ )_{\rm off}$ denotes the matrix without diagonal elements.

Noting that $b\zeta^T + \mu\xi^T = 0$ and $\Delta = \zeta\zeta^T$, we have

$$\begin{aligned} \tilde{b}\tilde{b}^T &= bb^T + 2\mu^2\xi^T\{2(\Delta + \mu\rho)^{-1}\Delta(\Delta - \mu\rho)^{-1} - (\Delta - \mu\rho)^{-1} - (\Delta + \mu\rho)^{-1}\}\xi \\ &= bb^T = 1. \end{aligned}$$

Moreover, it is easy to see that $\tilde{A} \in O(n)$ and $E_n\tilde{A} = b - 2b\zeta^T(\Delta + \mu\rho)^{-1}\zeta = \tilde{b}$. Hence, by Proposition 2.1, we have proved the following theorem.

Theorem 2.1. *Let* $\varphi : M^n(c) \to M^{2n-1}(c + \varepsilon^2)$ *be a local isometric immersion, and* Φ_λ *a solution of* (2.11). *Let* $\mu \in \mathbb{R} \setminus \{0\}$, *and* Q *be a real constant* $s \times (2n+1)$ *matrix satisfying* (2.13). *Set* $h = Q\tau\Phi_{i\mu} = (i\xi, i\eta, \zeta)$, $\tilde{\Phi}_\lambda = \Phi_\lambda D_\lambda$ *where* D_λ *is the Darboux matrix* (2.15) *determined by* h. *If* $\tilde{b}_j(0) \neq 0$ *for all* j, *then there exist an open neighborhood* U *at* $x = 0$ *and a local isometric immersion* $\tilde{\varphi} : U \to M^{2n+1}(c + \varepsilon^2)$ *such that* $\tilde{\mathbf{r}} = \mathbf{r}_c \circ i_0 \circ \tilde{\varphi}$ *is expressed expicitly via* $\mathbf{r} = \mathbf{r}_c \circ i_0 \circ \varphi$ *as*

$$\begin{aligned} \tilde{\mathbf{r}} = \tilde{\Phi}_{-\varepsilon}^{(1)} = \Phi_{-\varepsilon}D_{-\varepsilon}^{(1)} &= -\frac{2\mu}{\varepsilon^2 + \mu^2}\sum_j \frac{\zeta_j^T(\Delta - \mu\rho)^{-1}\xi}{b_j}\partial_j\mathbf{r}_j \\ &- \frac{2\mu}{\varepsilon^2 + \mu^2}\sum_j\Big\{\sum_k \frac{\partial_j b_k\zeta_k^T(\Delta - \mu\rho)^{-1}\xi}{b_j^2} + \mu\frac{\eta_j^T(\Delta - \mu\rho)^{-1}\xi}{b_j}\Big\}\mathbf{r}_j \\ &+ \Big\{1 - \frac{2\mu^2 c}{\varepsilon^2 + \mu^2}\xi^T(\Delta - \mu\rho)^{-1}\xi\Big\}\mathbf{r}, \end{aligned} \tag{2.18}$$

where $\zeta = (\zeta_1, \cdots, \zeta_n)$ *and* $\eta = (\eta_1, \cdots, \eta_n)$.

Remark 2.1. The above process $Q \to h \to D_\lambda \to \tilde{\Phi}_\lambda$ by making use of the Darboux transformation is a purely algebraic algorithm. Starting from a special solution Φ_λ of (2.11), even if $\Phi_{-\varepsilon}^{(1)} = r$ is degenerate, we can obtain a series of new solutions to (2.11): $\Phi_\lambda \to \tilde{\Phi}_\lambda \to \tilde{\tilde{\Phi}}_\lambda \to \cdots$ by iterating such construction, so that we obtain a series of local isometric immersions from $M^n(c)$ into $M^{2n-1}(c + \varepsilon^2)$.

The Darboux transformation can also be realized by solving directly the system (2.12) of ordinary differential equations with restricted conditions (2.14). It is equivalent to relinquishing the normalization condition in (2.11). The solutions obtained in such a way are the same as that in the above process when $c \neq 0$. In the case that $c = 0$, we may obtain more solutions.

§3. Local Isometric Immersions of Space Forms Derived from Trivial Solutions

We take a trivial solution of (2.9) as $F = 0,\ A = I_n,\ b = (0, 0, \cdots, 0, 1)$. Let

$$a_i = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & E_i \\ 0 & -E_i & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -cb & 0 \\ b^T & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where E_i is defined by (2.10). Then the system (2.11) can be written as

$$\begin{cases} \partial_\alpha \Phi_\lambda = \lambda \Phi_\lambda a_\alpha, \\ \partial_n \Phi_\lambda = \Phi_\lambda(\lambda a_n + B), \\ \Phi_\lambda(0) = J_c^2. \end{cases} \tag{3.1}$$

We solve (3.1) to get $\Phi_\lambda = J_c^2 \exp\Big\{\lambda \sum\limits_{i=1}^n a_i x_i + B x_n\Big\}$. Written concretely, it is

$$\Phi_\lambda = \begin{pmatrix} \frac{c^2}{\nu^2}(cX_n+\lambda^2) & 0 & \cdots 0 & -\frac{c}{\nu}Y_n & 0 & \cdots 0 & \frac{\lambda c}{\nu^2}(X_n-1) \\ 0 & X_1 & & & Y_1 & & \\ \cdot & & & & & & \\ 0 & & X_{n-1} & & & Y_{n-1} & \\ \frac{1}{\nu}Y_n & & & X_n & & & \frac{\lambda}{\nu}Y_n \\ 0 & -Y_1 & & & X_1 & & \\ \cdot & & & & & & \\ 0 & & -Y_{n-1} & & & X_{n-1} & \\ \frac{\lambda}{\nu^2}(X_n-1) & & & -\frac{\lambda}{\nu}Y_n & & & \frac{1}{\nu^2}(\lambda^2 X_n + c) \end{pmatrix}, \tag{3.2}$$

where

$\nu = \nu(\lambda) = \sqrt{\lambda^2 + c}, \quad X_\alpha = \cos(\lambda x_\alpha), \quad X_n = \cos(\nu x_n), \quad Y_\alpha = \sin(\lambda x_\alpha), \quad Y_n = \sin(\nu x_n).$

Choose $\mu \in \mathbb{R}$ and $l = (l_0, l_1, \cdots, l_{2n}) \in \mathbb{R}^{2n+1}$ such that

$$\sum_j l_{n+j}^2 = \sum_j l_j^2 + c l_0^2 \neq 0, \qquad l_{2n} + c^2 \mu l_0 = 0. \tag{3.3}$$

It is easily seen that $h = l\tau\Phi_{i\mu} = (l\tau\Phi_{i\mu}^{(1)}, l\tau\Phi_{i\mu}^{(2)}, l\tau\Phi_{i\mu}^{(3)}) = (i\xi, i\eta, \zeta)$ satisfies (2.12) and (2.14), where

$$\begin{aligned} \xi &= c^2 l_0 \cos(i\gamma x_n) - l_n \frac{i}{\gamma}\sin(i\gamma x_n), \\ \eta_\alpha &= -l_{n+\alpha}\mathrm{sh}(\mu x_\alpha) + l_\alpha \mathrm{ch}(\mu x_\alpha), \\ \eta_n &= l_{2n}\frac{i\gamma}{\mu}\sin(i\gamma x_n) + l_n \cos(i\gamma x_n), \\ \zeta_\alpha &= l_{n+\alpha}\mathrm{ch}(\mu x_\alpha) - l_\alpha \mathrm{sh}(\mu x_\alpha), \\ \zeta_n &= l_{2n}\cos(i\gamma x_n) + l_n \frac{i\mu}{\gamma}\sin(i\gamma x_n) = -\mu\xi, \\ \gamma &= \sqrt{\mu^2 - c} \ \in \mathbb{R} \text{ (for } \mu^2 > c) \text{ or } i\mathbb{R} \text{ (for } \mu^2 < c). \end{aligned} \tag{3.4}$$

Here when $\mu^2 = c$, i.e., $\gamma = 0$, we have

$$\xi = c^2 l_0, \quad \zeta_n = l_{2n} = -\mu\xi.$$

From (3.2), (2.15), (2.16) and (2.17) we know that

$$\begin{aligned} \mathbf{r} &= \Phi_{-\varepsilon}^{(1)} \\ &= \Big(\frac{c^2(c\cos(\nu(\varepsilon)x_n) + \varepsilon^2)}{(\nu(\varepsilon))^2}, 0, \cdots, 0, \frac{\sin(\nu(\varepsilon)x_n)}{\nu(\varepsilon)}, 0, \cdots, 0, \frac{\varepsilon(1-\cos(\nu(\varepsilon)x_n))}{(\nu(\varepsilon))^2}\Big)^T, \end{aligned}$$

$$D_{-\varepsilon}^{(1)} = \frac{1}{(\varepsilon^2+\mu^2)\Delta}\big((\varepsilon^2+\mu^2)\Delta - 2c\zeta_n^2, 2\mu\zeta_n\eta^T, 2\varepsilon\zeta_n\zeta^T\big)^T, \tag{3.5}$$

$$\tilde{A} = I - 2\frac{\zeta^T\zeta}{\Delta},$$
$$\tilde{b} = \frac{1}{\Delta}\left(-2\zeta_1\zeta_n, \cdots, -2\zeta_{n-1}\zeta_n, \sum_\alpha \zeta_\alpha^2 - \zeta_n^2\right). \tag{3.6}$$

It is obvious that $\mathbf{r}$ is degenerated as a curve in $M^{2n-1}(c+\varepsilon^2)$. So we can not use (2.18) to get a new immersion $\tilde{\mathbf{r}}$. We should use directly the formula $\tilde{\mathbf{r}} = \Phi_{-\varepsilon}D^{(1)}_{-\varepsilon}$ in a neighborhood U of the origin. Such $\tilde{\mathbf{r}}$ is nondegenerate only if there exists a point $\hat{x} \in U$ such that $\tilde{b}_j(\hat{x}) \neq 0$ for all j. For this aim, we need only to choose suitably l such that $\zeta_n\zeta_\alpha(\hat{x}) \neq 0$ and $\sum\limits_\alpha \zeta_\alpha^2(\hat{x}) - \zeta_n^2(\hat{x}) \neq 0$. Then there exists an open neighborhood $\tilde{U}$ of $\hat{x}$ such that $\tilde{\mathbf{r}}$ is nondegenerate in $\tilde{U}$, which implies that there is a local isometric immersion $\hat{\varphi} : \tilde{U} \to M^{2n-1}(c+\varepsilon^2)$. Moreover, by using Φ_λ, we can obtain a new solution $\tilde{\Phi}_\lambda$ of (2.11). Continuing this process, a series of immersions are obtained by an algebraic algorithm. In the following, we consider two cases respectively.

Case (i) $c \neq 0$, i.e., $c = \pm 1$.

For $\mu^2 \geq c$, we take $l = (-\mu^{-1}l_n, l_1, \cdots, l_{n-1}, \gamma\mu^{-1}l_n, l_1, \cdots, l_n)$ with $l_j \neq 0$ for all j. It is clear that (3.3) is satisfied. Then we have from (3.4) and (3.6)

$$\xi = -\frac{l_n}{\mu}e^{-\gamma x_n} = -\frac{1}{\mu}\zeta_n, \quad \eta_\alpha = l_\alpha e^{-\mu x_\alpha - \gamma x_n} = \zeta_\alpha,$$
$$\eta_n = \frac{\gamma}{\mu}l_n e^{-\gamma x_n}, \quad \tilde{b}_\alpha = -\frac{2}{\Delta}l_\alpha l_n e^{-\mu x_\alpha - \gamma x_n},$$
$$\tilde{b}_n = \frac{1}{\Delta}\Big(\sum_\alpha l_\alpha^2 e^{-2\mu x_\alpha} - l_n^2 e^{-2\gamma x_n}\Big),$$

where Δ is defined by (2.16). Since $l_j \neq 0$ for all j, we have $\tilde{b}_\alpha(0) \neq 0$. If l is chosen suitably such that $\sum\limits_\alpha l_\alpha^2 \neq l_n^2$, then $\tilde{b}_n(0) \neq 0$.

For $\mu^2 < c = 1$, we can take $l = (l_0, l_1, \cdots, l_{n-2}, 0, 0, l_1, \cdots, l_{n-2}, i\gamma l_0, -\mu l_0)$, where $l_{\alpha-1} \neq 0$ for all α, and $\sum\limits_{j=1}^{n-2} l_j^2 + (1-2\mu^2)l_0^2 \neq 0$. The remainder is similar to the above.

Example 3.1. $c = -1$, $\varepsilon = 1$. This is the isometric immersion $M^n(-1) \to \mathbb{R}^{2n-1}$ as in [12]. We have

$$\xi = -\frac{1}{\mu}l_n e^{-\sqrt{\mu^2+1}x_n} = -\frac{1}{\mu}\zeta_n, \quad \eta_\alpha = \zeta_\alpha = l_\alpha e^{-\mu x_\alpha},$$
$$\eta_n = \frac{\sqrt{\mu^2+1}}{\mu}l_n e^{-\sqrt{\mu^2+1}x_n}, \quad \Delta = \sum_\alpha l_\alpha^2 e^{-2\mu x_\alpha} + l_n^2 e^{-2\sqrt{\mu^2+1}x_n},$$

$$\tilde{\mathbf{r}} = \frac{1}{\Delta(\mu^2+1)} \begin{pmatrix} \sum_\alpha (\mu^2+1) l_\alpha^2 e^{-2\mu x_\alpha} + (\mu^2+3) l_n^2 e^{-2\sqrt{\mu^2+1}x_n} \\ 2l_1 l_n e^{-\mu x_1 - \sqrt{\mu^2+1}x_n} (\mu \mathrm{con} x_1 - \sin x_1) \\ \cdots \\ 2l_{n-1} l_n e^{-\mu x_{n-1} - \sqrt{\mu^2+1}x_{n-1}} (\mu \cos x_{n-1} - \sin x_{n-1}) \\ 2\sqrt{\mu^2+1} l_n^2 e^{-2\sqrt{\mu^2+1}x_n} \\ 2l_1 l_n e^{-\mu x_1 - \sqrt{\mu^2+1}x_n} (\cos x_1 + \mu \sin x_1) \\ \cdots \\ 2l_{n-1} l_n e^{-\mu x_{n-1} - \sqrt{\mu^2+1}x_n} (\cos x_{n-1} + \mu \sin x_{n-1}) \\ 2l_n^2 e^{-2\sqrt{\mu^2+1}x_n} \end{pmatrix},$$

$$\tilde{A} = \frac{1}{\Delta} \begin{pmatrix} \tilde{A}_{11} & -2l_1 l_2 e^{-\mu(x_1+x_2)} & \cdots & -2l_1 l_n e^{-\mu x_1 - \sqrt{\mu^2+1}x_n} \\ -2l_1 l_2 e^{-\mu(x_1+x_2)} & \tilde{A}_{22} & \cdots & -2l_2 l_n e^{-\mu x_2 - \sqrt{\mu^2+1}x_n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ -2l_1 l_{n-1} e^{-\mu(x_1+x_{n-1})} & . & \cdots & . \\ -2l_1 l_n e^{-\mu x_1 - \sqrt{\mu^2+1}x_n} & . & \cdots & \tilde{A}_{nn} \end{pmatrix}$$

with

$$\tilde{A}_{\alpha\alpha} = \sum_{\beta\neq\alpha} l_\beta^2 e^{-2\mu x_\beta} - l_\alpha^2 e^{-2\mu x_\alpha} + l_n^2 e^{-2\sqrt{\mu^2+1}x_n},$$

$$\tilde{A}_{nn} = \sum_{\beta} l_\beta^2 e^{-2\mu x_\beta} - l_n^2 e^{-2\sqrt{\mu^2+1}x_n}.$$

It is known that $\tilde{A}$ and $\tilde{F} = \frac{2\mu}{\Delta}(\zeta^T \eta)_{\mathrm{off}}$ satisfy the GSGE (generalized Sine-Gordon equation) (2.9).

In particular, if $n = 2$ and we set

$$\cos\phi = \frac{1}{\Delta}(l_1^2 e^{-2\mu x_1} - l_2^2 e^{-2\sqrt{\mu^2+1}x_2}), \quad \sin\phi = \frac{2}{\Delta} l_1 l_2 e^{-\mu x_1 - \sqrt{\mu^2+1}x_2},$$

then $\phi(x_1, x_2)$ satisfies the well-known Sine-Gordon equation $\phi_{x_1x_1} - \phi_{x_2x_2} = -\sin\phi\cos\phi$.

Example 3.2. $n = 2$. $c = \varepsilon = \mu = 1$. Then we have

$$\xi = -\zeta_2 = -l_2, \quad \eta_1 = \zeta_1 = l_1 e^{-x_1}, \quad \eta_2 = 0, \quad (l_1, l_2 \neq 0, \quad l_1^2 - l_2^2 \neq 0).$$

$$\tilde{\mathbf{r}} = \frac{1}{\Delta} \begin{pmatrix} \frac{-1}{2} l_2^2 (\cos\sqrt{2}x_2 - 1) + \frac{1}{2} l_1^2 e^{-2x_1} (\cos\sqrt{2}x_2 + 1) \\ l_1 l_2 e^{-x_1} (\cos x_1 - \sin x_1) \\ \frac{1}{\sqrt{2}} (\sin\sqrt{2}x_2)(l_1^2 e^{-2x_1} - l_2^2) \\ l_1 l_2 e^{-x_1} (\cos x_1 + \sin x_1) \\ \frac{1}{2} l_2^2 (\cos\sqrt{2}x_2 + 1) + \frac{1}{2} l_1^2 e^{-2x_1} (\cos\sqrt{2}x_2 - 1) \end{pmatrix},$$

$$\tilde{A} = \frac{1}{\Delta} \begin{pmatrix} l_2^2 - l_1^2 e^{-2x_1} & -2l_1 l_2 e^{-x_1} \\ -2l_1 l_2 e^{-x_1} & l_1^2 e^{-2x_1} - l_2^2 \end{pmatrix},$$

where $\Delta = l_1^2 e^{-2x_1} + l_2^2$. If we set $\cos\psi = \frac{1}{\Delta}(l_1^2 e^{-2x_1} - l_2^2)$, $\sin\psi = \frac{2}{\Delta} l_1 l_2 e^{-x_1}$, then $\psi(x_1, x_2)$ satisfies the sine-Gordon equation $\psi_{x_1x_1} - \psi_{x_2x_2} = \sin\psi\cos\psi$.

Case (ii) $c = 0$.

It follws from (3.3) and (3.4) that $\nu = \lambda$, $\gamma = \mu$, $l_{2n} = 0$ and

$$\zeta_n \zeta_\alpha = l_n \mathrm{sh}(\mu x_n)(l_{n+\alpha} \mathrm{ch}(\mu x_\alpha) - l_\alpha \mathrm{sh}(\mu x_\alpha)),$$

$$\sum_\alpha \zeta_\alpha^2 - \zeta_n^2 = \sum_\alpha (l_{n+\alpha} \mathrm{ch}(\mu x_\alpha) - l_\alpha \mathrm{sh}(\mu x_\alpha))^2 - l_n^2 \mathrm{sh}^2(\mu x_n).$$

Thus, we need only to take l such that $l_n \neq 0$ and $l_\alpha^2 + l_{n+\alpha}^2 \neq 0$ for all α. Then there exists a point $\hat{x}$ in a neighborhood of the origin such that $\tilde{b}_j(\hat{x}) \neq 0$ for all j.

On the other hand, we can directly solve systems (2.12) and (2.14) and obtain

$$\xi = -\frac{C_n}{\mu} e^{-\mu x_n}, \qquad \eta_j = \zeta_j = C_j e^{-\mu x_j},$$

where C_j $(1 \leq j \leq n)$ are real constants. If $C_j \neq 0$ for all j, and $\sum_\alpha C_\alpha^2 \neq C_n^2$, then $\tilde{b}_j(0) \neq 0$ for all j.

Example 3.3. $n = 2$. $c = 0$, $\varepsilon = \mu = 1$. We have

$$\tilde{\mathbf{r}} = \frac{1}{\Delta} \begin{pmatrix} 0 \\ C_1 C_2 e^{-(x_1+x_2)}(\cos x_1 - \sin x_1) \\ C_2^2 e^{-2x_2}(\cos x_2 - \sin x_2) + \Delta \sin x_2 \\ C_1 C_2 e^{-(x_1+x_2)}(\cos x_1 + \sin x_1) \\ C_2^2 e^{-2x_2}(\cos x_2 + \sin x_2) + \Delta(\sin x_2 - 1) \end{pmatrix},$$

$$\tilde{b} = \frac{1}{\Delta}(-2C_1 C_2 e^{-(x_1+x_2)}, C_1^2 e^{-2x_1} - C_2^2 e^{-2x_2}).$$

If we set

$$\cos\psi = \frac{-2}{\Delta} C_1 C_2 e^{-(x_1+x_2)}, \qquad \sin\psi = \frac{1}{\Delta}(C_1^2 e^{-2x_1} - C_2^2 e^{-2x_2}),$$

then $\psi(x_1, x_2)$ satisfies the homogeneous wave equation $\psi_{x_1x_1} - \psi_{x_2x_2} = 0$.

Example 3.4. $c = 0$. We take $b = \frac{1}{\sqrt{n}}(1, \cdots, 1)$, $A \in O(n)$ constant such that $b = E_n A$, and $F = 0$. Then we have

$$\Phi_\lambda = \begin{pmatrix} 0 & 0 & & & 0 & & \\ \frac{1}{\lambda\sqrt{n}}\sin\lambda x_1 & \cos\lambda x_1 & & & \sin\lambda x_1 & & \\ \cdot & & \cdot & & & \cdot & \\ \cdot & & & \cdot & & & \cdot \\ \frac{1}{\lambda\sqrt{n}}\sin\lambda x_n & & & \cos\lambda x_n & & & \sin\lambda x_n \\ \frac{1}{\lambda\sqrt{n}}\cos\lambda x_1 & -\sin\lambda x_1 & & & \cos\lambda x_1 & & \\ \cdot & & \cdot & & & \cdot & \\ \cdot & & & \cdot & & & \cdot \\ \frac{1}{\lambda\sqrt{n}}\cos\lambda x_n & & & -\sin\lambda x_n & & & \cos\lambda x_n \end{pmatrix}.$$

Set $\varepsilon = 1/\sqrt{n}$. Then $\mathbf{r} = (0, \sin\frac{x_1}{\sqrt{n}}, \cdots, \sin\frac{x_n}{\sqrt{n}}, -\cos\frac{x_1}{\sqrt{n}}, \cdots, -\cos\frac{x_n}{\sqrt{n}})^T$ is a standard n-torus T^n in $\mathbb{R}^{2n}$ and $T^n \to S^{2n-1}(\sqrt{n}) \subset \mathbb{R}^{2n}$.

A solution of (2.12) and (2.14) is

$$\xi = \frac{-1}{\mu\sqrt{n}} \sum_j l_j e^{-\mu x_j}, \ \eta_j = \zeta_j = l_j e^{-\mu x_j}.$$

By making the Darboux transformation, we get a new flat submanifold in $S^{2n-1}(\sqrt{n})$, of which the position vector is

$$\tilde{\mathbf{r}} = (0, \tilde{\mathbf{r}}_1, \cdots, \tilde{\mathbf{r}}_{2n}),$$

where

$$\tilde{\mathbf{r}}_j = \frac{2\kappa}{(1+n\mu^2)\Delta} l_j e^{-\mu x_j}\Big(\sqrt{n}\mu\cos\frac{x_j}{\sqrt{n}} - \sin\frac{x_j}{\sqrt{n}}\Big) + \sin\frac{x_j}{\sqrt{n}},$$

$$\tilde{\mathbf{r}}_{n+j} = \frac{2\kappa}{(1+n\mu^2)\Delta} l_j e^{-\mu x_j}\Big(\cos\frac{x_j}{\sqrt{n}} + \sqrt{n}\mu\sin\frac{x_j}{\sqrt{n}}\Big) - \cos\frac{x_j}{\sqrt{n}},$$

$$\kappa = \sum_k l_k e^{-\mu x_k}, \qquad \Delta = \sum_k l_k^2 e^{-2\mu x_k}.$$

On putting

$$\mu = \frac{1}{\sqrt{n}}, \quad y_j = \frac{x_j}{\sqrt{n}},$$

we have

$$\tilde{\mathbf{r}} = \Big(0, \frac{\kappa}{\Delta} l_1 e^{-y_1}(\cos y_1 - \sin y_1) + \sin y_1, \cdots, \frac{\kappa}{\Delta} l_1 e^{-y_1}(\cos y_1 + \sin y_1) - \cos y_1, \cdots\Big),$$

on which the induced metric is

$$ds^2 = \sum_j \Big(1 - \frac{2\kappa}{\Delta} l_j e^{-y_j}\Big)^2 (dy_j)^2.$$

References

[1] Cartan, E., Sur les variétés de courbure constante d'un espace euclidien ou noneuclidien, *Bull. Soc. Math. France*, **47**(1919), 132–208.

[2] Chern, S. S. & Kuiper, N. H., Some theorems on the isometric embedding of compact Riemannian manifolds in Euclidean space, *Ann. of Math.,* **56**(1952), 422–430.

[3] Ferus, D. & Pedit, F., Isometric immersions of space forms and soliton theorey, *Math. Ann.,* **305**(1996), 329–343.

[4] Gu, C. H. & Hu, H. S., Explicit solutions to the intrinsic generalization for the wave and sine-Gordon equations, *Lett. Math. Phys.,* **29**(1993), 1–11.

[5] Gu, C. H., Hu, H. S. & Zhou, Z. X., Darboux transformation in soliton theory and its geometric applications, Shanghai Sci. & Tech. Publ., Shanghai, 1999.

[6] Hilbert, D., Über Flächen von konstanter Gausscher Krümmung, *Trans. AMS,* **2**(1901), 87–99.

[7] Moore, J. D., Isometric immersions of space forms in space forms, *Pacific J. Math.,* **40**(1972), 157–166.

[8] Palais, R. S. & Terng, C. L., Critical point theory and submanifold geometry, Lect. Notes in Math. 1353, Springer-Verlag, 1988.

[9] Pedit, F., A non-immersion theorem for space forms, *Comment. Math. Helv.,* **63**(1988), 672–674.

[10] Tenenblat, K., Bäcklund's theorem for submanifolds of space forms and a generalized wave equation, *Bol. Soc. Brasil. Mat.,* **16**(1985), 67–92.

[11] Terng, C. L., Soliton equations and differential geometry, *J. Diff. Geom.,* **45**(1997), 407–445.

[12] Terng, C. L., A higher dimensional generalization of the sine-Gordon equation and its soliton theory, *Ann. of Math.,* **111**(1980), 491–510.

[13] Terng, C. L. & Uhlenbeck, K., Bäcklund transformations and loop group actions, *Communi. Pure and Appl. Math.,* **53**(2000), 1–75.

[14] Uhlenbeck, K., Harmonic maps into Lie groups (classical solutions of the Chiral model), *J. Diff. Geom.,* **30**(1989), 1–50.

[15] Xavier, F., A non-immersion theorem for hyperbolic manifolds, *Comment. Math. Helv.,* **60**(1985), 280–283.

[16] Zhou, Z. X., Darboux transformations for the twisted $so(p,q)$ system and local isometric immersion of space forms, *Inverse Problems,* **14**(1998), 1353–1370.