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ON SOLVABLE COMPLETE LIE SUPERALGEBRAS***

WANG Liyun* MENG Daoji**

Abstract

The authors discuss the properties of solvable complete Lie superalgebra, proving that solvable Lie superalgebras of maximal rank are complete.

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§1. Introduction

The Lie superalgebras and their representations emerged naturally as the fundamental algebraic structure behind several areas of mathematical physics, in 1970's (see [1, 2, 4, 11]). In [4], Kac gave a comprehensive presentation of the mathematical theory of Lie superalgebras and obtained an important classification theorem for finite-dimensional simple Lie superalgebras over algebraically closed fields of characteristic zero. Lie superalgebras closely depend on Lie algebras. In recent years, there have been many studies on Lie superalgebras. Some theories of complete Lie algebras have recently been developed by Meng et al. (see [5–10]). In [7, 8], Meng and Zhu developed a general theory on solvable complete Lie algebras. They proved that all solvable Lie algebras of maximal rank are complete, and they gave a classification theorem on solvable Lie algebras of maximal rank. Complete Lie superalgebras have received some attention since 1990's (see for example [3]).

Let $\mathbf{F}$ be a field, $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Z_2-graded algebra over $\mathbf{F}$.

We call $\mathfrak{g}$ a Lie superalgebra if the multiplication [,] satisfies the following identities:

(1) $[x_\alpha, x_\beta] = -(-1)^{\alpha\beta}[x_\beta, x_\alpha]$,

(2) $(-1)^{\alpha\gamma}[x_\alpha, [x_\beta, x_\gamma]] + (-1)^{\beta\alpha}[x_\beta, [x_\gamma, x_\alpha]] + (-1)^{\gamma\beta}[x_\gamma, [x_\alpha, x_\beta]] = 0$,

for all $x_\alpha \in \mathfrak{g}_\alpha, x_\beta \in \mathfrak{g}_\beta, x_\gamma \in \mathfrak{g}_\gamma$; $\alpha, \beta, \gamma \in \{\bar{0}, \bar{1}\}$.

A Lie superalgebra $\mathfrak{g}$ is called complete if its center is zero, and its derivations are inner.

As in the Lie algebra case, a Lie superalgebra is called nilpotent (resp. solvable) if the ideals in the lower (i.e., descending) central series (resp. in the derived series) vanish for sufficiently large indices.

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*Department of Mathematics, Shanghai Jiaotong University, Shnaghai 200030, China.
E-mail: xiaogu.shan@eyou.com.
**Department of Mathematics, Nankai University, Tianjin 300071, China.
E-mail: mengdj@nankai.edu.cn.

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§2. Main Results

Definition 2.1. *Let $\mathfrak{n}$ be a nilpotent Lie superalgebra, $\mathfrak{h}$ be an abelian subalgebra of* $\mathrm{Der}\mathfrak{n}$ *such that all elements of $\mathfrak{h}$ are semisimple linear transformations of $\mathfrak{n}$. Such $\mathfrak{h}$ is called a torus on $\mathfrak{n}$. When $\mathfrak{h}$ is a maximal torus on $\mathfrak{n}$, the dimension of $\mathfrak{h}$ is called the rank of $\mathfrak{n}$, denoted by* $\mathrm{rank}\mathfrak{n}$.

Therefore, if $\mathfrak{h}$ is a maximal torus of $\mathfrak{n}$, then $\dim\mathfrak{h} = \mathrm{rank}\,\mathfrak{n}$. The Lie superalgebra $\mathfrak{n}$ is called a nilpotent Lie superalgebra of maximal rank, if $\dim\mathfrak{h} = \dim\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$.

Lemma 2.1. *Let $\mathfrak{n}$ be a nilpotent Lie superalgebra and $\mathfrak{t}$ be a maximal torus of $\mathfrak{n}$. Then we have*

(1) $\mathfrak{g} = \mathfrak{t} + \mathfrak{n}$ *is a solvable Lie superalgebra;*

(2) *The root space decomposition of $\mathfrak{n}$ with respect to $\mathfrak{t}$ is $\mathfrak{n} = \sum\limits_{\alpha\in\mathfrak{t}^*} \mathfrak{n}_\alpha$, where $\mathfrak{n}_\alpha = \{x \in \mathfrak{n} \mid [t,x] = \alpha(t)x, \forall t \in \mathfrak{t}\ \}$;*

(3) *There exists a minimal system of generators $\{x_1,\cdots,x_n\}$ of $\mathfrak{n}$ such that $x_i \in \mathfrak{n}_{\alpha_i}, 1 \le i \le n$;*

(4) *Set $\Delta = \{\alpha \in \mathfrak{t}^* \mid \mathfrak{n}_\alpha \ne 0\}$, then $\dim\mathfrak{h} = \mathrm{rank}\,\Delta \le \dim\mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$.*

Proof. (1) and (2) are obvious.

(3) Obviously, $\mathfrak{n}$ is a completely reducible $\mathfrak{t}$-module, and $[\mathfrak{n},\mathfrak{n}]$ is a submodule. So there is a submodule $\mathfrak{c}$ such that $\mathfrak{n} = \mathfrak{c} + [\mathfrak{n},\mathfrak{n}]$. Then $\mathfrak{c}$ is also completely reducible. Therefore there exists a basis of $\mathfrak{c}$, i.e., a minimal system of generators $\{x_1,\cdots,x_n\}$ of $\mathfrak{n}$, such that (3) holds.

(4) If for all $x_i \in \{x_1,\cdots,x_n\}$, we have $[T,x_i] = 0, T \in \mathfrak{t}$, then $T = 0$. Therefore, $\dim\mathfrak{t} = \mathrm{rank}\,\{\alpha_1,\cdots,\alpha_n\} = \mathrm{rank}\,\Delta \le n$, i.e. (4) holds.

Lemma 2.2. *Let $\mathfrak{g}$ be a Lie superalgebra, and $\dim\mathfrak{g} > 1$. Then*

$$C(\mathfrak{g}) \subseteq \mathfrak{g}^{(1)} = [\mathfrak{g},\mathfrak{g}] \tag{2.1}$$

holds if and only if $\mathfrak{g}$ cannot be decomposed into the direct sum of an abelian ideal and another graded ideal of $\mathfrak{g}$.

Proof. Let $\mathfrak{a}$ be an abelian ideal, $\mathfrak{g}'$ be another graded ideal of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{g}'$. Then $\mathfrak{a} \subseteq C(\mathfrak{g})$. So $\mathfrak{g}^{(1)} = [\mathfrak{a}\oplus\mathfrak{g}',\mathfrak{a}\oplus\mathfrak{g}'] = (\mathfrak{g}')^{(1)} \subseteq \mathfrak{g}'$. This implies that (2.1) fails.

Conversely, suppose that (2.1) fails. Denote by $\mathfrak{a}$ the complementary subspace of $C(\mathfrak{g})\cap[\mathfrak{g},\mathfrak{g}]$ in $C(\mathfrak{g})$. So $\mathfrak{a}$ is a nonzero abelian ideal of $\mathfrak{g}$. Let $\mathfrak{l}$ be the complementary subspace of $\mathfrak{a}+[\mathfrak{g},\mathfrak{g}]$ in $\mathfrak{g}$ and $\mathfrak{g}' = \mathfrak{l}+[\mathfrak{g},\mathfrak{g}]$. Then $\mathfrak{g}'$ is a graded ideal of $\mathfrak{g}$, and $\mathfrak{g} = \mathfrak{a}\oplus\mathfrak{g}'$.

If $\mathfrak{g}' \ne 0$, then $\mathfrak{g}$ is decomposed into the direct sum of the abelian ideal $\mathfrak{a}$ and the graded ideal $\mathfrak{g}'$.

If $\mathfrak{g}' = 0$, then $\mathfrak{g}$ is an abelian Lie superalgebra. Since $\dim\mathfrak{g} > 1$, $\mathfrak{g}$ is decomposable, and any ideal is abelian.

Lemma 2.3. *Let $\mathfrak{g}$ be a Lie superalgebra with $C(\mathfrak{g}) \ne 0$, and $\mathfrak{a}$ be a graded ideal satisfying* $\mathrm{codim}\,\mathfrak{a} = 1, C(\mathfrak{g}) \subseteq \mathfrak{a}$. *Then $[\mathfrak{g},C(\mathfrak{a})]$ is a proper subset of $C(\mathfrak{a})$, i.e., $[\mathfrak{g},C(\mathfrak{a})] \subset C(\mathfrak{a})$.*

Proof. Since $\mathrm{codim}\mathfrak{a} = 1$, we can choose $e \in \mathfrak{g}$ so that $\mathfrak{g} = \mathbf{F}e\dot{+}\mathfrak{a}$. So $[\mathfrak{g},C(\mathfrak{a})] = [\mathbf{F}e,C(\mathfrak{a})]$. Since $C(\mathfrak{g}) \subseteq \mathfrak{a}$, we have $C(\mathfrak{g}) \subseteq C(\mathfrak{a})$. Therefore we have $\ker(ade|_{C(\mathfrak{a})}) \supseteq C(\mathfrak{g}) \ne 0$. Thus $\dim C(\mathfrak{a}) > \dim\,[e,C(\mathfrak{a})]$.

Lemma 2.4. *Let $\mathfrak{g}$ be a Lie superalgebra satisfying*

$$[\mathfrak{g},\mathfrak{g}]_{\bar{0}} \ne \mathfrak{g}_{\bar{0}}, \tag{2.2}$$

$$C(\mathfrak{g}) \ne 0. \tag{2.3}$$

Then $\mathrm{Der}\,\mathfrak{g} \ne \mathrm{ad}\,\mathfrak{g}$. *If $\mathfrak{g}$ has a decomposition into a direct sum of ideals as follows*

$$\mathfrak{g} = \mathbf{F}e\oplus\mathfrak{g}', \quad e \in \mathfrak{g}_{\bar{0}}, \tag{2.4}$$

$$[\mathfrak{g}',\mathfrak{g}'] = \mathfrak{g}', \tag{2.5}$$

$$C(\mathfrak{g}') = 0, \tag{2.6}$$

then there exist semisimple outer derivations of $\mathfrak{g}$. *One of* (2.5) *and* (2.6) *fails, there is an outer derivation* D *such that* $D^2 = 0$.

Proof. We first show that if $\mathfrak{g}$ satisfies (2.4), (2.5) and (2.6), then there exist semisimple outer derivations of $\mathfrak{g}$.

Let D be a linear transformation by setting $D(\lambda e + x) = \lambda e$ for $\lambda \in \mathbf{F},\ x \in \mathfrak{g}'$. Obviously, D is a semisimple outer derivation of $\mathfrak{g}$. In this case, clearly $\mathfrak{g}$ satisfies (2.2), (2.3) and $C(\mathfrak{g}) \not\subseteq \mathfrak{g}^{(1)}$.

In the general case, suppose $\mathfrak{g}$ satisfies (2.2), (2.3). We shall consider the following two cases.

(1) $C(\mathfrak{g}) \not\subseteq \mathfrak{g}^{(1)}$.

By Lemma 2.2, $\mathfrak{g}$ has the decomposition (2.4). If (2.5) fails, then we can choose D such that $D(\mathfrak{g}') = \mathbf{F}e,\ \ D(\mathbf{F}e + (\mathfrak{g}')^{(1)}) = 0$. Obviously, we have $D^2 = 0,\ [D(\mathfrak{g}),\ \mathfrak{g}] = D([\mathfrak{g}, \mathfrak{g}]) = 0$. Then $D \in \operatorname{Der}\mathfrak{g}$, but for $x,\ y \in \mathfrak{g}$, $\operatorname{ad} x(y) \in \mathfrak{g}^{(1)} = (\mathfrak{g}')^{(1)}$. Therefore $D \notin \operatorname{ad}\mathfrak{g}$.

If (2.6) fails, we fix $x_0 \in C(\mathfrak{g}'),\ x_0 \neq 0$. Define D by setting $D(\lambda e + x) = \lambda x_0,\ \forall x \in \mathfrak{g}',\ \lambda \in \mathbf{F}$. Obviously, $D^2 = 0$, and since $C(\mathfrak{g}') \subseteq C(\mathfrak{g}),\ \mathfrak{g}^{(1)} = (\mathfrak{g}')^{(1)}$, we have $D \in \operatorname{Der}\mathfrak{g}$. Clearly D is an outer derivation by the fact that $e \in C(\mathfrak{g})$.

(2) $C(\mathfrak{g}) \subseteq \mathfrak{g}^{(1)}$.

In this case, (2.2) implies that there exists a graded ideal $\mathfrak{a}$ of $\mathfrak{g}$ satisfying $\operatorname{codim}\mathfrak{a} = 1$. Hence $\mathfrak{a} \supseteq \mathfrak{g}^{(1)} \supseteq C(\mathfrak{g})$. By Lemma 2.3, we have $[\mathfrak{g}, C(\mathfrak{a})] \subset C(\mathfrak{a})$.

For all $x \in \mathfrak{g}$, we shall always write $x = x_0 + x_1$, where $x_0 \in \mathfrak{g}_{\bar{0}},\ x_1 \in \mathfrak{g}_{\bar{1}}$.

Since $\mathfrak{g}$ satisfies (2.2), we can choose $e \in \mathfrak{g}_{\bar{0}}$ such that $e \notin \mathfrak{a}$, and $z \in C(\mathfrak{a})\backslash[\mathfrak{g}, C(\mathfrak{a})]$ and we can define a linear transformation D such that $De = z,\ D(\mathfrak{a}) = 0$. Let $D_0(e) = z_0,\ D_1(e) = z_1$, where $D = D_0 + D_1$. Then $D^2 = 0$. For $x,\ y \in \mathfrak{a},\ \lambda,\ \mu \in \mathbf{F}$, we have $D([\lambda e + x, \mu e + y]) = 0$ and $[D_0(\lambda e + x), \mu e + y] + [\lambda e + x, D_0(\mu e + y)] = [\lambda z_0, \mu e + y] + [\lambda e + x, \mu z_0] = \lambda\mu[z_0, e] + \lambda\mu[e, z_0] = 0$, and

$$\begin{aligned}&[D_1(\lambda e + x_0), \mu e + y] + [\lambda e + x_0, D_1(\mu e + y)] + [D_1(x_1), \mu e + y] - [x_1, D_1(\mu e + y)]\\ &= [\lambda z_1, \mu e + y] + [\lambda e + x_0, \mu z_1] + [0, \mu e + y] - [x_1, \mu z_1]\\ &= \lambda\mu([z_1, e] + [e, z_1]) = 0.\end{aligned}$$

Therefore, $D \in \operatorname{Der}\mathfrak{g}$.

If $D \in \operatorname{ad}\mathfrak{g}$, then there exist $\lambda \in \mathbf{F}$ and $x \in \mathfrak{a}$ such that $D = \operatorname{ad}(\lambda e + x)$. But from $Dx = 0$, we obtain $\lambda[e, x] = [\lambda e + x, x] = Dx = 0$. If $\lambda \neq 0$, then $[e, x] = 0$. So $De = [\lambda e + x, e] = 0$. We arrive at a contradiction with $De = z \neq 0$. If $\lambda = 0$, then $D = \operatorname{ad} x$. Since $D(\mathfrak{a}) = 0$, we have $x \in C(\mathfrak{a})$. Then $z = De = [x, e] \in [\mathfrak{g}, C(\mathfrak{a})]$. Again, this gives us a contradiction with $z \in C(\mathfrak{a})\backslash[\mathfrak{g}, C(\mathfrak{a}]$. So D is an outer derivation of $\mathfrak{g}$.

Remark 2.1. Lemma 2.4 does not necessarily hold if the condition (2.2) is changed to $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$.

Example 2.1. Let $\mathfrak{g} = \mathbf{F}c \dot{+} \mathbf{F}e \dot{+} \mathbf{F}f$ be the three-dimensional Heisenberg superalgebra, where $c \in \mathfrak{g}_{\bar{0}},\ f,\ e \in \mathfrak{g}_{\bar{1}},\ [e, f] = c,\ [e, e] = [f, f] = [c, e] = [c, f] = 0$. Then $\mathfrak{g}$ is nilpotent, and $\mathfrak{g}^{(1)} = \mathbf{F}c = (\mathfrak{g}^{(1)})_{\bar{0}} = \mathfrak{g}_{\bar{0}}$.

Let $D \in (\operatorname{Der}\mathfrak{g})_{\bar{0}}$. Then we have $D(c) = \alpha c,\ D(e) = \beta e,\ D(f) = \gamma f$, where $\alpha = \beta + \gamma$. But $D^2 = 0$ if and only if $D = 0$. Let $D \in (\operatorname{Der}\mathfrak{g})_{\bar{1}}$. Then we have $D(c) = 0,\ D(e) = \alpha c,\ D(f) = \beta c,\ \alpha,\ \beta \in \mathbf{F}$. So $D = \operatorname{ad}(\alpha f + \beta e) \in \operatorname{ad}\mathfrak{g}$.

Hence, there does not exist an outer derivation D of $\mathfrak{g}$ such that $D^2 = 0$.

Theorem 2.1. *Let* $\mathfrak{g}$ *be a Lie superalgebra, and* $(\mathfrak{g}^{(1)})_{\bar{0}} \neq \mathfrak{g}_{\bar{0}}$. *Then* $\mathfrak{g}$ *is a complete Lie superalgebra if and only if* $\operatorname{Der}\mathfrak{g} = \operatorname{ad}\mathfrak{g}$.

Proof. By the definition of complete Lie superalgebras, $\mathfrak{g}$ is complete, then $\operatorname{Der}\mathfrak{g} = \operatorname{ad}\mathfrak{g}$.

Conversely, suppose $\operatorname{Der}\mathfrak{g} = \operatorname{ad}\mathfrak{g}$. If $C(\mathfrak{g}) \neq 0$, then by Lemma 2.4, $\operatorname{Der}\mathfrak{g} \neq \operatorname{ad}\mathfrak{g}$. This contradicts the hypothesis. So $\mathfrak{g}$ is a complete Lie superalgebra.

Let $\mathfrak{n}$ be a nilpotent Lie superalgebra, $\mathfrak{h}$ be a maximal torus on $\mathfrak{n}$.

Definition 2.2. *A minimal system of generators which consists of root vectors for $\mathfrak{h}$ is called a $\mathfrak{h}$-msg.*

Lemma 2.5. *Let $\mathfrak{l} = \mathfrak{h} + \mathfrak{n}$ be a solvable Lie superalgebra of maximal rank, and a direct sum of root spaces of $\mathfrak{l}$ for $\mathfrak{h}$ is $\mathfrak{l} = \mathfrak{h} + \sum\limits_{\alpha\in\Delta} \mathfrak{n}_\alpha$, where $\Delta \subset \mathfrak{h}^*$, $\mathfrak{n}_\alpha = \{x \in \mathfrak{n} \mid [h,x] = \alpha(h)x, \forall h \in \mathfrak{h}\}$. Then*

(1) *There exists a $\mathfrak{h}$-msg $\{x_1, \cdots, x_n\}$ of $\mathfrak{n}$ and a subset $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ of Δ such that $[h, x_i] = \alpha_i(h)x_i$, $\forall h \in \mathfrak{h}$.*

(2) *Π is a basis of $\mathfrak{h}^*$.*

(3) *If $\alpha \in \Delta$, then there is a unique n-tuple $\{k_1, \cdots, k_n\}$, $k_i \in \mathbf{Z}_+$, such that $\alpha = \sum\limits_{i=1}^{n} k_i\alpha_i$. Thus $\Delta \subset \mathfrak{h}^*\backslash\{0\}$.*

(4) *Let $|\alpha| = \sum\limits_{i=1}^{n} k_i$, p is the nilpotency of $\mathfrak{n}$ (i.e. p safeties $\mathfrak{n}^{p-1} \neq 0$, $\mathfrak{n}^p = 0$). Then $1 \leq |\alpha| \leq p$.*

(5) $\dim \mathfrak{n}_{\alpha_i} = 1, \quad i = 1, \cdots, n$.

Proof. (1) This results from Lemma 2.1.

(2) Since $\mathfrak{l}$ is of maximal rank, we have $\dim \mathfrak{h} = \dim \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$.

Because $\mathfrak{h}$ is a maximal torus, and $\{x_1, \cdots, x_n\}$ is a $\mathfrak{h}$–msg of $\mathfrak{n}$, we have $h \in \mathfrak{h}$, $h = 0$ if and only if $[h, x_i] = 0$, $1 \leq i \leq n$, if and only if $\alpha_i(h) = 0$, $1 \leq i \leq n$. So Π is a basis of $\mathfrak{h}^*$.

(2.3) and (2.5) hold since Π is a basis of $\mathfrak{h}^*$ and $\{x_1, \cdots, x_n\}$ is a $\mathfrak{h}$–msg.

(2.4) is known from the definition of the nilpotency of $\mathfrak{n}$.

Theorem 2.2. *Let $\mathfrak{n}$ be a nilpotent Lie superalgebra of maximal rank, $\mathfrak{h}$ be a maximal torus on $\mathfrak{n}$. Then $\mathfrak{l} = \mathfrak{h} + \mathfrak{n}$ is a solvable complete Lie superalgebra.*

Proof. By (2.3), (2.5) of Lemma 2.5, we have $0 \notin \Delta$, $\{\alpha_1, \cdots, \alpha_n\} \cap \Delta_0 \cap \Delta_1 = \emptyset$.

Therefore, $\mathfrak{l}$ satisfies conditions (2.1)–(2.5) of Theorem 1.5 in [13], so $\mathfrak{l}$ is a complete Lie superalgebra.

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