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THE TRACE SPACE INVARIANT AND UNITARY GROUP OF C^*-ALGEBRA**

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Abstract

Let A be a unital C^*-algebra, $n \in \mathbf{N} \cup \{\infty\}$. It is proved that the isomorphism $\Delta_n : U_0^n(A)/\overline{DU_0^n(A)} \mapsto AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))}$ is isometric for some suitable distances. As an application, the author has the split exact sequence $0 \mapsto AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))} \overset{i_A}{\mapsto} U^n(A)/\overline{DU^n(A)} \overset{\pi_A}{\mapsto} U^n(A)/U_0^n(A) \mapsto 0$ with i_A contractive (and isometric if $n = \infty$) under certain condition of A.

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§1. Introduction and Preliminary

For more than ten years much progross has been made in the classification of simple amenable separable C^*-algebras. Recently some work focus on the non-real rank zero case, in which the trace space (or the affine space on trace space) is suggested to be considered as an invariant. For this reason, the relations between the unitary group and the affine space on trace space have been studied and applied, but only in the stable case. For more application it is needed to consider the nonstale case (especially if the considered C^*-algebras are not the inductive limits). In this note we investigate the nonstable relations between the unitary group and the affine space on trace space by using some distances. The results in this note can also be considered as the nonstable similarity and extension of stable case.

Let A be a unital C^*-algebra. For each integer k, we denote the unitary group of $M_k(A)$ by $U^k(A)$, and the subgroup of $U^k(A)$ consisting of all elements connected to the unit of $M_k(A)$ by $U_0^k(A)$. Viewing $U^k(A)$ $(U_0^k(A))$as a subgroup of $U^{k+1}(A)$ $(U_0^{k+1}(A))$ by identifying $\operatorname{diag}(u,1)$ with u for any $u \in U^k(A)(U_0^k(A))$, we let $U^\infty(A) = \lim\limits_{k\to\infty} U^k(A)$ as a topological group with the inductive limit topology coming from the inclusion $U^k(A) \subseteq U^{k+1}(A)$, and similarly let $U_0^\infty(A) = \lim\limits_{k\to\infty} U_0^k(A)$ as a topological group with the inductive limit topology

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coming from the inclusion $U_0^k(A) \subseteq U_0^{k+1}(A)$. For any $n \in \mathbf{N} \cup \{\infty\}$, we let $DU^n(A)$ and $DU_0^n(A)$ denote the commutator subgroup of $U^n(A)$ and $U_0^n(A)$ respectively.

Let $AffT(A)$ denote the space of continuous affine real-valued function on the state space $T(A)$ of A. Let $\eta : [0,1] \mapsto U_0^n(A)$ $(n \in \mathbf{N} \cup \{\infty\})$ be a piecewise smooth path of unitaries from 1. We define $\Delta_n^1(\eta) \in AffT(A)$ by

$$\Delta_n^1(\eta)(\omega) = \frac{1}{2\pi i}\int_0^1 \omega(\eta'(t)\eta(t)^*)dt,\ \omega \in T(A).$$

By [1, Lemma 3] (see also [11, §3]), $\Delta_n^1(\eta)$ is homotopy invariant, and

$$\Delta_n^1(\eta_1\eta_2) = \Delta_n^1(\eta_1) + \Delta_n^1(\eta_2).$$

So Δ_n^1 defines a homomorphism

$$\Delta_n^0 : \pi_1(U_0^n(A)) \mapsto AffT(A),$$

where $\pi_1(U_0^n(A))$ is the first fundmental group of $U_0^n(A)$. In particular,

$$\Delta_\infty^0(\pi_1(U_0^\infty(A))) = \rho(K_0(A)),$$

where ρ is the canonical homomorphism from $K_0(A)$ to $AffT(A)$ (see [2, 10.10]). So Δ_n^1 defines a group homomorphism (called the de la Harpe-skandalis determinant, if $n = \infty$)

$$\Delta_n : U_0^n(A) \mapsto AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))}$$

by $\Delta_n(u) = q(\Delta_n^1(\eta_u))$, where q is the quotient map from $AffT(A)$ to

$$AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))}$$

and η_u is any piecewise smooth path in $U_0^n(A)$ from 1 to u.

Let $n \in \mathbf{N} \cup \{\infty\}$, q, q^0, q' be the quotient maps from $AffT(A)$, $U_0^n(A)$, and $U^n(A)$ to $AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))}$, $U_0^n(A)/\overline{DU_0^n(A)}$, and $U^n(A)/\overline{DU^n(A)}$ respectively. The distance D_A on $U_0^n(A)/\overline{DU_0^n(A)}$ is defined by

$$D_A(q^0(u), q^0(v)) = \inf\{\|uv^* - c\| :\ c \in \overline{DU_0^n(A)}\}$$

for any $u, v \in U_0^n(A)$, the distance D'_A on $U^n(A)/\overline{DU^n(A)}$ is defined by

$$D'_A(q'(u), q'(v)) = \inf\{\|uv^* - c\| :\ c \in \overline{DU^n(A)}\}$$

for any $u, v \in U^n(A)$, and the distance d_A on $AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))}$ is defined by

$$d_A(q(f), q(g)) = \begin{cases} 2, & \text{if } d(q(f), q(g)) \ge \frac{1}{2}, \\ |e^{2\pi i\ d(q(f),q(g))} - 1| & \text{if } d(q(f), q(g)) < \frac{1}{2} \end{cases}$$

for any $f, g \in AffT(A)$, where

$$d(q(f), q(g)) = \inf\{\|f - g - h\| : h \in \overline{\Delta_n^0(\pi_1(U_0^n(A)))}\}.$$

For $n < \infty$, there are standard contractive mappings

$$(i_A)_{n,m}(n < m) : \frac{U^n(A)}{\overline{DU^n(A)}} \mapsto \frac{U^m(A)}{\overline{DU^m(A)}}.$$

It is easy to see that $\frac{U^\infty(A)}{\overline{DU^\infty(A)}}$ is the inductive limit of $\left(\frac{U^n(A)}{\overline{DU^n(A)}}, (i_A)_{n,m}\right)$ and the quotient distance on $\frac{U^\infty(A)}{\overline{DU^\infty(A)}}$ defined above coincides with that defined by inductive limit. Similarly for $\frac{U_0^\infty(A)}{\overline{DU_0^\infty(A)}}$.

§2. Main Theorem

Lemma 2.1.[11,Theotem 3.2.] *For $n \in \mathbf{N} \cup \{\infty\}$, Δ_n induces a homeomorphic group isomorphism*

$$\Delta_n : \ U_0^n(A)/\overline{DU_0^n(A)} \mapsto AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))}.$$

In particular,

$$U_0^\infty(A)/\overline{DU_0^\infty(A)} \cong AffT(A)/\overline{\rho(K_0(A))}.$$

Proof. It is proved in [11]. For application, we write the inverse Φ_n of Δ_n as follows: By duality theorem, for any element ξ in $AffT(A)$, we have $a \in A_{sa}$ with $\xi(\tau) = \tau(a)$ for any $\tau \in T(A)$, and so we can denote ξ by $\hat{a}$. Then we define $\Phi_n(q(\xi)) = q^0(e^{2\pi i a})$ and this Φ_n does the job.

Note. By the proof of [11, Theorem 3.2] we also have that for every $a \in M_n(A)_{sa}$, $\Delta_n(q^0(e^{2\pi i a})) = q(\hat{a})$, where

$$\hat{a}(\omega) = (\mathrm{tr} \otimes \omega)(a) \ (\forall \omega \in T(A)), \quad \text{tr is the canonical trace state on } \ M_n.$$

Theorem 2.1. *Let $\Delta_n : U_0^n(A)/\overline{DU_0^n(A)} \mapsto AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))}$ be as above, where A is a unital C^*-algebra. Then, with the distances defined in §1, Δ_n is an isometric group isomorphism.*

Proof. The only part we need to prove is that Δ_n is isometric. By Lemma 2.1, the inverse map of Δ_n is Φ_n with $\Phi_n(q(\hat{a})) = q^0(e^{2\pi i a})$, where $\hat{a}(\omega) = \omega(a)$, $a = a^* \in A_{sa}$, $\omega \in T(A)$. Since

$$e^{ia}e^{ib}e^{-i(a+b)} = \lim_{n\to\infty} e^{ia}e^{ib}(e^{-ia/n}e^{-ib/n})^n \in \overline{DU_0(A)}$$

(see [11, §1]), i.e. $e^{ia}e^{ib} = e^{-i(a+b)}$ module $\overline{DU_0(A)}$, we have

$$\begin{aligned}
&D_A(q^0(e^{2\pi i a}), q^0(e^{2\pi i b}))\\
&= \inf\{\|e^{2\pi i a}e^{-2\pi i b} - u\| : \ u \in \overline{DU_0^n(A)}\}\\
&= \inf\{\|e^{2\pi i(a-b)}w - u\| : \ u \in \overline{DU_0^n(A)}\}\\
&\quad (\text{where } \ w \text{ is some element in } \overline{DU_0(A)})\\
&= \inf\{\|e^{2\pi i(a-b)}u - 1\| : \ u \in \overline{DU_0^n(A)}\}.
\end{aligned}$$

On the other hand, let $\sup\{|\omega(a)| : \ \omega \in T(A)\} = \|\hat{a}\|$ $(a \in A_{sa})$, and

$$A_0 = \Big\{x - y |\ \exists\{c_i\} \subseteq A \ s.t. \ x = \sum_i c_i c_i^*, \text{ and } y = \sum_i c_i^* c_i\Big\},$$

then $\|\hat{a}\| = \inf\{\|a - x\| \ |x \in A_0\}$ by the proof of [11, Lemma 3.1]. So, if $d(q(\hat{a}), q(\hat{b})) < 1/2$, then

$$\begin{aligned}
&d_A(q(\hat{a}), q(\hat{b})) = |e^{2\pi i d(q(\hat{a}),q(\hat{b}))} - 1| \\
&= \inf\{|e^{2\pi i\|\hat{a}-\hat{b}-\hat{c}\|} - 1| :\ c \in A_{sa} \text{ and } \hat{c} \in \overline{\Delta_n^0(\pi_1(U_0^n(A)))}\} \\
&= \inf\{|e^{2\pi i\|a-b-c-x\|} - 1| :\ c, x \in A_{sa},\ \hat{c} \in \overline{\Delta_n^0(\pi_1(U_0^n(A)))}, \\
&\quad x = \sum_i c_i c_i^* - \sum_i c_i^* c_i\} \\
&= \inf\{|e^{2\pi i\|a-b-d\|} - 1| :\ d \in A_{sa},\ \hat{d} \in \overline{\Delta_n^0(\pi_1(U_0^n(A)))}\}, \\
&\quad (\text{since } \hat{x} = 0, \hat{x} \in \overline{\Delta_n^0(\pi_1(U_0^n(A)))}) \\
&= \inf\{|e^{2\pi i\|a-b-d\|} - 1| :\ d \in A_{sa},\ e^{2\pi i d} \in \overline{DU_0^n(A)}\}, \\
&\quad (\text{since } \hat{d} \in \overline{\Delta_n^0(\pi_1(U_0^n(A)))} \text{ iff } e^{2\pi i d} \in \overline{DU_0^n(A)} \text{ by Lemma 2.1 }) \\
&= \inf\{\|e^{2\pi i(a-b-d)} - 1\| : d \in A_{sa},\ e^{2\pi i d} \in DU_0^n(A)\} \\
&\quad (\text{since } d(q(\hat{a}), q(\hat{b})) < 1/2, \text{and by the simple spectrum computation}).
\end{aligned}$$

So, if $d(q(\hat{a}), q(\hat{b})) < 1/2$, then for any $\varepsilon > 0$, we can take $d \in A_{sa}$ with $e^{2\pi i d} \in \overline{DU_0^n(A))}$ and

$$d_A(q(\hat{a}), q(\hat{b})) \geq \|e^{2\pi i(a-b-d)} - 1\| - \varepsilon.$$

Since there is $w \in \overline{DU_0(A)} \subseteq \overline{DU_0^n(A)}$ such that

$$e^{2\pi i(a-b)} e^{-2\pi i d} w = e^{2\pi i(a-b-d)},$$

and since $e^{2\pi i d} \in \overline{DU_0^n(A)}$, we have

$$\begin{aligned}
D_A(q^0(e^{2\pi i a}), q^0(e^{2\pi i b})) &\leq \|e^{2\pi i(a-b)} e^{-2\pi i d} w - 1\| = \|e^{2\pi i(a-b-d)} - 1\| \\
&\leq d_A(q(\hat{a}), q(\hat{b})) + \varepsilon.
\end{aligned}$$

Since

$$D_A(q^0(e^{2\pi i a}), q^0(e^{2\pi i b})) \leq 2, \quad D_A(q^0(e^{2\pi i a}), q^0(e^{2\pi i b})) \leq d_A(q(\hat{a}), q(\hat{b})).$$

If $D_A(q^0(e^{2\pi i a}),\ q^0(e^{2\pi i b})) < 2$, for any $0 < \varepsilon < 2 - D_A(q^0(e^{2\pi i a}), q^0(e^{2\pi i b}))$, there is $u_0 \in \overline{DU_0^n(A)}$ such that

$$\|e^{2\pi i(a-b)} u_0 - 1\| \leq D_A(q^0(e^{2\pi i a}), q^0(e^{2\pi i b})) + \varepsilon < 2.$$

Let $e^{2\pi i c} = e^{2\pi i(a-b)} u_0$ with $\|c\| < \frac{1}{2}$, and $d = a - b - c$ (the existence of c is given by the inequality above), where $c \in M_n(A)_{sa}$. Then

$$\begin{aligned}
q(\hat{d}) &= q(\hat{a} - \hat{b} - \hat{c}) = q(\widehat{a-b}) - q(\hat{c}) \\
&= \Delta_n(q^0(e^{-2\pi i c} e^{2\pi i(a-b)}) = \Delta_n(q^0(u_0^*)) \\
&= -\Delta_n(q^0(u_0)) = 0.
\end{aligned}$$

So $\hat{d} \in \overline{\Delta_n^0(\pi_1(U_0^n(A)))}$. Therefore

$$\begin{aligned}
d_A(q(\hat{a}), q(\hat{b})) &\leq \|e^{2\pi i(a-b-d)} - 1\| = \|e^{2\pi i(a-b)} u_0 - 1\| \\
&\leq D_A(q^0(e^{2\pi i a}), q^0(e^{2\pi i b})) + \varepsilon < 2,
\end{aligned}$$

since it implies that

$$d(q(\hat{a}), q(\hat{b})) \leq \|\widehat{(a-b)} - \hat{d}\| \leq \|a - b - d\| < \frac{1}{2}.$$

So if $D_A(q^0(e^{2\pi ia}), q^0(e^{2\pi ib})) < 2$, then

$$d_A(q(\hat{a}), q(\hat{b})) \le D_A(q^0(e^{2\pi ia}), q^0(e^{2\pi ib})).$$

If $D_A(q^0(e^{2\pi ia}), q^0(e^{2\pi ib})) = 2$, then $d_A(q(\hat{a}), q(\hat{b})) = 2$, since, otherwise,

$$D_A(q^0(e^{2\pi ia}), q^0(e^{2\pi ib})) \le d_A(q(\hat{a}), q(\hat{b})) < 2$$

by the discussion above.

Corollary 2.1. *With the notation as above, if the natural map from $\pi_1(U_0^n(A))$ to $\pi_1(U_0^\infty(A)) = K_0(A)$ is surjective, then, for each k with $k \ge n$ and $k \in \mathbf{N}$, the following natural maps are isometric group isomorphisms:*

$$\frac{U_0^n(A)}{\overline{DU_0^n(A)}} \cong \frac{U_0^k(A)}{\overline{DU_0^k(A)}} \cong \frac{AffT(A)}{\overline{\rho(K_0(A))}}.$$

Lemma 2.2. *With the notation as in §1, for any $n \in \mathbf{N} \cup \{\infty\}$, we have that*

$$DU_0^n(A) \subseteq DU^n(A) \cap U_0^n(A), \quad \overline{DU_0^n(A)} \subseteq \overline{DU^n(A)} \cap U_0^n(A);$$

and that

$$DU^n(A) \subseteq DU_0^{4n}(A), \quad \overline{DU^n(A)} \subseteq \overline{DU_0^{4n}(A)}.$$

In particular, we have

$$DU_0^\infty(A) = DU^\infty(A), \quad \textit{and} \;\; \overline{DU_0^\infty(A)} = \overline{DU^\infty(A)}.$$

Proof. Clearly $DU_0^n(A) \subseteq DU^n(A) \cap U_0^n(A)$ and

$$\overline{DU_0^n(A)} \subseteq \overline{DU^n(A)} \cap U_0^n(A)$$

by the definition. Let $u, v \in U^n(A)$. We have

$$\begin{aligned}
&\mathrm{diag}\,(u^{-1}v^{-1}uv, 1, 1, 1)\\
&= \mathrm{diag}\,(u^{-1}, 1, 1, 1)\mathrm{diag}\,(v^{-1}, 1, 1, 1)\mathrm{diag}\,(u, 1, 1, 1)\mathrm{diag}\,(v, 1, 1, 1)\\
&= \mathrm{diag}\,(u^{-1}, u, 1, 1)\mathrm{diag}\,(v^{-1}, 1, v, 1)\mathrm{diag}\,(u, u^{-1}, 1, 1)\mathrm{diag}\,(v, 1, v^{-1}, 1).
\end{aligned}$$

So $\mathrm{diag}\,(u^{-1}v^{-1}uv, 1, 1, 1) \in DU_0^{4n}(A)$, and therefore

$$DU^n(A) \subseteq DU_0^{4n}(A), \quad \text{and} \;\; \overline{DU^n(A)} \subseteq \overline{DU_0^{4n}(A)}.$$

Corollary 2.2. *With the notation as above, if the natural map from $\pi_1(U_0^n(A))$ to $\pi_1(U_0^\infty(A)) = K_0(A)$ is surjective, then, for each k, n with $k \ge n$ and $k \in \mathbf{N}$,*

$$U_0^k(A) \cap \overline{DU^k(A)} = \overline{DU_0^k(A)}.$$

Proof. It is enough to prove that

$$U_0^k(A) \cap \overline{DU^k(A)} \subseteq \overline{DU_0^k(A)}.$$

In fact, for any $x \in U_0^k(A) \cap \overline{DU^k(A)}$, $x \in \overline{DU_0^{4k}(A)} \cap U_0^k(A)$ by Lemma 2.2. Therefore $x \in \overline{DU_0^k(A)}$ by Corollary 2.1.

The following Lemma 2.3 is known to experts. Since no reference including its proof is known to me, and the concrete isomorphism is needed, we include its proof here.

Lemma 2.3. Let A be a unital C^*-algebra, SA be the suspension C^*-algebra of A. For any integer n,

$$\pi_1(U_0^n(A)) \cong U^n(\widetilde{SA})/U_0^n(\widetilde{SA}).$$

Proof. Let $F(n,A)=\{f:\mathbf{T}\to U^n(A)|f(1)=1_{M_n(A)}\}$. For any $f,\ g\in F(n,A)$, we define $f\sim g$ iff f is homotopic to g with $1\in\mathbf{T}$ fixed. Then $\sim$ is an equivalence relation, and $\pi_1(U_0^n(A))=F(n,A)/\sim$. Since A has a unit 1_A, it is easy to see

$$\widetilde{SA}=\{f\in C(\mathbf{T},A)|f(1)\in\mathbf{C}1_A\}$$

by identifying $g+\alpha 1$ with $g+\alpha e$, where $\alpha\in\mathbf{C}$, $g\in SA$, 1 is the added unit of $\widetilde{SA}$, and e is the unit of $C(\mathbf{T},A)$. So we view $\widetilde{SA}$ as a subset of $C(\mathbf{T},A)$ later. Now we define

$$\Psi:\pi_1(U^n(A))=F(n,A)/\sim\ \mapsto U^n(\widetilde{SA})/U_0^n(\widetilde{SA}),\quad \Psi([f])=[(x_{ij})],$$

where $f\in F(n,A)$, $(x_{ij})\in U^n(\widetilde{SA})$, $x_{ij}(t)=f(t)_{ij}$ (the (i,j)-th element of matrix $f(t)\in U^n(A)$). Since $x_{ij}(1)=f(1)_{ij}=1_A\delta_{ij},\ x_{ij}\in\widetilde{SA}$. Let $x'_{ij}(t)=(f(t)^*)_{ij}=(f(t)_{ji})^*$. Then

$$(x_{ij})(x'_{ij})=(x'_{ij})(x_{ij})=1_{M_n(\widetilde{SA})},$$

and so $(x_{ij})\in U^n(\widetilde{SA})$. First we note that Ψ is well-defined. In fact, let $[f]=[g]$, then there is an $F:[0,1]\times\mathbf{T}\mapsto U^n(A)$ which is continuous and satisfies

$$F(0,t)=f(t),\quad F(1,t)=g(t),\quad F(s,1)=1_{M_n(A)}(\forall s\in[0,1],t\in\mathbf{T}).$$

Let $(x'_{ij})\in U^n(\widetilde{SA})$, $x'_{ij}(t)=g(t)_{ij}$, and $(z^s_{ij})\in U^n(\widetilde{SA})$, $z^s_{ij}(t)=F(s,t)_{ij}$. Then $s\to(z^s_{ij})$ is continuous. Since F is continuous, therefore uniformly continuous on $[0,1]\times\mathbf{T}$, (z^s_{ij}) is a continuous path in $U^n(\widetilde{SA})$ connecting $(z^0_{ij})=(x_{ij})$ to $(z^1_{ij})=(x'_{ij})$. Therefore $[(x_{ij})]=[(x'_{ij})]$, i.e. $\Psi([f])=\Psi([g])$. It is well known that $[f]\times[g]=[f\cdot g]$, where $(f\cdot g)(t)=f(t)g(t)$, and $\times$ is the standard multiplication on elementary group by connecting two paths from and to 1 into one path from and to 1. From this, it is easy to see Ψ is a group homomorphism.

Let $(x_{ij})\in U^n(\widetilde{SA})$. Then $(x_{ij})=(y_{ij})+(\alpha_{ij}e)$, where $y_{ij}\in SA$, $(\alpha_{ij})\in U^n(\mathbf{C})$. Let $(\alpha_{ij})^*(y_{ij})=(z_{ij})$. Then

$$(x_{ij})=(\alpha_{ij})((z_{ij})+1_{M^n(\widetilde{SA})}).$$

Since $U^n(\mathbf{C})$ is path connected, $[(x_{ij})]=[(z_{ij})+1_{M^n(\widetilde{SA})}]$ in $U^n(\widetilde{SA})/U_0^n(\widetilde{SA})$. Now we define

$$\Phi:U^n(\widetilde{SA})/U_0^n(\widetilde{SA})\mapsto\pi_1(U^n(A))=F(n,A)/\sim,\quad \Phi([(x_{ij})])=[f],$$

where $f(t)=z_{ij}(t)+1_{M_n(A)}\in U^n(A)$. If $(x'_{ij})=(y'_{ij})+(\alpha'_{ij}e)\in U^n(\widetilde{SA})$ such that $[(x'_{ij})]=[(x_{ij})]$, then there is a continuous path $(x^s_{ij})\in U^n(\widetilde{SA})$ $(0\le s\le 1)$ from (x_{ij}) to (x'_{ij}). Let $(x^s_{ij})=(y^s_{ij})+(\alpha^s_{ij}e)$. Since $M_n(\mathbf{C})\cong M_n(\widetilde{SA})/M_n(SA)$, (α^s_{ij}) is the continuous path from (α_{ij}) to (α'_{ij}), and

$$(z^s_{ij})=(\alpha^s_{ij})^*(y^s_{ij})=(\alpha^s_{ij})^*((x^s_{ij})-(\alpha^s_{ij}e))$$

is the continuous path in $M_n(SA)$ from (z_{ij}) to (z'_{ij}). Let

$$F:[0,1]\times\mathbf{T}\to U^n(A),\quad F(s,t)=(z^s_{ij}(t))+1_{M_n(A)}).$$

Clearly F is continuous. Let $f'\in F(n,A)$, $f'(t)=z'_{ij}(t)+1_{M_n(A)}$. Then $F(0,\cdot)=f$, $F(1,\cdot)=f'$, so $[f]=[f']$. Therefore Φ is well-defined. It is not difficult to check that $\Psi\Phi=id$ and $\Phi\Psi=id$, which completes the proof.

Corollary 2.3. *Let A be a unital C^*-algebra, SA be the suspension C^*-algebra of A. If there is an integer $k>0$ such that*

$$U^k(\widetilde{SA})/U_0^k(\widetilde{SA})\mapsto K_1(SA)\ (\cong K_0(A))$$

is surjective, then for every $n \geq k$,

$$\pi_1(U_0^n(A)) \ (= \pi_1(U^n(A))) \mapsto \pi_1(U_0^\infty(A)) \ (\cong K_0(A))$$

is surjective.

Proof. Clearly for every $n \geq k$, the natural map

$$i_n : U^n(\widetilde{SA})/U_0^n(\widetilde{SA}) \mapsto K_1(SA)$$

is surjective. Let j_n be the homomorphism

$$\pi_1(U_0^n(A)) \ (= \pi_1(U^n(A))) \mapsto \pi_1(U_0^\infty(A)) \ (\cong K_0(A)).$$

Then, by the proof of Lemma 2.3, it is easy to check that $j_n = i_n\Psi$. Therefore j_n is surjective.

For a C^*-algebra A, if $n \geq csr(\widetilde{SA})$ (the central stable rank of $\widetilde{SA}$), by [8, 10.10], then the natural map of $GL(n-1,\widetilde{SA})/GL^0(n-1,\widetilde{SA})$ to $K_1(SA)(= K_1(\widetilde{SA}))$ is surjective, and also by [8, 10.12], if $n \geq Bsr(\widetilde{SA}) + 2$ (where $Bsr(A)$ is the Bass stable rank of A), then the natural map of $GL(n,\widetilde{SA})/GL^0(n,\widetilde{SA})$ to $K_1(SA)(= K_1(\widetilde{SA}))$ is isomorphic. It is well-known that

$$GL(n,\widetilde{SA})/GL^0(n,\widetilde{SA}) \cong U^n(\widetilde{SA})/U_0^n(\widetilde{SA}).$$

Since $csr(\widetilde{SA}) - 1 \leq Bsr(\widetilde{SA}) \leq str(\widetilde{SA}) = str(SA)$ by [8, 4.10], we have the following theorem by Corollary 2.3.

Theorem 2.2. *For a unital C^*-algebra A, if SA is of stable rank n, then we get that the natural map $\pi_1(U^n(A)) \mapsto K_1(SA) = K_0(A)$ is surjective, and the natural map $\pi_1(U^{n+2}(A)) \mapsto K_1(SA) = K_0(A)$ is isomorphic, therefore*

$$U_0^n(A) \cap \overline{DU^n(A)} = \overline{DU_0^n(A)}$$

by Corollary 2.2.

Theorem 2.3. (1) *For a unital C^*-algebra A with $U_0^n(A) \cap \overline{DU^n(A)} = \overline{DU_0^n(A)}$ $(n \in \mathbf{N})$ (In particular, when SA is of stable rank n by the Theorem* 2.2*), we have a split exact sequence*

$$0 \mapsto AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))} \overset{i_A}{\mapsto} U^n(A)/\overline{DU^n(A)} \overset{\pi_A}{\mapsto} U^n(A)/U_0^n(A) \mapsto 0,$$

where i_A is contractive with the quotient distance D'_A on $AffT(A)/\overline{\Delta_n^0(\pi_1(U_0^n(A)))} \cong U^n(A)/\overline{DU^n(A)}$ and the distance d_A on $U_0^n(A)/\overline{DU_0^n(A)}$.

(2) *For any unital C^*-algebra A, we have a split exact sequence*

$$0 \mapsto AffT(A)/\overline{\rho(K_0(A))} \overset{i_A}{\mapsto} U^\infty(A)/\overline{DU^\infty(A)} \overset{\pi_A}{\mapsto} K_1(A) \mapsto 0$$

with i_A isometric.

Proof. First let $n \in \mathbf{N}$. Since $e^{ia}e^{ib} = e^{i(a+b)}$ module $\overline{DU_0^n(A)}$ with $a, b \in M_n(A)_{sa}$, disscused as above, and any element in $U_0^n(A)$ is of form $e^{ia_1}e^{ia_2}\cdots e^{ia_n}$ with $a_1, a_2, \cdots, a_n \in M_n(A)_{sa}$, $\frac{U_0^n(A)}{\overline{DU_0^n(A)}}$ is a divisible subgroup of $\frac{U^n(A)}{\overline{DU^n(A)}}$ by

$$U_0^n(A) \cap \overline{DU^n(A)} = \overline{DU_0^n(A)}.$$

Therefore there is a split exact sequence

$$0 \mapsto \frac{U_0^n(A)}{\overline{DU^n(A)} \cap U_0^n(A)} \overset{j_A}{\mapsto} \frac{U^n(A)}{\overline{DU^n(A)}} \overset{\pi_A}{\mapsto} \frac{U^n(A)}{U_0^n(A)} \mapsto 0.$$

So, by $U_0^n(A)\cap\overline{DU^n(A)}=\overline{DU_0^n(A)}$ and Theorem 2.1 we get the desired split exact sequence with $i_A=j_A(\Delta_n)^{-1}$. It is clear that j_A is contractive, so is i_A. Now let $n=\infty$. Since $\frac{U^\infty(A)}{DU^\infty(A)}$ and $\frac{U_0^\infty(A)}{DU_0^\infty(A)}$ are the inductive limits of $\big(\frac{U^n(A)}{DU^n(A)},(i_A)_{n,m}\big)$ and $\big(\frac{U_0^n(A)}{DU_0^n(A)},(i_A)_{n,m}\big)$ respectively, $\frac{U_0^\infty(A)}{DU_0^\infty(A)}$ is a divisible subgroup of $\frac{U^\infty(A)}{DU^\infty(A)}$ by Lemma 2.2. Therefore, by Lemma 2.2 again, we have a split sequence

$$0\mapsto\frac{U_0^\infty(A)}{DU_0^\infty(A)}\overset{j_A}{\mapsto}\frac{U^\infty(A)}{DU^\infty(A)}\overset{\pi_A}{\mapsto}\frac{U^\infty(A)}{U_0^\infty(A)}\mapsto 0.$$

Similarly, by Theorem 2.1,

$$\frac{U^\infty(A)}{U_0^\infty(A)}\cong K_1(A)\quad\text{and}\quad\overline{\rho(K_0(A)}\cong\overline{\Delta_\infty^0(\pi_1(U_0^\infty(A)))},$$

we get the desired split exact sequence with $i_A=j_A(\Delta_\infty)^{-1}$ isometric.

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