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GEOMETRY OF COMPLETE HYPERSURFACES EVOLVED BY MEAN CURVATURE FLOW**

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Abstract

Some geometric behaviours of complete solutions to mean curvature flow before the singularities occur are studied. The author obtains the estimates of the rate of the distance between two fixed points and the derivatives of the second fundamental form. By use of a new maximum principle, some geometric properties at infinity are obtained.

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Let M be a complete n-dimensional manifold without boundary, and let $F_t : M^n \to R^{n+1}$ be a one-parameter family of smooth hypersurface immersions in Euclidean space. We say that $M_t = F_t(M^n)$ is a solution of the mean curvature flow (MCF) problem if F_t satisfies

$$\begin{aligned}\frac{\partial}{\partial t}F(X,t) &= -H(X,t)N(X,t), \quad X \in M, t \geq 0,\\ F(\cdot,0) &= F_0(\cdot),\end{aligned}$$

where $H(X,t)$ and $N(X,t)$ are the mean curvature and the unit normal vector field respectively and F_0 describes the immersion of some given initial hypersurface. It is well know[12,3] that for smooth closed initial hypersurface or for complete initial hypersurface with bounded second fundamental form $A = \{h_{ij}\}$ the solution of the MCF exists on a maximal time interval $[0,T)$, $0 < T \leq \infty$. If $T < \infty$, the curvature of the hypersurfaces M_t becomes unbounded for $t \to T$. One would like to understand the singular behaviour for $t \to T$ in detail. Here, by singularities of the MCF we mean solutions to the MCF with unbounded curvature. According to Huisken's report[15], we can discuss it like Hamilton has done on Ricci flow (see [7, 10] and [11]). First, try to analyze the singularities which develop in finite time well enough to enable one to perform geometric surgeries before the singularities occur, which will decompose the hypersurface, and then continue the solution.

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Second, classify solutions to the scaled MCF which exist for all time $t \in [0, \infty)$ and have uniformly bounded curvature.

In this article we are interested in the geometric behaviour of solutions before the singularities occur. We confine our attention to solutions which are smooth and compact, or smooth and complete with bounded second fundamental form. After recalling some notations, in Section 2, we show how the distance changes between two points under the MCF. In Section 3, we obtain the estimates of the derivatives of the second fundamental form from the bound of the second fundamental form. It is useful in discussing the existence of the long time solution and the classification of singularities. We also obtain a maximum principle for mean curvature flow which is more convenient than one in [3] to be used. In Section 4, we study the curvature of complete manifolds at infinity. In Section 5, we discuss the volume of the complete weakly convex solution to the MCF.

§1. Preliminaries

We recall the equations for some geometric quantities associated with the evolving hypersurface and other identities which we shall need in the sequel. We shall follow the notations of [4]; in particular $g = \{g_{ij}\}$ and $A = \{h_{ij}\}(i, j = 1, \cdots, n)$ will denote the metric tensor and the second fundamental form on M induced by the immersion, while $H = \operatorname{tr}(h_{ij})$ is the mean curvature. We also denote by dv the volume element on M. All these quantities depend on x, t (where x is a local coordinate on M), but this dependence will not be written explicitly unless necessary.

Lemma 1.1.[12] *If M_t is a solution of the MCF, we have*

(1) $\frac{\partial}{\partial t} g_{ij} = -2Hh_{ij}$,

(2) $\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2Hh_{il}g^{lm}h_{mj} + |A|^2 h_{ij}$,

(3) $\frac{\partial}{\partial t} H = \Delta H + |A|^2 H$,

(4) $\frac{\partial}{\partial t} |A|^2 = \Delta |A^2| - 2|DA|^2 + 2|A|^4$,

(5) $\frac{\partial}{\partial t} |D^m A|^2 = \Delta |D^m A|^2 - 2|D^{m+1}A|^2 + \sum\limits_{i+j+k=m} D^m A * D^i A * D^j A * D^k A$,

where $S * T$ denotes linear combinations of traces of S and T.

We can introduce the orthonormal frame $\{F_a\}(a = 1, \cdots, n)$, where $F_1, \cdots, F_n$ are tangent to M. We can take covariant derivatives $D = \{D_a\}$ in the frame coordinates. We also have the time-like vector field D_t on the frame bundle, which differentiates in the direction of the moving frame (cf. [8] for detail). We can compute the commutator of D_t and D_a, and obtain following lemma.

Lemma 1.2.[8] *For any function f,*

$$(D_t - \Delta)D_a f = D_a(D_t - \Delta)f + h_{ac}h_{cd}D_d f.$$

In these orthonormal frame coordinates, the evolution of the second fundamental form is particularly simple,

$$D_t h_{ab} = \Delta h_{ab} + |A|^2 h_{ab}.$$

§2. Bounds on Changing Distances

It is useful to see how the actual geometry changes under the mean curvature flow (MCF).

For this purpose we need to control the distance $d(P,Q,t)$ between two points P and Q at time t when P and Q are fixed but t increases.

We let

$$ds_t^2 = g_{ij}(X,t)dx^i dx^j, \quad 0 \le t \le T$$

and use D or D^t to denote the connection of ds_t^2, Δ or Δ_t the Laplacian operator of ds_t^2, and suppose $|A|^2 \le K$ holds on M_t for $0 \le t \le T$.

Lemma 2.1. *We have*

$$e^{-CKT} ds_0^2 \le ds_t^2 \le e^{CKT} ds_0^2, \quad 0 \le t \le T,$$

where the constant C depends only on n.

Proof. By use of the condition $|A|^2 \le K$ and the evolution equation of the metric g_{ij}, we can easily obtain the lemma.

Theorem 2.1. *There exists a constant C depending only on the dimension, such that if the square norm of the second fundamental form is bounded by a constant K, i.e., $|A|^2 \le K$, then*

$$e^{-CK(t_2-t_1)} d(P,Q,t_1) \le d(P,Q,t_2) \le e^{CK(t_2-t_1)} d(P,Q,t_1)$$

for any points P and Q and any times t_1 and t_2.

Proof. Let L be the length of a path γ in a hypersurface $M_t = F_t(M)$. Suppose T is the unit tangent vector to the path and s is the arc length along the path. We keep the path fixed. Then the length L evolves by the formula $\frac{\partial}{\partial t} L = -\int_\gamma Hh(T,T)ds$ under the MCF. The function $d(P,Q,t)$ is the least length L of all paths. In general it will not be smooth in t for fixed P and Q, but at least it will be Lipschitz continuous. Hence we can estimate its derivative above and below, in the sense of giving an upper bound on the lim sup of all forward difference quotients and lower bound on the lim inf of all forward difference quotients (see [5] for details). We have the estimate

$$-\inf_{\gamma\in\Gamma} \Big| \int_\gamma Hh(T,T)ds \Big| \le \frac{d}{dt} d(P,Q,t) \le \inf_{\gamma\in\Gamma} \Big| \int_\gamma Hh(T,T)ds \Big|,$$

where the inf is taken over the compact set Γ of all geodesics γ from P to Q realizing the distance as a minimal length.

Now we apply the bound

$$-CKL(\gamma) \le \int_\gamma Hh(T,T)ds \le CKL(\gamma)$$

to conclude

$$-CKd(P,Q,t) \le \frac{d}{dt} d(P,Q,t) \le CKd(P,Q,t),$$

that is,

$$-CK \le \frac{d}{dt} \log d(P,Q,t) \le CK;$$

all these inequalities are in the sense of [5]. Integrating the inequality we get the result.

§3. Derivative Estimates

In order to study the geometry of complete surfaces at infinity, we need a derivative estimate from the bound of the second fundamental form. In this section, we will let C

denote various constants which depend only on the dimension, the time interval T, and the bound K. With appropriate modifications, the proof of Lemma 7.1 in [18] yields the following derivative estimates of the second fundamental forms under the MCF.

Theorem 3.1. *Suppose we have a solution to the MCF for $0 \le t \le T$ which is complete with bounded second fundamental form $|A|^2 \le K$. Then there exist constants C_k for $k \ge 1$ depending only on k, n, K and T such that the covariant derivative of the second fundamental form is bounded*

$$|DA| \le C_1/\sqrt{t}$$

and the k-th covariant derivative of the curvature is bounded

$$|D^k A| \le C_k/t^{k/2}.$$

Corollary 3.1. *If the second fundamental form is bounded $|A|^2 \le K$ on $M \times [0,T]$, then the space-time derivatives are bounded*

$$|D_t^j D^k A| \le C_{j,k}/t^{(j+k/2)},$$

where the constant $C_{j,k}$ depends on j, k, n, K and T.

Proof. We can express $D_t A$ in terms of ΔA and $A * A$. Likewise we can differentiate this equation to express any space-time derivative $D_t^j D^k A$ just in terms of space derivatives, and recover the bound above.

Now we employ Hamilton's method in [6] to prove the following maximum principle.

Theorem 3.2. *Let $F(X,t)$ be a C^∞ function on $M \times [0,T]$ satisfying*

$$(D_t - \Delta)F \le Q(F, X, t)$$

on $M \times [0,T]$. Let $\rho(X,t)$ denote the distance between X and some fixed point O at time t. If $F(X,t)$ satisfies the following conditions:

(1) *$F(X,t) \le C\rho(X,t)^m$ for some positive integer m, constant $C > 0$;*

(2) *$F(X,0) \le 0$ holds on M;*

(3) *$Q(F,X,t) \le 0$ for $F(X,t) > 0$;*

then $F(X,t) \le 0$ holds on $M \times [0,T]$.

In order to prove this theorem, we may assume we are working on a closed time interval $0 \le t \le T$, for if we only start with $0 < t < T$, we can pass to $\varepsilon \le t \le T_\varepsilon$ and let $\varepsilon \to 0$.

Lemma 3.1. *There exists a function f such that $f \ge 1$ everywhere and $f(X) \to \infty$ as $X \to \infty$, but $|Df| \le C_0$ and $|\Delta f| \le C_0$ for a positive constant C_0 depending only on n, K and T for $t \in [0,T]$. (In case the hypersurface is compact, we take $f \equiv 1$.)*

Proof. From our assumption, at time zero, $|A|^2 \le K$, there must exist a constant $k > 0$, such that $\mathrm{Ric} \ge -k$. Then by a theorem of Schoen and Yau (see Chapter 1 of [17]) there exists a proper smooth function $\tilde{f}$, satisfying $|D\tilde{f}| \le C$, $\tilde{f} \ge C_1\rho$ and $|\Delta\tilde{f}| \le C$ for some constants C and C_1 depending on K, where ρ is a distance function from a fixed point.

In what follows, we will bound covariant derivatives of $\tilde{f}$ at time $t > 0$. By Theorem 2.1, we have $\tilde{f} \ge C\rho$ at $t \in [0,T]$. Choose a coordinate system such that $D_k g_{ij}(X,0) = 0$ at $X \in M$. We let D^t, Δ_t and $|\cdot|_t$ denote the covariant, the Laplacian and the norm at time

t. Then by the definition of covariant derivative, we have

$$D_i^t\tilde{f}(X) = D_i^0\tilde{f}(X),$$
$$\Delta_t\tilde{f}(X) = g^{ij}(X,t)D_i^0D_j^0\tilde{f}(X) - g^{ij}(X,t)\Gamma_{ij}^k(X,t)D_k^0\tilde{f}.$$

Since

$$\frac{\partial}{\partial t}\Gamma_{ij}^k = g^{kl}(D_l(Hh_{ij} - D_j(Hh_{il}) - D_i(Hh_{lj})),$$

we have

$$\frac{\partial}{\partial t}|\Gamma_{ij}^k|^2(X,t) = \frac{\partial}{\partial t}(g_{k\gamma}g^{i\alpha}g^{j\beta}\Gamma_{ij}^k\Gamma_{\alpha\beta}^\gamma) \le \frac{C}{\sqrt{t}}$$

by our assumption and Theorem 3.1. Then $|\Gamma_{ij}^k|^2 \le C$. We have

$$\begin{aligned}|D^t\tilde{f}(X)|^2 &= g^{ij}(X,t)D_i^t\tilde{f}(X)D_j^t\tilde{f}(X)\\ &\le Cg^{ij}(X,0)D_i^0\tilde{f}(X)D_j^0\tilde{f}(X)\\ &= C|D^0\tilde{f}(X)|^2 \le C\end{aligned}$$

and

$$\begin{aligned}|\Delta_t\tilde{f}(X)| &\le |g^{ij}(X,t)D_i^0D_j^0\tilde{f}(X)| + |g^{ij}(X,t)\Gamma_{ij}^k(X,t)D_k^0\tilde{f}|\\ &\le C|\Delta_0\tilde{f}(X)| + C|D^0\tilde{f}| \le C\end{aligned}$$

for some suitable constant C depending only on n, K and T. At last we let $f = 1 + \tilde{f}$, then f is desired.

Lemma 3.2. *Given any constant C, any $\eta > 0$ and any compact set K in space-time, we can find a function $\phi(X,t)$ such that*

(1) *$\phi \le \eta$ on the set K and $\phi \ge \epsilon$ for some $\epsilon > 0$, while $\phi(X,t) \to \infty$ if $X \to \infty$ in the sense that the sets $\phi \le N$ are all compact in space-time for $0 \le t \le T$;*

(2) *$(D_t - \Delta)\phi > C\phi$.*

Proof. Let

$$\phi(X,t) = \epsilon e^{Bt}f(X)$$

with f defined in Lemma 3.1 and constant B will be chosen later. Since $\Delta f \le C$ and $f \ge 1$, we get $\Delta\phi \le C\phi$, and to make (2) work we only need $D_t\phi > C\phi$ with a different C. This can be done by picking $B > C$. To prove (1) we need

$$\epsilon \le \eta e^{-BT}(\max_K f(X))^{-1},$$

which we can do.

Now we begin to prove Theorem 3.2. Let $G(X,t) = F(X,t) - \phi^{m+1}(X,t)$, where the function $\phi = \epsilon e^{Bt}f(X) = \epsilon e^{Bt}(1+\tilde{f})$ is given in Lemma 3.2 and Lemma 3.1. Then $G(X,0) = F(X,0) - \phi^{m+1}(X,0) < 0$ on M, and

$$G(X,t) \le C\rho^m(X,t) - (\epsilon C_1\rho)^{m+1} = (C - C'\rho)\rho^m < 0$$

outside a compact subset K. We want to show $G(X,t) < 0$ holds on $M \times [0,T]$.

If it is not true, we suppose that t_0 is the first time such that at some point $X_0 \in M$, $G(X_0,t_0) = 0$. Then at time t_0, X_0 is the maximal point of G on M and $F(X_0,t_0) =$

$\phi^{m+1}(X_0, t_0) > 0$. So at space-time point (X_0, t_0)

$$\begin{aligned} D_t G(X,t) &\leq \Delta F + Q(F,X,t) - (m+1)\phi^m D_t \phi \\ &= \Delta G + Q(F,X,t) - (m+1)\phi^m (D_t - \Delta)\phi + m(m+1)\phi^{m-1}|D\phi|^2 \\ &< (m+1)\phi^{m+1}(-C + mC_0) < 0, \end{aligned}$$

where the constant C comes from Lemma 3.2 and can be chosen large and C_0 comes from Lemma 3.1. Then at point X_0, $G(X_0, t) > 0$ for some time $t < t_0$. This is a contradiction. Then we have $G(X,t) < 0$ on $M \times [0,T]$. At last we only need to let $\eta \to 0$ in Lemma 3.2, and get $F(X,t) \leq 0$ on $M \times [0,T]$. This completes the proof of Theorem 3.2.

Now we can use Theorem 3.1 and Theorem 3.2 to prove following

Theorem 3.3. *If the square norm of the second fundamental form $|A|^2$ is bounded*

$$|A|^2 \leq K$$

up to time t with $0 < t \leq 1/K$, there exists a constant C depending only on the dimension such that the covariant derivative of the second fundamental form is bounded

$$|DA| \leq CK^{1/2}/t^{1/2}.$$

Proof. We have

$$\begin{aligned} D_t|A|^2 &= \Delta|A|^2 - 2|DA|^2 + 2|A|^4, \\ D_t|DA|^2 &\leq \Delta|DA|^2 - 2|D^2A|^2 + CK|DA|^2. \end{aligned}$$

Now let F be the function

$$F = t|DA|^2 + B|A|^2,$$

where B is a constant we shall choose in a minute. Then $D_tF \leq \Delta F + (CKt - 2B)|DA|^2 + 2B|A|^4$. We assume $tK \leq 1$. Then if we take $B \geq C$, we get $D_tF \leq \Delta F + CK^2$ for some constant C. Also the inequality $F \leq CK$ at $t = 0$ together with Theorem 3.2 implies $F \leq CK + CK^2 t$. Now as long as $tK \leq 1$ this gives $F \leq CK$ for some constant C, and $t|DA|^2 \leq F \leq CK$ yields $|DA| \leq CK^{1/2}/t^{1/2}$ for some constant C.

§4. Geometry of Complete Surfaces at Infinity

Theorem 4.1. *Suppose we have a complete solution to the MCF with bounded second fundamental form. Let s denote the distance from a fixed point on the complete hypersurface. If $|A| \to 0$ as $s \to \infty$ at $t = 0$, this remains true for $t \geq 0$.*

Proof. Suppose $|A|^2 \leq K$ for some constant K. At $t = 0$, for every $\varepsilon > 0$ we can find $\sigma < \infty$, such that $|A|^2 \leq \varepsilon$ for $s \geq \sigma$. We have the evolution equation

$$D_t|A|^2 = \Delta|A|^2 - 2|DA|^2 + 2|A|^4$$

and an estimate

$$D_t|A|^2 \leq \Delta|A|^2 + 2|A|^4.$$

For any $\sigma > 0$ choose

$$\rho = \sigma + (K - \varepsilon)/\delta$$

and choose the continuous function

$$\psi = \begin{cases} K, & \text{if } s \le \sigma, \\ K - \delta(s-\sigma) = \varepsilon + \delta(\rho - \varepsilon), & \text{if } \sigma \le s \le \rho, \\ \varepsilon, & \text{if } s \ge \rho, \end{cases}$$

where s is the distance from some origin at $t = 0$. Then ψ is Lipschitz continuous since s is, and since $|Ds| \le 1$ almost everywhere, we also have $|D\psi| \le \delta$ almost everywhere.

Now we can smooth ψ locally and patch together with a partition of unity to get a function $\tilde{\psi}$ which is smooth and satisfies

$$-\varepsilon \le \tilde{\psi} \le K + \varepsilon \quad \text{and} \quad |D\tilde{\psi}| \le 2\delta \quad \text{everywhere}$$

and

$$\tilde{\psi} \ge K - \varepsilon, \quad \text{if } s \le \sigma,$$
$$\tilde{\psi} \le \varepsilon, \quad \text{if } s \ge \rho.$$

Lastly take $\phi = \tilde{\psi} + 2\varepsilon$. Then

$$\varepsilon \le \phi \le K + 3\varepsilon \quad \text{and} \quad |D\phi| \le 2\delta \quad \text{everywhere}$$

and

$$\phi \ge K \quad \text{if } s \le \sigma \quad \text{and} \quad \phi \le 3\varepsilon \quad \text{if } s \ge \rho.$$

Now define ϕ for $t \ge 0$ by solving the scalar heat equation $\frac{\partial \phi}{\partial t} = \Delta\phi$ in the Laplacian of the metric evolving by the MCF. By the maximum principle we still have $\varepsilon \le \phi \le K + 3\varepsilon$ everywhere for $t \ge 0$. The derivative $D_a\phi$ evolves in an evolving orthonormal frame by Lemma 1.2

$$D_t D_a \phi = \Delta D_a \phi + h_{ac} h_{cd} D_d \phi$$

and hence

$$\begin{aligned} \frac{\partial}{\partial t}|D\phi|^2 &= \Delta|D\phi|^2 - 2|D^2\phi|^2 + 2h_{ab}h_{bc}D_a\phi D_c\phi \\ &\le \Delta|D\phi|^2 + CK|D\phi|^2 \end{aligned}$$

for some constant C depending only on the dimension. By the maximum principle, we have $|D\phi|^2 \le 4\delta^2 e^{CKt}$ for $t \ge 0$.

The second derivative $D_aD_b\phi$ evolves by the formula

$$\begin{aligned} D_t D_a D_b \phi = {} & \Delta D_a D_b \phi + 2(h_{ab}h_{cd} - h_{ad}h_{bc}) D_c D_d \phi + h_{ac}h_{cd} D_d D_b \phi \\ & + h_{bc}h_{cd} D_a D_d \phi + 2h_{cd} D_a h_{bc} D_d \phi. \end{aligned}$$

Then we can obtain

$$D_t|D^2\phi|^2 \le \Delta|D^2\phi|^2 + CK|D^2\phi|^2 + CK^{1/2}|DA||D\phi||D^2\phi|.$$

By Theorem 3.3 we have for $0 < t \le 1/K$

$$D_t|D^2\phi|^2 \le \Delta|D^2\phi|^2 + CK|D^2\phi|^2 + CK|D\phi||D^2\phi|/t^{1/2},$$

where C depends only on the dimension. Let us put $F = t|D^2\phi|^2 + |D\phi|^2$ and compute

$$\frac{\partial}{\partial t}F \le \Delta F - |D^2\phi|^2 + CKt|D^2\phi|^2 + CKt^{1/2}|D\phi||D^2\phi| + CK|D\phi|^2.$$

As $2t^{1/2}|D\phi||D^2\phi| \le t|D^2\phi|^2 + |D\phi|^2$, we have

$$\frac{\partial}{\partial t}F \le \Delta F - |D^2\phi|^2 + CKt|D^2\phi|^2 + CK|D\phi|^2.$$

Then if $t \le C_0/K$ where $C_0 = 1/C$ depends only on the dimension, we have

$$\frac{\partial}{\partial t}F \le \Delta F + CK|D\phi|^2 \le \Delta F + CK\delta^2 e^{CKt}.$$

Now in the time interval $0 < t \le \min\{\frac{C_0}{K}, \frac{1}{K}\}$, we have

$$\frac{\partial}{\partial t}F \le \Delta F + C\delta^2 K,$$

where C depends only on the dimension. By the maximum principle (Theorem 3.2)

$$F(t) \le F(0) + C\delta^2 Kt \le 4\delta^2 + C\delta^2 Kt$$

for $0 < t \le \min\{\frac{C_0}{K}, \frac{1}{K}\}$. Then $F(t) \le C\delta^2$, and $t|D^2\phi|^2 \le C\delta^2$. Thus

$$|D^2\phi| \le C\delta/\sqrt{t}, \quad \text{for} \quad 0 < t \le \min\Big\{\frac{C_0}{K}, \frac{1}{K}\Big\}.$$

Since $|\Delta\phi|^2 \le n|D^2\phi|^2$ and ϕ solves the heat equation,

$$\Big|\frac{\partial\phi}{\partial t}\Big| \le C\delta/\sqrt{t} \quad \text{for} \quad 0 < t \le \min\Big\{\frac{C_0}{K}, \frac{1}{K}\Big\},$$

where C depends only on the dimension n. For all point X,

$$|\phi(X,t) - \phi(X,0)| \le 2C\delta\sqrt{t}, \quad \text{for} \quad 0 < t \le \min\Big\{\frac{C_0}{K}, \frac{1}{K}\Big\}.$$

Since $\delta > 0$ is arbitrarily small, we can take

$$\delta \le \frac{\varepsilon\sqrt{K}}{2C\sqrt{C_0}}$$

so that $2C\delta\sqrt{t} \le \varepsilon$, for $0 < t \le \min\{\frac{C_0}{K}, \frac{1}{K}\}$. Then $\phi \le 4\varepsilon$ for $t \le \min\{\frac{C_0}{K}, \frac{1}{K}\}$ on the set where $s \ge \rho$ at $t = 0$. Now distances can expand, but only at an exponential rate governed by K. In particular if $s = s(X, O, t)$ is the distance between a point X and the origin O at time t, we have $\frac{\partial s}{\partial t} \le CKs$ and $s(t) \le s(0)e^{CKt}$. This gives us a constant C depending only on the dimension such that if $s \ge C\rho$ at X at time $t \le \min\{\frac{C_0}{K}, \frac{1}{K}\}$, then $s \ge \rho$ at X at $t = 0$, and $\phi \le 4\varepsilon$ at X at time t.

Now at $t = 0$, we have

$$|A|^2 \le K \le \phi, \quad \text{if } s \le \sigma,$$
$$|A|^2 \le \varepsilon \le \phi, \quad \text{if } s \ge \sigma.$$

So $|A|^2 \le \phi$ everywhere at $t = 0$. Since

$$\frac{\partial}{\partial t}|A|^2 \le \Delta|A|^2 + 2|A|^4,$$

we have

$$\frac{\partial}{\partial t}|A|^2 \le \Delta|A|^2 + 2K|A|^2,$$

while

$$\frac{\partial}{\partial t}(e^{2Kt}\phi) = \Delta(e^{2Kt}\phi) + 2K(e^{2Kt}\phi).$$

So $|A|^2 \le e^{2Kt}\phi$ by Theorem 3.2, for $t \le \min\{\frac{C_0}{K}, \frac{1}{K}\}$. This gives $|A|^2 \le C\phi$ for some constant C depending only on the dimension. Hence at time t we have

$$|A|^2 \le C\varepsilon \quad \text{for } s \ge C\rho,$$

where the constant C depends only on the dimension and independent of ε. Thus $|A| \to 0$ for $t \le \min\{\frac{C_0}{K}, \frac{1}{K}\}$ also as $s \to \infty$. Since the time interval can always be advanced by $\min\{\frac{C_0}{K}, \frac{1}{K}\}$ as long as $|A|^2 \le K$, we get the result until $|A|$ becomes unbounded or $t \to \infty$.

§5. Asymptotic Volume Ratio

Next we introduce the concept asymptotic volume ratio which was first defined by Hamilton (see [7]). Let s denote the distance to an origin O in a complete manifold of dimension n, let B_s denote the ball of radius s around the origin, and let $V(B_s)$ be its volume. If the manifold has weakly positive Ricci curvature, then the standard volume comparison theorem tells us that $V(B_s)/s^n$ is monotone decreasing in s. We define the asymptotic volume ratio

$$\nu = \lim_{s\to\infty} V(B_s)/s^n.$$

In Euclidean space ν is the volume $\bar{\nu}$ of the unit ball, otherwise $\nu \le \bar{\nu}$. For all s, $V(B_s) \ge \nu s^n$. It is clear that the value of ν is independent of the choice of the origin. Hence the lower bound holds on any ball around any point P,

$$V(B_s(P)) \ge \nu s^n.$$

Of course we also have $V(B_s(P)) \le \bar{\nu} s^n$.

Theorem 5.1. *Suppose we have a complete weakly convex solution to the MCF with bounded second fundamental form, where $|A|^2 \to 0$ as $s \to \infty$ (a condition preserved by the flow). Then the asymptotic volume ratio ν is constant.*

Proof. Let γ be a small constant we shall choose soon, and consider the annulus

$$N_\sigma = \{\gamma\sigma \le s \le \sigma\}.$$

Since $N_\sigma = B_\sigma - B_{\gamma\sigma}$, we have

$$V(N_\sigma) = V(B_\sigma) - V(B_{\gamma\sigma}).$$

If the asymptotic volume ratio is at least ν, then

$$V(N_\sigma) \ge (\nu - \gamma^n\bar{\nu})\sigma^n,$$

where γ is small, $\nu - \gamma^n\bar{\nu}$ is nearly ν and most of the ball is in the annulus.

The volume of the annulus changes at a rate

$$\frac{d}{dt}V(N_\sigma) = -\int_{N_\sigma} H^2 dv.$$

For every ε and every γ we can find σ_0 so that if $\sigma \ge \sigma_0$ then $|A|^2 \le \varepsilon$ on N_σ. This makes

$$\left|\frac{d}{dt}V(N_\sigma)\right| \le n\varepsilon V(N_\sigma).$$

If $V_1(N_\sigma)$ is the volume at time t_1 and $V_2(N_\sigma)$ is the volume at time t_2, we have

$$V_2(N_\sigma) \ge e^{-n\varepsilon|t_2-t_1|}V_1(N_\sigma).$$

Let ν_i be the asymptotic volume ratio at time t_i $(i = 1, 2)$. Then

$$V_1(N_\sigma) \ge (\nu_1 - \gamma^n\bar{\nu})\sigma^n$$

for all σ and all $\gamma > 0$. If $V_2(B_\sigma)$ is the volume of B_σ at time t_2, then

$$V_2(B_\sigma) \geq V_2(N_\sigma).$$

Together these make

$$V_2(B_\sigma) \geq e^{-n\varepsilon|t_2-t_1|}(\nu_1 - \gamma^n \bar{\nu})\sigma^n.$$

Fix $\gamma > 0$ and let $\sigma \to \infty$. Then $\varepsilon \to 0$ and

$$\nu_2 = \lim_{\sigma \to \infty} \frac{V_2(B_\sigma)}{\sigma^n} \geq \nu_1 - \gamma^n \bar{\nu}.$$

Since this is true for all $\gamma > 0$, $\nu_2 \geq \nu_1$. But we can switch t_1 and t_2, so $\nu_1 = \nu_2$ and ν is constant.

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