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SPHERICAL MEANS, DISTRIBUTIONS AND CONVOLUTION OPERATORS IN CLIFFORD ANALYSIS

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Abstract

New higher dimensional distributions are introduced in the framework of Clifford analysis. They complete the picture already established in previous work, offering unity and structural clarity. Amongst them are the building blocks of the principal value distribution, involving spherical harmonics, considered by Horváth and Stein.

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§1. Introduction

In a previous paper[2] we introduced in m-dimensional Euclidean space, three sets of distributions $T_{\lambda,p}, U_{\lambda,p}$ and $V_{\lambda,p}$, with $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$, in the framework of Clifford analysis. Clifford analysis may be regarded as a direct and elegant generalization to higher dimensions of the theory of holomorphic functions in the complex plane, centred around the notion of monogenic function, i.e. a null solution of the Dirac operator $\underline{\partial}$. Those distributions were defined using the spherical co-ordinates, the fundamental distribution "finite parts" Fp on the real line, the so-called generalized spherical means, i.e. integrals over the unit sphere S^{m-1} of test functions and the like, and the inner and outer spherical monogenics, i.e. restrictions to the unit sphere of monogenic homogeneous polynomials and functions respectively. The spherical co-ordinates $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} \in S^{m-1}$ really play a fundamental rôle: firstly, they reflect the "spherical" philosophy of our approach encompassing all dimensions at once as opposed to a cartesian or tensorial approach with products of one dimensional phenomena, and secondly, they enable designing a highly efficient technique where the explicit calculations are carried out in one dimension and then exported to the original setting of Euclidean space.

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As we consider Clifford algebra valued test functions and as the multiplication in a Clifford algebra is non-commutative, distributions may be regarded as left as well as right linear functionals on these spaces of test functions. The above mentioned three families of distributions are interrelated by the action of the Dirac operator $\underline{\partial}$, which, again due to the non-commutativity, may act from the left or from the right on those distributions. These relationships are expressed in the general formulae:

$$\underline{\partial}\, T_{\lambda,p} = \lambda\, U_{\lambda-1,p}, \quad T_{\lambda,p}\, \underline{\partial} = \lambda\, V_{\lambda-1,p},$$
$$\underline{\partial}\, U_{\lambda,2k} = V_{\lambda,2k}\, \underline{\partial} = -(\lambda + m - 1 + 4k)\, T_{\lambda-1,2k},$$
$$\underline{\partial}\, U_{\lambda,2k+1} = V_{\lambda,2k+1}\, \underline{\partial} = -(\lambda + m - 1 + 4k)\, T_{\lambda-1,2k+1},$$

disregarding some exceptional values of the parameters $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$.

For specific values of the parameters $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$ these distributions turn into known kernel functions in harmonic and Clifford analysis, thus illustrating the unifying character of our approach: up to constants, $U_{-m,0}$ reduces to $Pv\frac{\underline{\omega}}{r^m}$, the higher dimensional analogue of the "Principal Value" distribution on the real line and the kernel for the Hilbert transform (see [3]); $U_{-m+1,0}$ reduces to a fundamental solution of the Dirac operator $\underline{\partial}$; $T_{-m+2,0}$ reduces to a fundamental solution of the Laplace operator; for $\lambda = -p$ we obtain the inner and outer spherical monogenics $P_p(\underline{\omega})$, $P_p(\underline{\omega})\underline{\omega}$ and $\underline{\omega}P_p(\underline{\omega})$; etc.

By the same procedure but with a new kind of generalized spherical mean, we introduce in this paper a fourth family of Clifford distributions $W_{\lambda,p}$. Our motivation is threefold. The three generalized spherical means underlying the definition of the $T_{\lambda,p}, U_{\lambda,p}$ and $V_{\lambda,p}$ distributions, were constructed as integrals over the unit sphere S^{m-1} of products of scalar valued test functions with the spherical monogenics $P_p(\underline{\omega})$, $\underline{\omega}P_p(\underline{\omega})$ and $P_p(\underline{\omega})\underline{\omega}$, where $\underline{\omega}$ may be regarded as the higher dimensional counterpart to the "signum"-distribution on the real line. The picture of generalized spherical means is now "symmetrically" completed by considering the ultimate combination $\underline{\omega}P_p(\underline{\omega})\underline{\omega}$. The second motivation is again a completeness argument but now for the action of the Dirac operator. If $\underline{\partial}$ acts from the right on $U_{\lambda,p}$ and from the left on $V_{\lambda,p}$, then the distributions of type $W_{\lambda,p}$ are automatically gener ated. Finally it will be shown that, while $U_{-m-p,p}$ and $V_{-m-p,p}$ are building blocks of the principal value distribution $Pv\ \frac{S_p(\underline{\omega})}{r^m}$, $S_p(\underline{\omega})$ being a spherical harmonic, considered by Horváth in [7] and by Stein-Weiß in [SW], the distribution $W_{-m-p,p}$ is the missing term in a similar symmetric decomposition of the principal value distribution $Pv\frac{\underline{\omega}S_p(\underline{\omega})}{r^m}$. Morover the distributions $W_{\lambda,p}$ intervene in the construction of new distributions generalizing the above mentioned principal value distributions.

§2. Clifford Analysis

For the sake of completeness we first recall some basic notions and results in Clifford analysis. Clifford analysis offers a function theory which is a higher dimensional analogue of the theory of holomorphic functions of one complex variable. For more details concerning this function theory and its applications to harmonic analysis we refer the reader to [1, 4, 6].

Let $\mathbb{R}^{0,m}$ be the real vector space $\mathbb{R}^m$, endowed with a non-degenerate quadratic form of signature $(0, m)$, let $(e_1, \cdots, e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m}$, and let $\mathbb{R}_{0,m}$ be the universal Clifford algebra constructed over $\mathbb{R}^{0,m}$. The non-commutative multiplication in $\mathbb{R}_{0,m}$ is then governed by the rules

$$e_i^2 = -1, \quad i = 1, 2, \cdots, m,$$
$$e_i e_j + e_j e_i = 0, \quad i \neq j.$$

For a set $A = \{i_1, \cdots, i_h\} \subset \{1, \cdots, m\}$ with $1 \leq i_1 < i_2 < \cdots < i_h \leq m$, we put

$$e_A = e_{i_1} e_{i_2} \cdots e_{i_h},$$
$$e_\phi = 1,$$

the latter being the identity element; then $(e_A : A \subset \{1, \cdots, m\})$ is a basis for the Clifford algebra $\mathbb{R}_{0,m}$. Any $a \in \mathbb{R}_{0,m}$ may thus be written as

$$a = \sum_A a_A\, e_A, \qquad a_A \in \mathbb{R},$$

or still as $a = \sum_{k=0}^{m} [a]_k$, where $[a]_k = \sum_{|A|=k} a_A\, e_A$ is a so-called k-vector $(k = 0, 1, \cdots, m)$.

If we denote the space of k-vectors by $\mathbb{R}^k_{0,m}$, then $\mathbb{R}_{0,m} = \sum_{k=0}^{m} \oplus \mathbb{R}^k_{0,m}$, leading to the identification of $\mathbb{R}$ and $\mathbb{R}^{0,m}$ with respectively $\mathbb{R}^0_{0,m}$ and $\mathbb{R}^1_{0,m}$.

We will also identify an element $\underline{x} = (x_1, \cdots, x_m) \in \mathbb{R}^m$ with the one-vector (or vector for short)

$$\underline{x} = \sum_{j=1}^{m} x_j\, e_j.$$

For any two vectors $\underline{x}$ and $\underline{y}$ we have $\underline{x}\,\underline{y} = -\langle\, \underline{x}, \underline{y}\,\rangle + \underline{x} \wedge \underline{y}$, where

$$\langle\, \underline{x}, \underline{y}\,\rangle = \sum_{j=1}^{m} x_j y_j = -\frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\underline{x})$$

is a scalar and

$$\underline{x} \wedge \underline{y} = \sum_{i<j} e_{ij}(x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\underline{x})$$

is a 2-vector, also called bivector.

In particular

$$\underline{x}^2 = -\langle\, \underline{x}, \underline{x}\,\rangle = -|\underline{x}|^2 = -\sum_{j=1}^{m} x_j^2.$$

Conjugation in $\mathbb{R}_{0,m}$ is defined as the anti-involution for which

$$\overline{e}_j = -e_j, \qquad j = 1, \cdots, m.$$

In particular for a vector $\underline{x}$ we have $\overline{\underline{x}} = -\underline{x}$. The Dirac operator in $\mathbb{R}^m$ is the first order vector-valued differential operator

$$\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j},$$

its fundamental solution being given by

$$E_m(\underline{x}) = \frac{1}{a_m}\, \frac{\overline{\underline{x}}}{|\underline{x}|^m}$$

with a_m the area of the unit sphere S^{m-1} in $\mathbb{R}^m$.

Considering functions defined in $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$, we say that the function f is left-monogenic in the open region Ω of $\mathbb{R}^m$ iff f is continuously differentiable in Ω and satisfies in Ω :

$$\underline{\partial}\, f = 0.$$

As $\overline{\underline{\partial} f} = \overline{f}\,\overline{\underline{\partial}} = -\overline{f}\underline{\partial}$, a function f is left monogenic in Ω iff $\overline{f}$ is right monogenic in Ω.

As moreover the Dirac operator factorizes the Laplace operator

$$-\underline{\partial}^2 = \underline{\partial}\,\overline{\underline{\partial}} = \overline{\underline{\partial}}\,\underline{\partial} = \Delta,$$

where $\Delta = \sum_{j=1}^{m} \partial_{x_j}^2$, a monogenic function in Ω is harmonic and hence C_∞ in Ω.

Introducing spherical co-ordinates

$$\underline{x} = r\underline{\omega}, \quad r = |\underline{x}|, \quad \underline{\omega} \in S^{m-1},$$

the Dirac operator $\underline{\partial}$ may be written as

$$\underline{\partial} = \underline{\omega}\partial_r + \frac{1}{r}\partial_{\underline{\omega}} = \underline{\omega}\Big(\partial_r - \frac{1}{r}\,\underline{\omega}\,\partial_{\underline{\omega}}\Big),$$

while the Laplace operator takes the form

$$\Delta = \partial_r^2 + \frac{m-1}{r}\,\partial_r + \frac{1}{r^2}\,\Delta^*,$$

Δ^* being the Laplace-Beltrami operator on S^{m-1}.

In the definition of our Clifford distributions a fundamental role is played by the so-called inner and outer spherical monogenics.

Start with a homogeneous polynomial $P_p(\underline{x})$ of degree p which we take to be vector-valued and left (and hence also right) monogenic.

Then the following formulae hold in $\mathbb{R}^m$:

$$\begin{aligned} \underline{\partial}P_p(\underline{x}) &= P_p(\underline{x})\underline{\partial} = 0, \\ \underline{\partial}(\underline{x}\,P_p(\underline{x})) &= (P_p(\underline{x})\,\underline{x})\underline{\partial} = -(m+2p)P_p(\underline{x}), \\ \underline{\partial}(P_p(\underline{x})\,\underline{x}) &= (\underline{x}\,P_p(\underline{x}))\underline{\partial} = (m-2)P_p(\underline{x}), \qquad p \neq 0, \end{aligned}$$

and

$$\Delta\ P_p(\underline{x}) = \Delta(\underline{x}\,P_p(\underline{x})) = \Delta(P_p(\underline{x})\,\underline{x}) = 0,$$

since the Dirac operator $\underline{\partial}$ factorizes the Laplace operator in $\mathbb{R}^m$.

These specific polynomials $P_p(\underline{x})$ may be realized under the action of the Dirac operator on real-valued harmonic homogeneous polynomials S_{p+1} of degree $(p+1)$:

$$P_p(\underline{x}) = \underline{\partial}\,S_{p+1}(\underline{x}).$$

We then have

$$\begin{aligned} \underline{x}\,P_p(\underline{x}) &= -\langle\, \underline{x}, \underline{\partial}\,\rangle S_{p+1}(\underline{x}) + \underline{x} \wedge \underline{\partial}\,S_{p+1}(\underline{x}), \\ P_p(\underline{x})\,\underline{x} &= -\langle\, \underline{x}, \underline{\partial}\,\rangle S_{p+1}(\underline{x}) - \underline{x} \wedge \underline{\partial}\,S_{p+1}(\underline{x}), \end{aligned}$$

from which it follows that

$$\underline{x}\,P_p(\underline{x}) + P_p(\underline{x})\,\underline{x} = -2(p+1)S_{p+1}(\underline{x})$$

is scalar valued.

We also have

$$S_{p+1}(\underline{x}) = R_{p+1}(\underline{x}) - \frac{1}{m+2k}\,\underline{x}\,P_p(\underline{x}),$$

where

$$R_{p+1}(\underline{x}) = -\,\frac{1}{2p+2}\Big(\frac{m-2}{m+2p}\,\underline{x}\,P_p(\underline{x}) + P_p(\underline{x})\,\underline{x}\Big)$$

is a left monogenic homogeneous polynomial of degree $(p+1)$.

By taking restrictions to the unit sphere S^{m-1} of the polynomials $P_p(\underline{x})$, we obtain so-called inner spherical monogenics $P_p(\underline{\omega})$, for which the following formulae hold:

$$\begin{aligned}
\underline{\partial}\, P_p(\underline{\omega}) &= -\frac{p}{r}\,\underline{\omega}\, P_p(\underline{\omega}), \quad P_p(\underline{\omega})\underline{\partial} = -\frac{p}{r}\, P_p(\underline{\omega})\,\underline{\omega},\\
\partial_{\underline{\omega}}\, P_p(\underline{\omega}) &= -p\,\underline{\omega}\, P_p\,(\underline{\omega}), \quad P_p(\underline{\omega})\,\partial_{\underline{\omega}} = -p\, P_p(\underline{\omega})\,\underline{\omega},\\
\underline{\omega}\,\partial_{\underline{\omega}}\, P_p(\underline{\omega}) &= P_p(\underline{\omega})\partial_{\underline{\omega}}\,\underline{\omega} = p\, P_p(\underline{\omega}),\\
\partial_{\underline{\omega}}(\underline{\omega}\, P_p(\underline{\omega})) &= (P_p(\underline{\omega})\underline{\omega})\partial_{\underline{\omega}} = -(m+p-1)P_p(\underline{\omega}),\\
\partial_{\underline{\omega}}(P_p(\underline{\omega})\,\underline{\omega}) &= (\underline{\omega}\, P_p(\underline{\omega}))\partial_{\underline{\omega}} = (m-2)P_p(\underline{\omega}) - (p+1)\underline{\omega}\, P_p(\underline{\omega})\,\underline{\omega}, \quad p \neq 0,\\
\partial^2_{\underline{\omega}}\, P_p(\underline{\omega}) &= P_p(\underline{\omega})\,\partial^2_{\underline{\omega}} = p(m+p-1)\, P_p(\underline{\omega}),\\
\Delta^*\, P_p(\underline{\omega}) &= (-p)(p+m-2)\, P_p(\underline{\omega}).
\end{aligned}$$

Given an inner spherical monogenic $P_p(\underline{\omega})$, then obviously $r^p\, P_p(\underline{\omega}) = P_p(\underline{x})$ is a left and right monogenic homogeneous polynomial, the restriction to the unit sphere of which is precisely $P_p(\underline{\omega})$.

At the same time the functions

$$\begin{aligned}
\frac{1}{r^{m+p-1}}\,\underline{\omega}\, P_p(\underline{\omega}) &= \frac{1}{r^{m+2p}}\,\underline{x}\, P_p(\underline{x}) = Q_p^{(l)}(\underline{x}),\\
\frac{1}{r^{m+p-1}}\, P_p(\underline{\omega})\,\underline{\omega} &= \frac{1}{r^{m+2p}}\, P_p(\underline{x})\,\underline{x} = Q_p^{(r)}(\underline{x})
\end{aligned}$$

are left, respectively right, monogenic homogeneous functions of order $-(m+p-1)$ in the complement of the origin. Their restrictions to the unit sphere S^{m-1}:

$$Q_p^{(l)} = \underline{\omega}\, P_p(\underline{\omega}) \quad \text{and} \quad Q_p^{(r)} = P_p(\underline{\omega})\,\underline{\omega}$$

are called outer spherical monogenics.

With the above notations we have

$$\begin{aligned}
\underline{\omega}\, P_p(\underline{\omega}) + P_p(\underline{\omega})\,\underline{\omega} &= -2(p+1)\, S_{p+1}(\underline{\omega}),\\
S_{p+1}(\underline{\omega}) &= R_{p+1}(\underline{\omega}) - \frac{1}{m+2p}\,\underline{\omega}\, P_p(\underline{\omega}).
\end{aligned}$$

Also note that the inner spherical monogenics $P_p(\underline{\omega})$ and the outer spherical monogenics $\underline{\omega}\, P_p(\underline{\omega})$ and $P_p(\underline{\omega})\,\underline{\omega}$ are special cases of spherical harmonics.

Denoting by $\mathcal{D}$ and $\mathcal{S}$ the space of the compactly supported, respectively rapidly decreasing, real-valued test functions in $\mathbb{R}^m$, we will consider the modules of testfunctions $\prod\limits_{A\subset\{1,\cdots,m\}} \mathcal{D}$ and $\prod\limits_{A\subset\{1,\cdots,m\}} \mathcal{S}$; any such test function φ may be written as

$$\varphi = \sum_{A\subset\{1,\cdots,m\}} e_A\,\phi_A, \quad \phi_A \in \mathcal{D} \text{ or } \mathcal{S}.$$

In most cases we will even use real-valued test functions.

A left Clifford distribution $\mathcal{T}^{(l)}$ is then a bounded left $\mathbb{R}_{0,m}$-linear functional for which there exist bounded real-linear functionals $\mathcal{T}_B (B \subset \{1,\cdots,m\})$ such that

$$\langle\, \varphi, \mathcal{T}^{(l)}\,\rangle = \sum_{A,B} e_A\, e_B \langle\, \mathcal{T}_B, \phi_A \rangle,$$

and a similar definition for a right Clifford distribution $\mathcal{T}^{(r)}$:

$$\langle\, \mathcal{T}^{(r)}, \varphi\,\rangle = \sum_{A,B} e_B\, e_A \langle\, \mathcal{T}_B, \phi_A\,\rangle.$$

§3. The Picture Thus Far

In [2] we already introduced in m-dimensional Euclidean space, three families of Clifford ditributions: $T_{\lambda,p}, U_{\lambda,p}$ and $V_{\lambda,p}$, with $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$. In order to make this paper selfcontained we recall their construction and basic properties.

Let μ be a complex parameter, let x be a real variable and consider the function

$$x_+^\mu = \begin{cases} x^\mu, & x > 0, \\ 0, & x < 0. \end{cases}$$

For $\mathbb{R}e\mu > -1$, this function x_+^μ is a regular distribution. For each $n \in \mathbb{N}$ and for $\mu \in \mathbb{C}$ such that $-n-1 < \mathbb{R}e\mu < -n$, the classical one-dimensional distribution $Fp\, x_+^\mu$—where Fp stands for "finite part"—is defined by

$$\begin{aligned}\langle\, Fp\, x_+^\mu, \phi \,\rangle &= \int_0^{+\infty} x^\mu \Big(\phi(x) - \phi(0) - \frac{\phi'(0)}{1!}\, x - \cdots - \frac{\phi^{(n-1)}(0)}{(n-1)!}\, x^{n-1}\Big) dx \\ &= \lim_{\substack{\varepsilon \to 0 \\ >}} \int_\varepsilon^{+\infty} \Big(x^\mu \phi(x) + \phi(0) \frac{\varepsilon^{\mu+1}}{\mu+1} + \cdots + \frac{\phi^{(n-1)}(0)}{(n-1)!}\, \frac{\varepsilon^{\mu+n}}{\mu+n}\Big)\, dx.\end{aligned}$$

As a function of μ, x_+^μ is holomorphic in $\mathbb{R}e\mu > -1$, and by analytic continuation $Fp\, x_+^\mu$ is holomorphic in $\mathbb{C}\backslash\{-1,-2,-3,\cdots\}$; the singular points $-n, n \in \mathbb{N}$, are simple poles with residue $\frac{(-1)^{n-1}}{(n-1)!}\, \delta_x^{(n-1)}$.

The derivative of $Fp\, x_+^\mu$ is given by

$$\frac{d}{dx}\, Fp\, x_+^\mu = \mu\, Fp\, x_+^{\mu-1}, \quad \mu \neq 0, -1, -2, -3, \cdots,$$

and multiplication with powers of the variable x follows the rule

$$x\, Fp\, x_+^\mu = Fp\, x_+^{\mu+1}, \quad \mu \neq -1, -2, -3, \cdots.$$

By slightly changing the above definition of $Fp\, x_+^\mu$, it may be defined for negative entire exponents, leading to the so-called monomial pseudofunctions $Fp\, x_+^{-n}$, $n \in \mathbb{N}$, (see e.g. [8,5]) given by

$$\begin{aligned}&\langle\, Fp\, x_+^{-n}, \phi(x) \,\rangle \\ &= \lim_{\substack{\varepsilon \to 0 \\ >}} \int_\varepsilon^{+\infty} \Big(x^{-n}\phi(x)\, dx + \phi(0) \frac{\varepsilon^{-n+1}}{-n+1} + \cdots + \frac{\phi^{(n-2)}(0)}{(n-2)!}\, \frac{\varepsilon^{-1}}{(-1)} + \frac{\phi^{(n-1)}(0)}{(n-1)!} \ln \varepsilon\Big) dx.\end{aligned}$$

Their derivatives are given by

$$\frac{d}{dx}\, Fp\, x_+^{-n} = (-n) Fp\, x_+^{-n-1} + (-1)^n \frac{1}{n!}\, \delta_x^{(n)}\,, \quad n \in \mathbb{N},$$

and they satisfy the multiplication rule

$$\begin{aligned}&x\, Fp\, x_+^{-1} = Y(x), \\ &x\, Fp\, x_+^{-n} = Fp\, x_+^{-n+1}, \quad n = 2, 3, 4, \cdots,\end{aligned}$$

where $Y(x)$ stands for the Heaviside distribution, which is identified with $Fp\, x_+^0$.

In the sequel we will also make use of the following technical lemma.

Lemma 3.1. *If the test function ϕ is such that $\phi(0) = \phi'(0) = \cdots = \phi^{(k-1)}(0) = 0$, then*

$$\begin{aligned}\langle\, x_+^\mu, \frac{1}{x^k}\phi(x) \,\rangle &= \langle\, x_+^{\mu-k}, \phi(x) \rangle, \quad \mu \in \mathbb{C}\backslash\{k-1, k-2, k-3, \cdots\}, \\ \langle\, Fp\, x_+^n, \frac{1}{x^k}\phi(x) \rangle &= \langle\, Fp\, x_+^{n-k},\, \phi(x) \rangle, \qquad n = k-1, k-2, k-3, \cdots.\end{aligned}$$

Next define the so-called generalized spherical means $\Sigma_p^{(0)}[\phi]$, $\Sigma_p^{(1)}[\phi]$ and $\Sigma_p^{(3)}[\phi]$ as follows (see also [10]).

Let $\phi(\underline{x})$ be a scalar valued test function in $\mathbb{R}^m$, and let $P_p(\underline{x})$ be a vector valued, monogenic, homogeneous polynomial of degree $p \neq 0$ as introduced in the previous section. Then

(i) $\Sigma_{2k}^{(0)}[\phi] = \Sigma^{(0)}[P_{2k}(\underline{\omega})\phi(\underline{x})] = \dfrac{1}{a_m} \displaystyle\int_{S^{m-1}} P_{2k}(\underline{\omega})\phi(\underline{x})\, dS(\underline{\omega})$,

(ii) $\Sigma_{2k+1}^{(0)}[\phi] = \Sigma^{(0)}[r\, P_{2k+1}(\underline{\omega})\phi(\underline{x})] = \dfrac{r}{a_m} \displaystyle\int_{S^{m-1}} P_{2k+1}(\underline{\omega})\phi(\underline{x})\, dS(\underline{\omega})$,

(iii) $\Sigma_{2k}^{(1)}[\phi] = \Sigma^{(0)}[\underline{\omega}\, P_{2k}(\underline{\omega})\phi(\underline{x})] = \dfrac{1}{a_m} \displaystyle\int_{S^{m-1}} \underline{\omega}\, P_{2k}(\underline{\omega})\phi(\underline{x})\, dS(\underline{\omega})$,

(iv) $\Sigma_{2k+1}^{(1)}[\phi] = \Sigma^{(0)}[r\, \underline{\omega}\, P_{2k+1}(\underline{\omega})\phi(\underline{x})] = \dfrac{r}{a_m} \displaystyle\int_{S^{m-1}} \underline{\omega}\, P_{2k+1}(\underline{\omega})\phi(\underline{x})\, dS(\underline{\omega})$,

(v) $\Sigma_{2k}^{(3)}[\phi] = \Sigma^{(0)}[P_{2k}(\underline{\omega})\, \underline{\omega}\, \phi(\underline{x})] = \dfrac{1}{a_m} \displaystyle\int_{S^{m-1}} P_{2k}(\underline{\omega})\, \underline{\omega}\, \phi(\underline{x})\, dS(\underline{\omega})$,

(vi) $\Sigma_{2k+1}^{(3)}[\phi] = \Sigma^{(0)}[P_{2k+1}(\underline{\omega})\, r\, \underline{\omega}\, \phi(\underline{x})] = \dfrac{r}{a_m} \displaystyle\int_{S^{m-1}} P_{2k+1}(\underline{\omega})\, \underline{\omega}\, \phi(\underline{x})\, dS(\underline{\omega})$.

Finally define the distributions $T_{\lambda,p}, U_{\lambda,p}$ and $V_{\lambda,p}$, where $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$, as follows.

Let ϕ be a scalar valued test funtion, let $\mu = \lambda + m - 1$ and introduce the notation p_e ("even part of p") by $p_e = p$ if p is even, and $p_e = p - 1$ if p is odd; then put

(i) $\langle\, T_{\lambda,p}, \phi \,\rangle = a_m \,\langle\, Fp\, r_+^{\mu+p_e}, \Sigma_p^{(0)}[\phi] \,\rangle$,

(ii) $\langle\, U_{\lambda,p}, \phi \,\rangle = a_m \langle\, Fp\, r_+^{\mu+p_e}, \Sigma_p^{(1)}[\phi] \,\rangle$,

(iii) $\langle\, V_{\lambda,p}, \phi \,\rangle = a_m \langle\, Fp\, r_+^{\mu+p_e}, \Sigma_p^{(3)}[\phi] \,\rangle$.

These three families of distributions are interrelated by the action of the Dirac operator. In the next proposition we only mention the general case; for the exceptional cases we refer the reader to [2].

Proposition 3.1. *For $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$ such that $\lambda + m - 1 + p_e \neq 0, -1, -2, -3, \cdots$, we have*

(i) $\underline{\partial}\, T_{\lambda,p} = \lambda\, U_{\lambda-1,p}$,

(ii) $T_{\lambda,p}\, \underline{\partial} = \lambda\, V_{\lambda-1,p}$,

(iii) $\underline{\partial}\, U_{\lambda,p} = V_{\lambda,p}\, \underline{\partial} = -(\lambda + m - 1 + 2p_e)\, T_{\lambda-1,p}$.

§4. The Generalized Spherical Mean $\Sigma_p^{(2)}$

We introduce the new generalized spherical mean $\Sigma_p^{(2)}$ in which the higher dimensional "signum distribution" $\underline{\omega}$ plays a symmetrical rôle, in contrast to the generalized spherical means $\Sigma_p^{(1)}$ nad $\Sigma_p^{(3)}$ where the position of $\underline{\omega}$ is non-symmetrical. It is defined by

$$\Sigma_{2k}^{(2)}[\phi] = \frac{1}{a_m} \int_{S^{m-1}} \underline{\omega}\, P_{2k}(\underline{\omega})\, \underline{\omega}\, \phi(\underline{x})\, dS(\underline{\omega}), \quad k = 1, 2, 3, \cdots,$$

$$\Sigma_{2k+1}^{(2)}[\phi] = \frac{r}{a_m} \int_{S^{m-1}} \underline{\omega}\, P_{2k+1}(\underline{\omega})\, \underline{\omega}\, \phi(\underline{x})\, dS(\underline{\omega}) \quad , \quad k = 0, 1, 2, \cdots,$$

ϕ being a scalar valued test function.

The generalized spherical mean $\Sigma_p^{(2)}$ is vector valued, it is an even function of r with all odd order derivatives vanishing at the origin $r = 0$. As to the even order derivatives we have the following proposition.

Proposition 4.1. *For a scalar valued test function we have*

$$\begin{aligned}
\left\{\partial_r^{2l}\,\Sigma_{2k}^{(2)}[\phi]\right\}_{r=0} &= \frac{(2l)!}{(2k+2l+2)!}\,\frac{(-1)^{k+l+1}}{C(k+l+1)}\left\{\underline{\partial}^{2k+2l+2}(\underline{x}\ P_{2k}(\underline{x})\ \underline{x}\ \phi(\underline{x}))\right\}_{\underline{x}=0}\\
&= \frac{(2l)!}{(2k+2l+2)!}\,\frac{1}{C(k+l+1)}\left\{\Delta_m^{k+l+1}(\underline{x}\ P_{2k}(\underline{x})\ \underline{x}\ \phi(\underline{x}))\right\}_{\underline{x}=0}\\
&= \frac{(2l)!}{(2k+2l+2)!}\,\frac{1}{C(k+l+1)}\langle\ \underline{x}\ P_{2k}(\underline{x})\ \underline{x}\ \Delta_m^{k+l+1}\ \delta(\underline{x})\ ,\ \phi(\underline{x}))\rangle,\\
\left\{\partial_r^{2l}\,\Sigma_{2k+1}^{(2)}[\phi]\right\}_{r=0} &= \frac{(2l)!}{(2k+2l+2)!}\,\frac{(-1)^{k+l+1}}{C(k+l+1)}\left\{\underline{\partial}^{2k+2l+2}(\underline{x}\ P_{2k+1}(\underline{x})\ \underline{x}\ \phi(\underline{x}))\right\}_{\underline{x}=0}\\
&= \frac{(2l)!}{(2k+2l+2)!}\,\frac{1}{C(k+l+1)}\left\{\Delta_m^{k+l+1}(\underline{x}\ P_{2k+1}(\underline{x})\ \underline{x}\ \phi(\underline{x}))\right\}_{\underline{x}=0}\\
&= \frac{(2l)!}{(2k+2l+2)!}\,\frac{1}{C(k+l+1)}\langle\ \underline{x}\ P_{2k+1}(\underline{x})\ \underline{x}\ \Delta_m^{k+l+1}\ \delta(\underline{x})\ ,\ \phi(\underline{x}))\rangle,
\end{aligned}$$

where the constants $C(l),\ l=0,1,2,\dots$ are given by

$$C(l)=\frac{2^{2l}l!}{(2l)!}\left(\frac{m}{2}+l-1\right)\dots\left(\frac{m}{2}\right).$$

Proof. First consider the case where $p=2k$ and observe that

$$\Sigma^{(0)}[\underline{x}\ P_{2k}(\underline{x})\ \underline{x}\ \phi(\underline{x})]=r^{2k+2}\ \Sigma^{(0)}[\underline{\omega}\ P_{2k}(\underline{\omega})\ \underline{\omega}\ \phi]=r^{2k+2}\ \Sigma_{2k}^{(2)}[\phi],$$

whence

$$\left\{\partial_r^{2k+2+j}\Sigma^{(0)}[\underline{x}\ P_{2k}(\underline{x})\ \underline{x}\ \phi(\underline{x})]\right\}_{r=0}=\frac{(2k+2+j)!}{j!}\left\{\partial_r^{j}\Sigma_{2k}^{(2)}[\phi]\right\}_{r=0}.$$

If $j=2l+1$, then

$$\left\{\partial_r^{2l+1}\ \Sigma_{2k}^{(2)}[\phi]\right\}_{r=0}=0,$$

while for $j=2l$ we get

$$\begin{aligned}
\left\{\partial_r^{2l}\Sigma_{2k}^{(2)}[\phi]\right\}_{r=0} &= \frac{(2l)!}{(2k+2l+2)!}\left\{\partial_r^{2k+2l+2}\Sigma^{(0)}[\underline{x}\ P_{2k}(\underline{x})\ \underline{x}\ \phi(\underline{x})]\right\}_{r=0}\\
&= \frac{(2l)!}{(2k+2l+2)!}\,\frac{(-1)^{k+l+1}}{C(k+l+1)}\left\{\underline{\partial}^{2k+2l+2}(\underline{x}\ P_{2k}(\underline{x}))\ \underline{x}\ \phi(\underline{x})\right\}_{x=0}.
\end{aligned}$$

In the case where $p=2k+1$, start with

$$\Sigma^{(0)}[\underline{x}\ P_{2k+1}(\underline{x})\ \underline{x}\ \phi(\underline{x})]=r^{2k+2}\ \Sigma^{(0)}[r\ \underline{\omega}\ P_{2k+1}(\underline{\omega})\ \underline{\omega}\ \phi(\underline{x})]=r^{2k+2}\ \Sigma_{2k+1}^{(2)}[\phi]$$

to obtain, in a similar way, the desired result.

We expect the newly introduced generalized spherical mean $\Sigma_p^{(2)}$ to be related to the other generalized spherical means $\Sigma_p^{(0)}$, $\Sigma_p^{(1)}$ and $\Sigma_p^{(2)}$, by an action of the Dirac operator. This is indeed the case as shown in the next proposition.

Proposition 4.2. *For a scalar valued test function ϕ we have*

$$\begin{aligned}
r\ \Sigma_p^{(1)}[\underline{\partial}\ \phi] &= (r\ \partial_r+(m+p_e))\,\Sigma_p^{(2)}[\phi]-(m-2)\ \Sigma_p^{0}[\phi],\\
r\ \Sigma_p^{(3)}[\phi\ \underline{\partial}] &= (r\ \partial_r+(m+p_e))\,\Sigma_p^{(2)}[\phi]-(m-2)\ \Sigma_p^{0}[\phi].
\end{aligned}$$

Proof. We only prove the first formula in the case where $p=2k$, the proofs of the other formulae running along similar lines.

We have consecutively

$$
\begin{aligned}
r\,\Sigma_{2k}^{(1)}[\underline{\partial}\;\phi] &= \frac{r}{a_m}\int_{S^{m-1}} \underline{\omega}\;P_{2k}(\underline{\omega})\;\underline{\partial}\phi\;dS(\underline{\omega})\\
&= \frac{1}{a_m}\;r\;\partial_r\int_{S^{m-1}} \underline{\omega}\;P_{2k}(\underline{\omega})\;\underline{\omega}\;\phi\;dS(\underline{\omega}) + \frac{1}{a_m}\int_{S^{m-1}} \underline{\omega}\;P_{2k}(\underline{\omega})\;\partial_{\underline{\omega}}\phi\;dS(\underline{\omega})\\
&= r\;\partial_r\Sigma_{2k}^{(2)}[\phi] + \frac{1}{a_m}\int_{S^{m-1}} (\underline{\omega}\;P_{2k}(\underline{\omega})\;\phi)\,\partial_{\underline{\omega}}\;dS(\underline{\omega})\\
&\quad - \frac{1}{a_m}(m-2)\int_{S^{m-1}} P_{2k}(\underline{\omega})\;\phi\;dS(\underline{\omega})\\
&\quad + \frac{1}{a_m}(2k+1)\int_{S^{m-1}} \underline{\omega}\;P_{2k}(\underline{\omega})\;\underline{\omega}\;\phi\;dS(\underline{\omega}).
\end{aligned}
$$

But

$$
\begin{aligned}
&\frac{1}{a_m}\int_{S^{m-1}} (\underline{\omega}\;P_{2k}(\underline{\omega})\;\phi(\underline{x}))\,\partial_{\underline{\omega}}\;dS(\underline{\omega}) = \Sigma^{(0)}[(\underline{\omega}\;P_{2k}(\underline{\omega})\;\phi)\,\partial_{\underline{\omega}}]\\
&= (m-1)\;\Sigma^{(0)}[\underline{\omega}\;P_{2k}(\underline{\omega})\;\underline{\omega}\;\phi] = (m-1)\;\Sigma_{2k}^{(2)}[\phi],
\end{aligned}
$$

and hence

$$
r\;\Sigma_{2k}^{(1)}[\underline{\partial}\phi] = r\;\partial_r\;\Sigma_{2k}^{(2)}[\phi] + (m+2k)\;\Sigma_{2k}^{(2)}[\phi] - (m-2)\;\Sigma_{2k}^{(0)}[\phi].
$$

§5. The Distributions $W_{\lambda,p}$

The definition of the distributions $W_{\lambda,p},\ \lambda\in\mathbb{C},\ p\in\mathbb{N}$, is similar to that of the distributions $T_{\lambda,p}, U_{\lambda,p}$ and $V_{\lambda,p}$:

$$
\begin{aligned}
\langle\; W_{\lambda,2k},\phi\;\rangle &= a_m\;\langle\; Fp\;r_+^{\mu+2k},\Sigma_{2k}^{(2)}[\phi]\;\rangle,\\
\langle\; W_{\lambda,2k+1},\phi\;\rangle &= a_m\;\langle\; Fp\;r_+^{\mu+2k},\Sigma_{2k+1}^{(2)}[\phi]\;\rangle,
\end{aligned}
$$

where we have put $\mu=\lambda+m-1$. Note that the distributions $W_{\lambda,p}$ are vector valued.

We have three motivations for introducing the generalized spherical mean $\Sigma_p^{(2)}$ and the corresponding distributions $W_{\lambda,p}$. The first motivation is a symmetry and completeness argument. In [2], considering spherical monogenics $P_p(\underline{\omega})$ led to the generalized spherical mean $\Sigma_p^{(0)}$ and the corresponding distributions $T_{\lambda,p}$. Multiplication of P_p with the higher dimensional "signum distribution" $\underline{\omega}$ at the left hand side gave rise to the generalized spherical mean $\Sigma_p^{(1)}$ and the distributions $U_{\lambda,p}$, whereas multiplication at the right hand side led to the generalized spherical mean $\Sigma_p^{(3)}$ and the distributions $V_{\lambda,p}$ (see Section 3). Now multiply $\underline{\omega}\;P_p(\underline{\omega})$ by $\underline{\omega}$ at the right, or multiply $P_p(\underline{\omega})\;\underline{\omega}$ by $\underline{\omega}$ at the left to obtain the "symmetric form" $\underline{\omega}\;P_p(\underline{\omega})\;\underline{\omega}$, which is at the basis of the definition of $\Sigma_p^{(2)}$ and $W_{\lambda,p}$. As $\underline{\omega}^2=-1$ no other combinations are possible so that the list of generalized spherical means and distributions of the type under consideration is complete now.

The second motivation points at the action of the Dirac operator. In [2] the action of $\underline{\partial}$ from the left on $U_{\lambda,p}$ considered as a left distribution and from the right on $V_{\lambda,p}$ considered as a right distribution, yielded, in the general case and up to constants, the distribution $T_{\lambda-1,p}$ (see Proposition 3.1). The following propositions in this section will show that the action of the Dirac operator from the right on $U_{\lambda,p}$ and from the left on $V_{\lambda,p}$ automatically generates, next to the distributions $T_{\lambda-1,p}$, the new distributions $W_{\lambda-1,p}$.

The third motivation for introducing the distributions $W_{\lambda,p}$ is fully explained in the next section 6.

Proposition 5.1. *For $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$ such that $\lambda + m - 1 + p_e \neq 0, -1, -2, -3, \cdots$ we have*

$$U_{\lambda,p}\ \underline{\partial} = \underline{\partial}\ V_{\lambda,p} = (\lambda-1)\ W_{\lambda-1,p} + (m-2)\ T_{\lambda-1,p}.$$

Proof. In the case where $p = 2k$ we have for a scalar valued test function ϕ:

$$\begin{aligned}
\langle\ U_{\lambda,2k}\ \underline{\partial}\ ,\ \phi\ \rangle &= -\ \langle\ U_{\lambda,2k}\ ,\ \underline{\partial}\phi\rangle \\
&= -\ a_m\langle\ Fp\ r_+^{\mu+2k-1}\ ,\ r\ \Sigma_{2k}^{(1)}[\underline{\partial}\ \phi]\rangle \\
&= -\ a_m\langle\ Fp\ r_+^{\mu+2k-1}\ ,\ (r\partial_r + (m+2k))\ \Sigma_{2k}^{(2)}[\phi] - (m-2)\ \Sigma_{2k}^{(0)}[\phi]\rangle \\
&= a_m\langle\ \partial_r\ Fp\ r_+^{\mu+2k}\ ,\ \Sigma_{2k}^{(2)}[\phi]\ \rangle - (m+2k)a_m\langle\ Fp\ r_+^{\mu+2k-1}\ ,\ \Sigma_{2k}^{(2)}[\phi]\rangle \\
&\quad + (m-2)a_m\langle\ Fp\ r_+^{\mu+2k-1}\ ,\ \Sigma_{2k}^{(0)}[\phi]\rangle,
\end{aligned}$$

whence

$$U_{\lambda,2k}\ \underline{\partial} = (\lambda-1)\ W_{\lambda-1,2k} + (m-2)\ T_{\lambda-1,2k}.$$

In the case where $p = 2k+1$ we get

$$\begin{aligned}
\langle\ U_{\lambda,2k+1}\ \underline{\partial}\ ,\ \phi\ \rangle &= -\ \langle\ U_{\lambda,2k+1}\ ,\ \underline{\partial}\phi\rangle \\
&= -\ a_m\langle\ Fp\ r_+^{\mu+2k-1}\ ,\ r\ \Sigma_{2k+1}^{(1)}[\underline{\partial}\ \phi]\rangle \\
&= -\ a_m\langle\ Fp\ r_+^{\mu+2k-1}\ ,\ (r\partial_r + (m+2k))\ \Sigma_{2k+1}^{(2)}[\phi] - (m-2)\ \Sigma_{2k+1}^{(0)}[\phi]\rangle \\
&= a_m\langle\ \partial_r\ Fp\ r_+^{\mu+2k}\ ,\ \Sigma_{2k+1}^{(2)}[\phi]\ \rangle - (m+2k)a_m\langle\ Fp\ r_+^{\mu+2k-1}\ ,\ \Sigma_{2k+1}^{(2)}[\phi]\rangle \\
&\quad + (m-2)a_m\langle\ Fp\ r_+^{\mu+2k-1}\ ,\ \Sigma_{2k+1}^{(0)}[\phi]\rangle,
\end{aligned}$$

whence

$$U_{\lambda,2k+1}\ \underline{\partial} = (\lambda-1)\ W_{\lambda-1,2k+1} + (m-2)\ T_{\lambda-1,2k+1}.$$

The calculations for the distribution $V_{\lambda,p}$ are completely similar.

Example 5.1. We verify the above formula in the specific case where $\lambda = -2k$, $p = 2k$ and thus $\lambda + m - 1 + p_e = m - 1$, for which

$$\begin{aligned}
T_{-2k-1,2k} &= \frac{1}{r}\ P_{2k}(\underline{\omega}), \\
W_{-2k-1,2k} &= \frac{1}{r}\ \underline{\omega}\ P_{2k}(\underline{\omega})\ \underline{\omega}, \\
U_{-2k,2k} &= \underline{\omega}\ P_{2k}(\underline{\omega}).
\end{aligned}$$

A direct calculation yields

$$U_{-2k,2k}\ \underline{\partial} = (m-2)\ \frac{1}{r}\ P_{2k}(\underline{\omega}) - (2k+1)\ \frac{1}{r}\ \underline{\omega}\ P_{2k}(\underline{\omega})\ \underline{\omega},$$

while

$$(\lambda-1)\ W_{\lambda-1,2k} + (m-2)\ T_{\lambda-1,2k} = -\ (2k+1)\ \frac{1}{r}\ \underline{\omega}\ P_{2k}(\underline{\omega})\ \underline{\omega} + (m-2)\ \frac{1}{r}\ P_{2k}(\underline{\omega}).$$

A similar verification goes through in the case where $\lambda = -2k-1$ and $p = 2k+1$.

Proposition 5.2. *For $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$ such that $\lambda + m - 1 + p_e = 0$, we have*

$$\begin{aligned} U_{-m-2k+1,2k}\ \underline{\partial} &= \underline{\partial}\ V_{-m-2k+1,2k} \\ &= (m-2)\ T_{-m-2k,2k} - (m+2k)\ W_{-m-2k,2k} \\ &\quad +\ a_m\ \frac{1}{(2k+2)!}\ \frac{1}{C(k+1)}\ \underline{x}\ P_{2k}(\underline{x})\ \underline{x}\ \Delta^{k+1}\ \delta(\underline{x}), \\ U_{-m-2k+1,2k+1}\ \underline{\partial} &= \underline{\partial}\ V_{-m-2k+1,2k+1} \\ &= (m-2)\ T_{-m-2k,2k+1} - (m+2k)\ W_{-m-2k,2k+1} \\ &\quad +\ a_m\ \frac{1}{(2k+2)!}\ \frac{1}{C(k+1)}\ \underline{x}\ P_{2k+1}(\underline{x})\ \underline{x}\ \Delta^{k+1}\ \delta(\underline{x}). \end{aligned}$$

Proof. In the case where $p = 2k$ we have

$$\begin{aligned} \langle\ U_{-m-2k+1,2k}\ \underline{\partial}\ ,\ \phi\ \rangle &= -\ \langle\ U_{-m-2k+1,2k}\ ,\ \underline{\partial}\ \phi\ \rangle \\ &= -\ a_m\ \langle\ Fp\ r_+^{-1}\ ,\ r\ \Sigma_{2k}^{(1)}\ [\underline{\partial}\ \phi]\ \rangle \\ &= -\ a_m\ \langle\ Fp\ r_+^{-1}\ ,\ (r\ \partial_r + (m+2k))\ \Sigma_{2k}^{(2)}\ [\phi] - (m-2)\ \Sigma_{2k}^{(0)}\ [\phi]\ \rangle \\ &= a_m\ \langle\ \delta(r)\ ,\ \Sigma_{2k}^{(2)}\ [\phi]\ \rangle - (m+2k)\ a_m\ \langle\ Fp\ r_+^{-1}\ ,\ \Sigma_{2k}^{(2)}\ [\phi]\ \rangle \\ &\quad +\ (m-2)\ a_m\ \langle\ Fp\ r_+^{-1}\ ,\ \Sigma_{2k}^{(0)}\ [\phi]\ \rangle \\ &= a_m\ \left\{\Sigma_{2k}^{(2)}\ [\phi]\right\}_{r=0} - (m+2k)\ \langle\ W_{-m-2k,2k}\ ,\ \phi\ \rangle \\ &\quad +\ (m-2)\ \langle\ T_{-m-2k,2k}\ ,\ \phi\ \rangle, \end{aligned}$$

from which the formula follows in view of the result of Proposition 4.1 for $l = 0$.

In the case where $p = 2k+1$ we have

$$\begin{aligned} &\langle\ U_{-m-2k+1,2k+1}\ \underline{\partial}\ ,\ \phi\ \rangle \\ &= -\ \langle\ U_{-m-2k+1,2k+1}\ ,\ \underline{\partial}\ \phi\ \rangle \\ &= -\ a_m\ \langle\ Fp\ r_+^{-1}\ ,\ r\ \Sigma_{2k+1}^{(1)}\ [\underline{\partial}\ \phi]\ \rangle \\ &= -\ a_m\ \langle\ Fp\ r_+^{-1}\ ,\ (r\ \partial_r + (m+2k))\ \Sigma_{2k+1}^{(2)}\ [\phi] - (m-2)\ \Sigma_{2k+1}^{(0)}\ [\phi]\ \rangle \\ &= a_m\ \langle\ \delta(r)\ ,\ \Sigma_{2k+1}^{(2)}\ [\phi]\ \rangle - (m+2k)\ a_m\ \langle\ Fp\ r_+^{-1}\ ,\ \Sigma_{2k+1}^{(2)}\ [\phi]\ \rangle \\ &\quad +\ (m-2)\ a_m\ \langle\ Fp\ r_+^{-1}\ ,\ \Sigma_{2k+1}^{(0)}\ [\phi]\ \rangle \\ &= a_m\ \left\{\Sigma_{2k+1}^{(2)}\ [\phi]\right\}_{r=0} - (m+2k)\ \langle\ W_{-m-2k,2k+1}\ ,\ \phi\ \rangle \\ &\quad +\ (m-2)\ \langle\ T_{-m-2k,2k+1}\ ,\ \phi\ \rangle, \end{aligned}$$

from which the formula follows in view of the result of Proposition 4.1 for $l = 0$.

The formulae for the distributions $V_{\lambda,p}$ are proved in a completely similar manner.

Proposition 5.3. *For $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$ such that $\lambda + m - 1 + p_e = -s, s = 1, 2, 3, \cdots$, we have*

(i) *for $l = 1, 2, 3, \cdots$,*

$$\begin{aligned} U_{-m-2k-2l+1,2k}\ \underline{\partial} &= \underline{\partial}\ V_{-m-2k-2l+1,2k} \\ &= (m-2)\ T_{-m-2k-2l,2k} - (m+2k+2l)\ W_{-m-2k-2l,2k} \\ &\quad +\ a_m\ \frac{1}{(2k+2l+2)!}\ \frac{1}{C(k+l+1)}\ \underline{x}\ P_{2k}(\underline{x})\ \underline{x}\ \Delta^{k+l+1}\ \delta(\underline{x}), \end{aligned}$$

(ii) *for* $l = 0, 1, 2, \cdots$,

$$U_{-m-2k-2l,2k}\ \underline{\partial} = \underline{\partial}\ V_{-m-2k-2l,2k}$$
$$= (m-2)\ T_{-m-2k-2l-1,2k} - (m+2k+2l+1)\ W_{-m-2k-2l-1,2k},$$

(iii) *for* $l = 1, 2, 3, \cdots$,

$$U_{-m-2k-2l+1,2k+1}\ \underline{\partial} = \underline{\partial}\ V_{-m-2k-2l+1,2k+1}$$
$$= (m-2)\ T_{-m-2k-2l,2k+1} - (m+2k+2l)\ W_{-m-2k-2l,2k+1}$$
$$+\ a_m\ \frac{1}{(2k+2l+2)!}\ \frac{1}{C(k+l+1)}\ \underline{x}\ P_{2k+1}(\underline{x})\ \underline{x}\ \Delta^{k+l+1}\ \delta(\underline{x}),$$

(iv) *for* $l = 0, 1, 2, \cdots$,

$$U_{-m-2k-2l,2k+1}\ \underline{\partial} = \underline{\partial}\ V_{-m-2k-2l,2k+1}$$
$$= (m-2)\ T_{-m-2k-2l-1,2k+1} - (m+2k+2l+1)\ W_{-m-2k-2l-1,2k+1}.$$

Proof. We only prove the formulae for the distributions $U_{\lambda,p}$ in the case where $p = 2k$, the proofs of the other formulae running along similar lines.

We have consecutively

$$\langle\ U_{-m-2k+1-s,2k}\ \underline{\partial}\ ,\ \phi\ \rangle$$
$$= -\ \langle\ U_{-m-2k+1-s,2k}\ ,\ \underline{\partial}\ \phi\ \rangle$$
$$= -\ a_m\ \langle\ Fp\ r_+^{-s-1}\ ,\ r\ \Sigma_{2k}^{(1)}\ [\underline{\partial}\ \phi]\ \rangle$$
$$= -\ a_m\ \langle\ Fp\ r_+^{-s-1}\ ,\ (r\ \partial_r + (m+2k))\ \Sigma_{2k}^{(2)}\ [\phi] - (m-2)\ \Sigma_{2k}^{(0)}\ [\phi]\ \rangle$$
$$= a_m\ \langle\ -s\ Fp\ r_+^{-s-1} + (-1)^s\ \frac{1}{s!}\ \delta^{(s)}(r)\ ,\ \Sigma_{2k}^{(2)}\ [\phi]\ \rangle$$
$$-\ (m+2k)\ a_m\ \langle\ Fp\ r_+^{-s-1},\ \Sigma_{2k}^{(2)}\ [\phi]\ \rangle +\ (m-2)\ a_m\ \langle\ Fp\ r_+^{-s-1}\ ,\ \Sigma_{2k}^{(2)}\ [\phi]\ \rangle$$
$$= -\ (s+m+2k)\ a_m\ \langle\ Fp\ r_+^{-s-1}\ ,\ \Sigma_{2k}^{(2)}\ [\phi]\ \rangle\ +\ a_m\ (-1)^s\ \frac{1}{s!}\langle\ \delta^{(s)}\ ,\ \Sigma_{2k}^{(2)}\ \rangle$$
$$+\ a_m\ (m-2)\ \langle\ Fp\ r_+^{-s-1}\ ,\ \Sigma_2 k^{(0)}\ \rangle,$$

from which the formulae (i) and (ii) follow in view of the results of Proposition 4.1.

§6. Convolution Operators

Let $k(\underline{\omega})$, $\underline{\omega} \in S^{m-1}$ be either of the following functions: $P_p(\underline{\omega})$ $(p = 1, 2, \dots)$; $\underline{\omega}\ P_p(\underline{\omega})$ and $P_p(\underline{\omega})\ \underline{\omega}$ $(p = 0, 1, 2, \cdots)$; $\underline{\omega}\ P_p(\underline{\omega})\ \underline{\omega}$ $(p = 1, 2, \cdots)$. Then $k \in L_2(S^{m-1})$ and

$$\int_{S^{m-1}}\ k(\underline{\omega})\ dS(\underline{\omega}) = 0.$$

As a consequence, the distributions

$$T_{-m-p,p} = Fp\ \frac{1}{r}\ P_p(\underline{\omega}) = Pv\ \frac{P_p(\underline{\omega})}{r^m},$$
$$U_{-m-p,p} = Fp\ \frac{1}{r}\ \underline{\omega}\ P_p(\underline{\omega}) = Pv\ \frac{\underline{\omega}\ P_p(\underline{\omega})}{r^m},$$
$$V_{-m-p,p} = Fp\ \frac{1}{r}\ P_p(\underline{\omega})\ \underline{\omega} = Pv\ \frac{P_p(\underline{\omega})\ \underline{\omega}}{r^m},$$
$$W_{-m-p,p} = Fp\ \frac{1}{r}\ \underline{\omega}\ P_p(\underline{\omega})\ \underline{\omega} = Pv\ \frac{\underline{\omega}\ P_p(\underline{\omega})\ \underline{\omega}}{r^m}$$

are so-called principal value distributions (see [7, 9]) leading to convolution operators of the form

$$K * \phi = \lim_{\varepsilon \underset{>}{\to} 0} \int_{\mathbb{R}^m \setminus B(0,\varepsilon)} K(\underline{y} - \underline{x})\ \phi(\underline{x})\ dV(\underline{x}), \quad \phi \in \mathcal{S}(\mathbb{R}^m),$$

where $K(x) = \frac{k(\underline{\omega})}{r^m}$ (see [9, Theorem VI.3.1]).

Note that for $p = 0$ and $P_0(\underline{\omega}) = 1$ the distributions $U_{-m,0} = V_{-m,0} = Pv\ \frac{\underline{\omega}}{r^m}$ are nothing else but the higher dimensional "Principal Value"-distribution studied by Delanghe in [3].

In [2] we showed that

$$U_{-m-p,p} + V_{-m-p,p} = Pv\ \frac{\underline{\omega}\ P_p(\underline{\omega})}{r^m} + Pv\ \frac{P_p(\underline{\omega})\ \underline{\omega}}{r^m} = -2(p+1)\ Pv\ \frac{S_{p+1}(\underline{\omega})}{r^m},$$

where $S_{p+1}(\underline{\omega})$ is the spherical harmonic for which

$$\underline{\omega}\ P_p(\underline{\omega}) + P_p(\underline{\omega})\ \underline{\omega} = -\ 2(p+1)\ S_{p+1}(\underline{\omega})$$

(see Section 2). This means that the $\mathbb{R} \oplus \mathbb{R}^{(2)}_{0,m}$-valued distributions $U_{-m-p,p}$ and $V_{-m-p,p}$ are building blocks for the scalar valued principal value distribution $Pv\ \frac{1}{r^m}\ S_{p+1}(\underline{\omega})$ considered in [7] and [9].

As

$$S_{p+1}(\underline{\omega}) = R_{p+1}(\underline{\omega}) - \frac{1}{m+2p}\ \underline{\omega}\ P_p(\underline{\omega})$$

(see Section 2) we could also have decomposed $Pv\ \frac{1}{r^m}\ S_{p+1}(\underline{\omega})$ using the distributions $T_{-m-p-1,p+1}$ and $U_{-m-p,p}$ but then vector valued and $\mathbb{R} \oplus \mathbb{R}^{(2)}_{0,m}$-valued distributions would get mixed up, different spherical monogenics would be involved and the symmetry in the indices would be broken down.

The above symmetrical decomposition inspires the definition of more general, scalar valued, distributions $T^{(h)}_{\lambda,p}$ for $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$:

$$T^{(h)}_{\lambda,p} = -\ \frac{1}{2p}\ (U_{\lambda+1,p-1} + V_{\lambda+1,p-1}) = Fp\ r^{\lambda+p}\ S_p(\underline{\omega}),$$

which in terms of a scalar valued test function ϕ respectively read:

$$\langle\ T^{(h)}_{\lambda,2k}\ ,\ \phi\ \rangle = -\ \frac{a_m}{4k}\ \langle\ Fp\ r_+^{\lambda+m+2k-2}\ ,\ \Sigma^{(1)}_{2k-1}[\phi] + \Sigma^{(3)}_{2k-1}[\phi] \rangle,$$

and

$$\langle\ T^{(h)}_{\lambda,2k+1}\ ,\ \phi\ \rangle = -\ \frac{a_m}{2(2k+1)}\ \langle\ Fp\ r_+^{\lambda+m+2k}\ ,\ \Sigma^{(1)}_{2k}[\phi] + \Sigma^{(3)}_{2k}[\phi] \rangle.$$

They include the above scalar valued principal value distribution for $\lambda = -m-p$:

$$T^{(h)}_{-m-p,p} = Pv\ \frac{S_p(\underline{\omega})}{r^m}.$$

Now as

$$-\ 2p\ \underline{\omega}\ S_p(\underline{\omega}) = \underline{\omega}\ P_{p-1}(\underline{\omega})\ \underline{\omega} - P_{p-1}(\underline{\omega}),$$

we obtain that

$$Pv\ \frac{\underline{\omega}\ S_p(\underline{\omega})}{r^m} = -\ \frac{1}{2p}\ \Big(Pv \frac{\underline{\omega} P_{p-1}(\underline{\omega})\ \underline{\omega}}{r^m} - Pv\ \frac{P_{p-1}(\underline{\omega})}{r^m} \Big),$$

or

$$Pv\ \frac{\underline{\omega}\ S_p(\underline{\omega})}{r^m} = -\ \frac{1}{2p}\ \left(W_{-m-p+1,p-1} - T_{-m-p+1,p-1}\right).$$

This means that the distributions $W_{-m-p+1,p-1}$ and $T_{-m-p+1,p-1}$ are building blocks for the vector valued principal value distribution $Pv\ \frac{\underline{\omega}\ S_p(\underline{\omega})}{r^m}$.

Note that as

$$\underline{\omega}\ S_p(\underline{\omega}) = \underline{\omega}\ R_p(\underline{\omega}) + \frac{1}{m+2p-2}\ P_{p-1}(\underline{\omega}),$$

we also could have decomposed $Pv\ \frac{\underline{\omega}\ S_p(\underline{\omega})}{r^m}$ by means of the distributions $T_{-m-p+1,p-1}$ and $U_{-m-p,p}$ however showing again the non-symmetrical and non-elegant features mentioned above.

In its turn the symmetrical decomposition inspires the definition of the more general, vector valued, distributions $U^{(h)}_{\lambda,p}$ for $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}$:

$$U^{(h)}_{\lambda,p} = -\frac{1}{2p}\ (W_{\lambda+1,p-1} - T_{\lambda+1,p-1}) = Fp\ r^{\lambda+p}\ \underline{\omega}\ S_p(\underline{\omega}),$$

which in terms of a scalar valued test function ϕ read

$$\langle\ U^{(h)}_{\lambda,2k}\ ,\ \phi\ \rangle = -\frac{a_m}{4k}\ \langle\ Fp\ r_+^{\lambda+m+2k-2}\ ,\ \Sigma^{(2)}_{2k-1}[\phi] - \Sigma^{(0)}_{2k-1}[\phi]\rangle$$

and

$$\langle\ U^{(h)}_{\lambda,2k+1}\ ,\ \phi\ \rangle = -\frac{a_m}{2(2k+1)}\ \langle\ Fp\ r_+^{\lambda+m+2k}\ ,\ \Sigma^{(2)}_{2k}[\phi] + \Sigma^{(0)}_{2k}[\phi]\rangle.$$

Amongst them is the above vector valued principal value distribution for $\lambda = -m-p$:

$$U^{(h)}_{-m-p,p} = Pv\ \frac{\underline{\omega}\ S_p(\underline{\omega})}{r^m}.$$

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