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INEQUALITY OF KORN'S TYPE ON COMPACT SURFACES WITHOUT BOUNDARY

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Abstract

Inequalities of Korn's type on a surface with boundary have been proved in many papers using different techniques (see e.g. [4], [5] [11]). The author proves here an inequality of Korn's type on a compact surface without boundary. The idea is to use a finite number of maps for defining the surface and the inequality of Korn's type without boundary conditions for every map and to recast these in a general functional analysis setting about quotient spaces.

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§1. Introduction

"Compact surfaces without boundary" were already considered by Şlicaru [sli] in his doctoral dissertation, where he studied the asymptotic behaviour of thin elastic shells with such middle surfaces.

For defining a surface with boundary, one mapping is usually enough. In fact, this is the way we define the surface, as the image of that map. It is easily seen that this is no longer possible in the case of surfaces without boundary. It suffices to consider the case of the sphere: it is well known that a sphere is not homeomorphic with a part of a plane. So the study we want to make here seems to be more complicated. However, this is compensated by the fact that the functional analysis which is behind is simplier than in the case of surfaces with boundary. That is why, when possible, we will recast our problems in a general functional analysis settong about quotient spaces. In this setting, we establish some general theorems which will give in particular the desired inequality of Korn's type.

We start with some definitions and results about surfaces and Sobolev spaces on surfaces. These are needed for justifying the inequality of Korn's type in the case of compact surfaces without boundary.

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§1. Some Elements of Surface Theory

Let there be given a three-dimensional vector space, in which we fix an origin O and a basis; in this way the three-dimensional vector space is identified with $\mathbb{R}^3$.

In this paper we use the classical definition of a regular surface as found, for instance, in [2] or [9].

Definition 1.1. *A connected subset $S \subset \mathbb{R}^3$ is a regular surface of class C^k if, for each point $p \in S$, there exists a neighborhood V of p in $\mathbb{R}^3$ and a map $\theta : U \to V \cap S$ of the open set $U \subset \mathbb{R}^2$ onto $V \cap S \subset \mathbb{R}^3$ such that*

(1) *θ is of class C^k.*

(2) *θ is a homeomorphism (the topology of S is the induced topology of the usual topology on $\mathbb{R}^3$).*

(3) *For each $q \in U$, the differential $d\theta_q : \mathbb{R}^2 \to \mathbb{R}^3$ is one-to-one.*

The third condition is equivalent to the fact that the vectors $\partial_\alpha \theta$ ($\alpha \in \{1, 2\}$) are linearly independent at all points $q \in U$. This means that $\theta : U \to \mathbb{R}^3$ is an immersion.

Another observation is that we can replace the neighborhood V appearing in Definition 1.1 with an open neighborhood $V' \subset \mathbb{R}^3$. Indeed, since V is a neighborhood of p, there exists an open neighborhood $V' \subset V$ of p. Now, since $\theta : U \to V \cap S$ is continuous, $U' := \theta^{-1}(V' \cap S)$ is an open set in U, hence in $\mathbb{R}^2$ (because U is open in $\mathbb{R}^2$). Now we can see that the map $\theta : U' \to V' \cap S$ also satisfies the conditions of Definition 1.1.

Conversely, if we have a collection of maps $(\theta_t)_{t \in A}$ (where A is an arbitrary set of indices) and a family $(\omega_t)_{t \in A}$ of open sets of $\mathbb{R}^2$ such that:

(1) $\theta_t : \omega_t \to \mathbb{R}^3$ is of class C^k,

(2) $\theta_t : \omega_t \to \theta_t(\omega_t)$ is a homeomorphism,

(3) For each $q \in \omega_t$, the differential $d\theta_{t_{|q}}$ is one-to-one,

(4) $\theta_t(\omega_t)$ is open in $\cup_{t \in A} \theta_t(\omega_t)$,

(5) $\cup_{t \in A} \theta_t(\omega_t)$ is connected,

then $S := \displaystyle\bigcup_{t \in A} \theta_t(\omega_t)$ is a regular surface of classe C^k. In particular, this shows that Definition 1.1 is equivalent to the definition of an embedded surface as found in Definition 6.1.1 of [9].

The second viewpoint is the analog in the general case of a surface defined as the image of a single map (obviously, the fourth condition is satisfied in the case of a single map).

Before passing to the case of compact surfaces, we recall some important results about general surfaces:

1. Change of parameters. If $\theta : U \to S$ and $\psi : V \to S$ are two parameterizations of a regular surface of class C^k such that $\theta(U) \cap \theta(V) = W \neq \emptyset$, then the "change of coordinates" $\theta^{-1} \circ \psi : \psi^{-1}(W) \to \theta^{-1}(W)$ is of class C^k. For a proof, see for instance [2], §2.3, Proposition 1.

This shows that a regular surface of class C^k is, in particular, a differentiable manifold of class C^k. So we can use all the results on differentiable manifolds and especially those concerning tensor fields.

2. Using the same technique as in the proof of Proposition 1 of [2], we can show that conditions (1) and (3) (here, we suppose that $k \geq 1$), together with the fact that θ_t is one-to-one, imply that $\theta_t : \omega_t \to \theta_t(\omega_t)$ is a homeomorphism.

In what follows, a regular surface of class C^k will be defined as a connected subset of $\mathbb{R}^3$ that can be writen as $S := \bigcup_{t\in A} \theta_t(\omega_t)$, where

$$
\begin{aligned}
&(1) \quad \omega_t \subset \mathbb{R}^2 \text{ are open sets,}\\
&(2) \quad \theta_t : \omega_t \to \mathbb{R}^3 \text{ is an immersion of class } C^k \text{ over } \omega_t,\\
&(3) \quad \theta_t(\omega_t) \text{ is open in } S.
\end{aligned}
\tag{1.1}
$$

If S is in addition compact, then we can assume that A is a finite set. More specifically, we will use the following property of compact surfaces, which allow to apply on those parts of the surface that are images of a single map the results already known for surfaces defined by a single map.

Theorem 1.1. *Let S be a compact regular surface of class C^k. Then there exists a finite number of maps $(\theta_t, \omega_t)_{t=1}^N$ such that $S = \bigcup_{t=1}^N \theta_t(\omega_t) = \bigcup_{t=1}^N \theta_t(\overline{\omega}_t)$, where*

(1) $\omega_t \subset \mathbb{R}^2$ are open, bounded and connected sets with Lipschitz-continuous boundary,

(2) $\theta_t : \overline{\omega}_t \to \mathbb{R}^3$ are injective, C^k-differentiable immersions,

(3) $\theta_t(\omega_t)$ are open in S.

Here, we consider that a function is of class C^k over an arbitrary nonempty set, if it is the restriction to that set of a function of class C^k on a larger open set.

Proof. Since S is a compact regular surface of class C^k, there exist $(\theta_t, D_t)_{t=1}^N$, where $\theta_t : D_t \to \mathbb{R}^3$, such that $S = \bigcup_{t=1}^N \theta_t(D_t)$, where

(1) $D_t \subset \mathbb{R}^2$ are open sets,

(2) $\theta_t : D_t \to \mathbb{R}^3$ are injective, of class C^k over D_t, and $\partial_\alpha \theta_t$ are linearly independent at all points of D_t,

(3) $\theta_t(D_t)$ are open in S.

Now we will use the following theorem due to Lebesgue: Let K be a compact metric space and let $K = \bigcup_{t=1}^N V_t$, where $V_t \subset K$ are open sets. Then there exists an $\varepsilon > 0$ such that for all $x \in K$, there exists $t_x \in \{1, \dots, N\}$ such that $B(x, \varepsilon) \subset V_{t_x}$ (here, $B(x, \varepsilon)$ denotes the open ball centered at x with radius ε).

We use this theorem with $K = S$ and $V_t = \theta_t(D_t)$. We consider that S is endowed with the metric induced by the Euclidian metric on $\mathbb{R}^3$. With ε given by Lebesgue's theorem, define $V_t' := \{x \in V_t\,;\ d(x, K \setminus V_t) > \frac{\varepsilon}{2}\}$. Obviously, V_t' is open in K. We will show that $K = \cup_{t=1}^N V_t'$.

Let $x \in K$. Then, by Lebesgue's Theorem, the ball $B(x, \varepsilon)$ is included in some V_{t_x}, which implies that $d(x, K \setminus V_{t_x}) \geq \varepsilon > \frac{\varepsilon}{2}$. Consequently, $x \in V_{t_x}'$, so that $x \in \bigcup_{t=1}^N V_t'$.

Since $\overline{V_t'} = \{x \in V_t\,;\ d(x, K \setminus V_t) \geq \frac{\varepsilon}{2}\}$, we have $V_t' \subset \overline{V_t'} \subset V_t$. Define $\omega_t' := \theta_t^{-1}(V_t')$ and note that ω_t' is open. Since θ_t is a homeomorphism (as observed above), we have $V_t' = \theta_t(\omega_t')$ and $\overline{\omega}_t' = \theta_t^{-1}(\overline{V_t'}) \subset \theta_t^{-1}(V_t) = D_t$. We also see that the sets ω_t' are bounded (indeed, θ_t is a homeomorphism and $\overline{V_t'}$ is a compact set, so $\overline{\omega}_t'$ is compact and therefore bounded). Since $\overline{\omega}_t' \subset D_t$, there exists a bounded open set ω_t with Lipschitz-continuous boundary, such that $\overline{\omega}_t' \subset \omega_t \subset \overline{\omega}_t \subset D_t$. So $\theta_t(\overline{\omega}_t) \subset S$ and $\theta_t(\omega_t)$ is open in S, because $\theta_t(\omega_t) \subset \theta_t(D_t)$ and $\theta_t : D_t \to \theta_t(D_t)$ is a homeomorphism.

Now it is clear that θ_t satisfies the regularity conditions on $\overline{\omega}_t$ (since it satisfies these conditions on D_t). Finally, we have

$$S=\bigcup_{t=1}^{N} V_t'=\bigcup_{t=1}^{N}\theta_t(\omega_t')\subset\bigcup_{t=1}^{N}\theta_t(\omega_t)\subset\bigcup_{t=1}^{N}\theta_t(\overline{\omega}_t)\subset S.$$

To conclude the proof, we observe that we can assume that the sets ω_t are connected: otherwise, we take the connected components of ω_t and use the compactness of S to again obtain a finite number of maps.

Before passing to the next section, let us introduce some classical elements of a surface, which will be used in the present paper. Here and in the sequel, Greek indices and exponents (except ε and ν) take their values in the set $\{1,2\}$, Latin indices and exponents (except t) take their values in the set $\{1,2,3\}$, and the summation convention with respect to repeted indices and exponents is used. The Euclidian scalar and vector products are denoted by $\mathbf{a}\cdot\boldsymbol{b}$ and $\mathbf{a}\wedge\boldsymbol{b}$, respectively. We consider a regular surface of class C^k $(k\geq 2)$ denoted by S. Let $p\in S$ be a point of the surface and let (θ,ω) be a local map at p with $\theta(x)=p$. We define the following elements of either p or x by the same symbol (we write this dependence explicitly only for the first definition) in applications, we will consider functions of either p or x, depending on whether we work on the surface or on the set ω defining the surface through the map θ:

$\mathbf{a}_\alpha(p)=\mathbf{a}_\alpha(x):=\partial_\alpha\theta(x)$ are the vectors of the covariant basis of T_pS (the tangent space to S at p) associated with the map θ,

$\mathbf{a}^\alpha$ are the vectors of the contravariant basis of T_pS associated with the map θ, and they are defined by the relations $\mathbf{a}^\alpha\cdot\mathbf{a}_\beta=\delta^\alpha_\beta$, where δ^α_β designates the Kronecker's delta,

$\mathbf{a}_3=\mathbf{a}^3:=\dfrac{\boldsymbol{a}_1\wedge\boldsymbol{a}_2}{|\boldsymbol{a}_1\wedge\boldsymbol{a}_2|}$ is the unit normal vector to S at p,

$a_{\alpha\beta}:=\mathbf{a}_\alpha\cdot\mathbf{a}_\beta$ are the covariant components of the metric tensor,

$a^{\alpha\beta}:=\mathbf{a}^\alpha\cdot\mathbf{a}^\beta$ are the contravariant components of the metric tensor,

$a:=\det(a_{\alpha\beta})$ is the square of the surface element,

$\Gamma^\sigma_{\alpha\beta}:=\mathbf{a}^\sigma\cdot\partial_\beta\mathbf{a}_\alpha=\mathbf{a}^\sigma\cdot\partial_{\alpha\beta}\theta$ are the Christoffel symbols,

$b_{\alpha\beta}:=\mathbf{a}^3\cdot\partial_\beta\mathbf{a}_\alpha=\mathbf{a}^3\cdot\partial_{\alpha\beta}\theta$ are the covariant components of the curvature tensor,

$b^\beta_\alpha:=a^{\beta\sigma}b_{\sigma\alpha}$ are the mixed components of the curvature tensor,

$ds:=\sqrt{a}\,dx$ is the area element on S.

§2. Sobolev Spaces on Surfaces

Let S be a regular surface of class C^k and let $f:S\to\mathbb{R}$ be a real valued function on S. We say that $f\in H^m(S)$, $(m\leq k)$ if, for every $p\in S$ and for a map (θ,ω) such that $p=\theta(x)\in\theta(\omega)$, the derivatives of $f\circ\theta$ of order $\leq m$ are in $L^1_{\rm loc}(\omega)$ and the following expression is finite:

$$\|f\|_{H^m(S)}:=\Big(\int_S f^2\,ds+\sum_{l=1}^m\int_S a^{\alpha_1\beta_1}\dots a^{\alpha_l\beta_l}f_{|\alpha_1\dots\alpha_l}f_{|\beta_1\dots\beta_l}\,ds\Big)^{\frac12},\tag{2.1}$$

where $f_{|\alpha_1\dots\alpha_l}$ are the l-covariant derivatives of f.

It is easily seen that such a property of weak-differentiability does not depend on the map we choose; this comes from the fact that S is a C^k-differentiable manifold (see Section 1).

As regards the expression in (2.1), we know from tensor theory that it is intrinsic, which means that it does not depend on the maps we choose to calculate it.

In the case of a compact surface, we can replace these norms by the following equivalent norms which are simplier: for a fixed collection of maps $(\theta_t, \omega_t)_{t=1}^N$ as those in Theorem 1.2, define

$$\Big(\sum_{t=1}^N \|f\circ\theta_t\|^2_{H^m(\omega_t)}\Big)^{\frac12}.$$

These norms are no more intrinsic, but this is not inconvenient for our analysis, since they are equivalent (for a fixed collection of maps) with the intrinsic norms defined in (2.1).

Let us show this assertion for $m = 1, 2$ (in this paper, we will use only the $H^1(S)$ and $H^2(S)$-norms). First of all, we have

$$\|f\|_{H^1(S)} := \Big(\int_S f^2\,ds + \int_S a^{\alpha\beta} f_{|\alpha} f_{|\beta}\,ds\Big)^{\frac12}, \tag{2.2}$$

$$\|f\|_{H^2(S)} := \Big(\int_S f^2\,ds + \int_S a^{\alpha\beta} f_{|\alpha} f_{|\beta}\,ds + \int_S a^{\alpha\beta}a^{\sigma\tau} f_{|\alpha\sigma} f_{|\beta\tau}\,ds\Big)^{\frac12}, \tag{2.3}$$

where $f_{|\alpha}(p) = f_{|\alpha}(x) := \frac{\partial(f\circ\theta)}{\partial x_\alpha}(x)$ and $f_{|\alpha\beta}(p) = f_{|\alpha\beta}(x) := \frac{\partial^2(f\circ\theta)}{\partial x_\alpha \partial x_\beta}(x) - \Gamma^\sigma_{\alpha\beta}(x) f_{|\sigma}(x)$, with $p = \theta(x)$, are the first and second covariant derivatives of the function f in p.

For the $H^1(S)$-norm, thanks to the positive definitness of $(a^{\alpha\beta})$, we have $\tilde c\sum_\alpha (f_{|\alpha})^2 \le a^{\alpha\beta} f_{|\alpha} f_{|\beta} \le \tilde C \sum_\alpha (f_{|\alpha})^2$ for some positive constants $\tilde c$ and $\tilde C$. We multiply this relation by $\sqrt{a}$, add $(f\circ\theta_t)^2\sqrt{a}$, then integrate over ω_t. Using in addition the fact that a is a stictly positive function on $\bar\omega_t$, we obtain that there exist two constants $c > 0$ and $C > 0$ such that

$$c\|f\circ\theta_t\|^2_{H^1(\omega_t)} \le \int_{\theta_t(\omega_t)} (f^2 + a^{\alpha\beta} f_{|\alpha} f_{|\beta})\,ds \le C\|f\circ\theta_t\|^2_{H^1(\omega_t)}. \tag{2.4}$$

Since the function $f^2 + a^{\alpha\beta} f_{|\alpha} f_{|\beta}$ is positive on S, we have the following inequalities

$$\int_{\theta_t(\omega_t)} (f^2 + a^{\alpha\beta} f_{|\alpha} f_{|\beta})\,ds \le \int_S (f^2 + a^{\alpha\beta} f_{|\alpha} f_{|\beta})\,ds \le \sum_{t=1}^N \int_{\theta_t(\omega_t)} (f^2 + a^{\alpha\beta} f_{|\alpha} f_{|\beta})\,ds.$$

Taking the sum with respect to t in (2.4) and using the last inequalities, we obtain

$$\frac{c}{N}\sum_{t=1}^N \|f\circ\theta_t\|^2_{H^1(\omega_t)} \le \|f\|^2_{H^1(S)} \le C\sum_{t=1}^N \|f\circ\theta_t\|^2_{H^1(\omega_t)}\,,$$

which is the sought equivalence for $m = 1$.

Now, let us prove the equivalence for the $H^2(S)$-norm. It suffices to find two constants $c > 0$ and $C > 0$ such that

$$c\sum_{t=1}^N \|f\circ\theta_t\|^2_{H^2(\omega_t)} \le \int_S a^{\alpha\beta}a^{\sigma\tau} f_{|\alpha\sigma} f_{|\beta\tau}\,ds \le C\sum_{t=1}^N \|f\circ\theta_t\|^2_{H^2(\omega_t)}. \tag{2.5}$$

Using in particular Theorem 3.3-2 of [cia], which states that $(a^{\alpha\beta}a^{\sigma\tau})$ is uniformly positive definite, we obtain on the one hand that

$$\tilde c\sum_{\alpha,\beta} (f_{|\alpha\beta})^2 \le a^{\alpha\beta}a^{\sigma\tau} f_{|\alpha\sigma} f_{|\beta\tau} \le \tilde C\sum_{\alpha,\beta} (f_{|\alpha\beta})^2 \tag{2.6}$$

for some constants $\tilde{c} > 0$ and $\tilde{C} > 0$. On the other hand, since the functions $\Gamma^\sigma_{\alpha\beta}$ are bounded on ω_t, we deduce from the definition of $f_{|\alpha\beta}$ that there exist two constants $c_1 > 0$ and $C_1 > 0$ such that

$$c_1\Big\{\big(\partial_{\alpha\beta}(f\circ\theta_t)\big)^2+\sum_\sigma\big(\partial_\sigma(f\circ\theta_t)\big)^2\Big\}\le (f_{|\alpha\beta})^2\le C_1\Big\{\big(\partial_{\alpha\beta}(f\circ\theta_t)\big)^2+\sum_\sigma\big(\partial_\sigma(f\circ\theta_t)\big)^2\Big\}.$$

We introduce this inequality in (2.6), then multiply the result by $\sqrt{a}$ (which is a strictly positive and bounded function on $\bar{\omega}_t$) and finally integrate on ω_t. This gives

$$\begin{aligned}&c_2\Big(\sum_{\alpha,\beta}\|\partial_{\alpha\beta}(f\circ\theta_t)\|^2_{L^2(\omega_t)}+\sum_\sigma\|\partial_\sigma(f\circ\theta_t)\|^2_{L^2(\omega_t)}\Big)\\&\le\int_{\theta_t(\omega_t)}a^{\alpha\beta}a^{\sigma\tau}f_{|\alpha\sigma}f_{|\beta\tau}\,ds\le C_2\Big(\sum_{\alpha,\beta}\|\partial_{\alpha\beta}(f\circ\theta_t)\|^2_{L^2(\omega_t)}+\sum_\sigma\|\partial_\sigma(f\circ\theta_t)\|^2_{L^2(\omega_t)}\Big),\end{aligned}$$

for some positive constants c_2 and C_2. Since the function $a^{\alpha\beta}a^{\sigma\tau}f_{|\alpha\sigma}f_{|\beta\tau}$ is positive on S, we can use the same method than that used for the $H^1(S)$-norm in order to obtain (2.5).

We say that a spatial vector field $\boldsymbol{\eta}$ (which means that with each $p\in S$, we associate a vector $\boldsymbol{\eta}(p)$ in space, not necessarily in the tangent space T_pS) is in the space $\boldsymbol{H}^m(S)$ if all its components in a fixed basis of $\mathbb{R}^3$ belong to $H^m(S)$.

Now, let us consider the tangential and the normal components of $\boldsymbol{\eta}$. More specifically, let $\boldsymbol{\eta}=\boldsymbol{\eta}_\tau+\boldsymbol{\eta}_\nu$, where $\boldsymbol{\eta}_\tau(p)\in T_pS$ and $\boldsymbol{\eta}_\nu(p)$ is parallel to the normal to S at the point p. We say that $\boldsymbol{\eta}\in\boldsymbol{H}^1_\tau(S)\oplus\boldsymbol{H}^2_\nu(S)$ if $\boldsymbol{\eta}_\tau\in\boldsymbol{H}^1(S)$ and $\boldsymbol{\eta}_\nu\in\boldsymbol{H}^2(S)$.

§3. Koiter's Model for a Linearly Elastic Shell

Throughout this paragraph, S is a compact regular surface of class C^3. Throughout the sequel, the points $p\in S$ and $x\in\omega$ (or ω_t) are related by the relation $p=\theta(x)$. Our aim is to establish an inequality, thereafter called inequality of Korn's type, which eventually will imply the existence (and uniqueness) of a solution to Koiter's model for a linearly elastic shell with a compact regular middle surface. To establish such an inequality, we make an analogy with the case of one-mapping surfaces (surfaces which are parameterized by a single map) and we retain from this case only the intrinsic quantities. To begin with, let us define the two-dimensional Koiter equations for a linearly elastic shell. We consider a shell with middle surface S and thickness 2ε, subjected to applied body forces. In Koiter's model, the unknown is the displacement field $\boldsymbol{\zeta}^\varepsilon_K : S\to\mathbb{R}^3$ of the middle surface of the shell. In the case where the surface is defined by a single map (θ,ω) satisfying properties (1) and (2) (with $k=3$) of Theorem 1.2, the problem under consideration is the following:

$$\left\{\begin{aligned}&\text{Find } A\vec{\zeta}^\varepsilon_K\in\boldsymbol{V}_K(\omega)\text{ such that}\\&\int_\omega\Big\{\varepsilon a^{\alpha\beta\sigma\tau,\varepsilon}\gamma_{\sigma\tau}(A\vec{\zeta}^\varepsilon_K)\gamma_{\alpha\beta}(A\vec{\eta})+\frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau,\varepsilon}\rho_{\sigma\tau}(A\vec{\zeta}^\epsilon_K)\rho_{\alpha\beta}(A\vec{\eta})\Big\}\sqrt{a}\,dx\\&=\int_\omega\tilde{\boldsymbol{f}}^\varepsilon\cdot A\vec{\eta}\sqrt{a}\,dx\text{ for all }A\vec{\eta}=(\tilde{\eta}_i)\in\boldsymbol{V}_K(\omega),\end{aligned}\right.\tag{3.1}$$

where $V_K(\omega)$ is a closed subspace ofof the space $H^1(\omega)\times H^1(\omega)\times H^2(\omega)$ considered with the usual norm, $\tilde{\boldsymbol{f}}^\varepsilon=(\tilde{f}^{\varepsilon,i})\in L^2(\omega)\times L^2(\omega)\times L^2(\omega)$ (where $\boldsymbol{f}^\varepsilon=\tilde{f}^{\varepsilon,i}ba_i$ account for the applied body forces) and

$$
\begin{aligned}
a^{\alpha\beta\sigma\tau,\varepsilon}(p) &= a^{\alpha\beta\sigma\tau,\varepsilon}(x) := \frac{4\lambda^\varepsilon\mu^\varepsilon}{\lambda^\varepsilon+2\mu^\varepsilon}a^{\alpha\beta}a^{\sigma\tau}+2\mu^\varepsilon(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}),\\
\gamma_{\alpha\beta}(\boldsymbol{\eta})(p) &= \gamma_{\alpha\beta}(A\vec{\eta})(x) := \frac{1}{2}(\partial_\beta\eta cdot\mathbf{a}_\alpha+\partial_\alpha\eta cdot\mathbf{a}_\beta)\\
&= \frac{1}{2}(\partial_\beta\tilde{\eta}_\alpha+\partial_\alpha\tilde{\eta}_\beta)-\Gamma^\sigma_{\alpha\beta}\tilde{\eta}_\sigma-b_{\alpha\beta}\tilde{\eta}_3,\\
\rho_{\alpha\beta}(\boldsymbol{\eta})(p) &= \rho_{\alpha\beta}(A\vec{\eta})(x) := (\partial_{\alpha\beta}\boldsymbol{\eta}-\Gamma^\sigma_{\alpha\beta}\partial_\sigma\boldsymbol{\eta})\cdot\mathbf{a}_3\\
&= \partial_{\alpha\beta}\tilde{\eta}_3-\Gamma^\sigma_{\alpha\beta}\partial_\sigma\tilde{\eta}_3-b^\sigma_\alpha b_{\sigma\beta}\tilde{\eta}_3\\
&\quad + b^\sigma_\alpha(\partial_\beta\tilde{\eta}_\sigma-\Gamma^\tau_{\beta\sigma}\partial_\sigma\tilde{\eta}_\tau)+b^\tau_\beta(\partial_\alpha\tilde{\eta}_\tau-\Gamma^\sigma_{\alpha\tau}\partial_\sigma\tilde{\eta}_\sigma)\\
&\quad + (\partial_\alpha b^\tau_\beta+\Gamma^\tau_{\alpha\sigma}b^\sigma_\beta-\Gamma^\sigma_{\alpha\beta}b^\tau_\sigma)\tilde{\eta}_\tau,
\end{aligned}
$$

where $\lambda^\varepsilon>0$ and $\mu^\varepsilon>0$ are the Lamé constants of the elastic material constituting the shell, and $\boldsymbol{\eta}(p)=\tilde{\eta}_i(x)\mathbf{a}^i(p)$. Then, in Koiter's model, the displacement field of the middle surface of the shell is given by $\boldsymbol{\zeta}^\varepsilon_K=\zeta^\varepsilon_{K,i}\mathbf{a}^i$. Throughout this section, we denote by the same symbol a function of p (defined on the surface) or of x (provided that $\theta(x)=p$). We also make the convention that the two functions are equal. Recall that we have already used this convention in Section 2. For further details about Koiter's model, see [4].

Let us notice that matrices $\big(\gamma_{\alpha\beta}(\boldsymbol{\eta})(p)\big)$ and $\big(\rho_{\alpha\beta}(\boldsymbol{\eta})(p)\big)$ are symmetric and that the functions $\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})$ and $\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})$ belong to $L^2(\omega)$, since $\tilde{\boldsymbol{\eta}}\in H^1(\omega)\times H^1(\omega)\times H^2(\omega)$. As long as we are only interested in the existence and uniqueness of a solution to problem (3.1), the coefficients ε and $\frac{\varepsilon^3}{3}$ are not really relevant, so we make the convention that $\varepsilon=1$ and $\frac{\varepsilon^3}{3}=1$ in (3.1). Accordingly, the expression appearing in (3.1) becomes:

$$A(\boldsymbol{\zeta},\boldsymbol{\eta}) := a^{\alpha\beta\sigma\tau}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta})+a^{\alpha\beta\sigma\tau}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta}),$$

where $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$ are two spatial vector fields. We already know that $(a^{\alpha\beta\sigma\tau})$ are the contravariant components of a tensor field of rank 4 (the two-dimensional elasticity tensor), that $\big(\gamma_{\alpha\beta}(\boldsymbol{\eta})\big)$ are the covariant components of a tensor field of rank 2 (the linearized change of metric tensor), and that $\big(\rho_{\alpha\beta}(\boldsymbol{\zeta})(\rho_{\sigma\tau}(\boldsymbol{\eta})\big)$ are the covariant components of a tensor field of rank 4. Finally, by inner multiplication, we see that the expression $A(\boldsymbol{\zeta},\boldsymbol{\eta})$ is a tensor field of rank 0, i. e., a function. This means that this expression does not depend on the choice of maps, but only on the vector fields $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$ defining the tensor fields $\big(\gamma_{\alpha\beta}(\boldsymbol{\zeta})\big)$, $\big(\gamma_{\alpha\beta}(\boldsymbol{\eta})\big)$ and $\big(\rho_{\alpha\beta}(\boldsymbol{\zeta})(\rho_{\sigma\tau}(\boldsymbol{\eta})\big)$.

As we shall see later in this paper, the fact that $A\vec{\eta}\in H^1(\omega)\times H^1(\omega)\times H^2(\omega)$ is equivalent with $\eta\in\boldsymbol{H}^1_\tau(S)\oplus\boldsymbol{H}^2_\nu(S)$. Now, problem (3.1) (with the simplifying convention that ε have been replaced by one and $\frac{\varepsilon^3}{3}$ have been replaced by one) takes the folowing intrinsic form:

$$
\begin{aligned}
&\boldsymbol{\zeta}\in\boldsymbol{H}^1_\tau(S)\oplus\boldsymbol{H}^2_\nu(S),\\
&\int_S A(\boldsymbol{\zeta},\boldsymbol{\eta})\,ds=\int_S \boldsymbol{f}\cdot\eta,ds \text{ for all } \eta\in\boldsymbol{H}^1_\tau(S)\oplus\boldsymbol{H}^2_\nu(S).
\end{aligned}
\tag{3.2}
$$

Note that this form of Koiter's model can be transposed verbatim in the case where the surface S is compact.

Let us consider the bilinear form

$$B(\boldsymbol{\zeta},\boldsymbol{\eta}) := \int_S A(\boldsymbol{\zeta},\boldsymbol{\eta})\,ds \tag{3.3}$$

defined over the space $\boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$, and the linear form

$$L(\boldsymbol{\eta}) := \int_S \boldsymbol{f} \cdot \eta, ds$$

defined over the same space. Then we rewrite problem (3.2) in a functional form, that is:

$$\begin{gathered}\text{Find } \boldsymbol{\zeta} \in E := \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S) \text{ such that} \\ B(\boldsymbol{\zeta}, \boldsymbol{\eta}) = L(\boldsymbol{\eta}) \text{ for all } \eta \in E.\end{gathered} \tag{3.4}$$

We are interested in proving the existence and uniqueness of the solution to this problem. To this end, we will use the Lax-Milgram theorem; naturally, proving the ellipticity of the bilinear form is the only difficulty. The object of the following section is to establish an inequality which allow to prove the ellipticity of the bilinear form B.

§4. Korn Inequality for Compact Surfaces

Let us begin by making some observations on the bilinear form B. We can verify that $\eta \mapsto \sqrt{B(\boldsymbol{\eta}, \boldsymbol{\eta})}$ is a seminorm on E. This comes from the fact that B is a symmetric bilinear form which satisfies $B(\boldsymbol{\eta}, \boldsymbol{\eta}) \geq 0$ for all $\boldsymbol{\eta} \in E$. To prove this last property of B, we use the fact that $a^{\alpha\beta\sigma\tau} t_{\sigma\tau} t_{\alpha\beta} \geq c \sum\limits_{\alpha,\beta} |t_{\alpha\beta}|^2$ for all symmetric matrices $(t_{\alpha\beta})$ (for a proof, see [4, Theorem 3.3-2]). We shall see later that $\eta \mapsto \sqrt{B(\boldsymbol{\eta}, \boldsymbol{\eta})}$ is not a norm on E.

We consider the following framework, which is well suited to our problem: let $(E, \|\cdot\|)$ be a Banach space, let $L \in E'$ be a linear form over E, and let B be a symmetric bilinear form over E which satisfies $B(x, x) \geq 0$ for all $x \in E$. Consider the problem

$$\text{Find } x \in E \text{ such that } B(x, y) = L(y) \text{ for all } y \in E. \tag{4.1}$$

Define the seminorm $x \mapsto |x| := \sqrt{B(x, x)}$ and the set $F := \{x \in E\,;\, |x| = 0\}$. We can easily verify the following properties:

1. F is a vector space.

2. We must have $L(y) = 0$ for all $y \in F$ if we wish that problem (4.1) would have solutions. Indeed, since $B(x, x) \geq 0$ for all $x \in E$, we have the Cauchy-Schwarz inequality: $|B(x, y)| \leq \sqrt{B(x, x)}\sqrt{B(y, y)} = |x||y|$. So, if $x \in E$ and $y \in F$, we have $|B(x, y)| \leq |x||y| = 0$. In other words,

$$B(x, y) = 0 \text{ for all } x \in E,\ y \in F. \tag{4.2}$$

Therefore, if x is a solution of (4.1) and $y \in F$, then $B(x, y) = 0 = L(y)$.

Remark 4.1. Property (4.2) says that $F = \ker B$, where

$$\ker B := \{y \in E\,;\ B(x, y) = 0 \text{ for all } x \in E\}.$$

3. If x is a solution of (4.1) and $\tilde{x} \in F$, then $x + \tilde{x}$ is also a solution of (4.1).

So, if we wish that problem (4.1) be well posed (in the sense that it has one and only one solution), we have to impose the condition

$$L_{|F} = 0 \tag{4.3}$$

and try to solve the problem over the quotient space E/F, not over E. Note that L is well defined on E/F, thanks to the compatibility condition (4.3). The same remark holds for B, thanks to relation (4.2). Now, the problem we want to solve reads:

$$\text{Find } \hat{x} \in E/F \ \text{ such that } \ B(\hat{x}, \hat{y}) = L(\hat{y}) \ \text{ for all } \ \hat{y} \in E/F. \tag{4.4}$$

The following abstract result gives the ellipticity of B on E/F under some additional assumptions. This is equivalent to saying that the seminorm induced by B is a norm equivalent to the norm of E/F. Applying the following theorem to our case gives us the Korn inequality for compact surfaces. Note that some ideas of the proof are close to those used by Duvaut and Lions in [7, Chapter 3], where they have studied the three-dimensional elasticity problem without boundary conditions.

Theorem 4.1. *Let $(E, \|\cdot\|)$ be a Banach space, let $|\cdot|$ be a seminorm on E and let $(\tilde{E}, \|\cdot\|_0)$ be a larger normed space ($E \subset \tilde{E}$) such that*

(i) *There exists $c > 0$ such that $|x| \le c\|x\|$ for all $x \in E$,*

(ii) *The inclusion $(E, \|\cdot\|) \hookrightarrow (\tilde{E}, \|\cdot\|_0)$ is compact,*

(iii) *There exists $c_0 > 0$ such that $\|x\| \le c_0(\|x\|_0 + |x|)$ for all $x \in E$.*

Then there exists $C > 0$ such that $\|\hat{x}\|_{E/F} \le C|\hat{x}|_{E/F}$ for all $\hat{x} \in E/F$, where $F := \{x \in E\,;\, |x| = 0\}$, $\|\hat{x}\|_{E/F} := \inf\{\|x\|\,;\, x \in \hat{x}\}$ and $|\hat{x}|_{E/F} := \inf\{|x|\,;\, x \in \hat{x}\}$.

In the case of problem (3.4), (i) means the continuity of the quadratic form $\eta \mapsto B(\boldsymbol{\eta}, \boldsymbol{\eta})$ and will be given by the continuity of the bilinear form B, (ii) will be the compact inclusion of $\boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$ in $\boldsymbol{L}^2_\tau(S) \oplus \boldsymbol{H}^1_\nu(S)$ and (iii) will be an inequality of Korn's type without boundary conditions.

Proof. We argue by contradiction. If the announced inequality is false, then there exists a sequence $(\hat{x}_n)_{n\in\mathbb{N}}$ in E such that $\|\hat{x}_n\|_{E/F} = 1$ and $|\hat{x}_n|_{E/F} \to 0$ when $n \to +\infty$.

For each $\hat{x}_n$ we choose a representative x_n such that $\|x_n\| \le 2$. The inclusion $(E, \|\cdot\|) \hookrightarrow (\tilde{E}, \|\cdot\|_0)$ is compact, so there exists a subsequence (also denoted by (x_n) for simplicity of notations) such that (x_n) converges in the norm $\|\cdot\|_0$ to some element of $\tilde{E}$. In particular, (x_n) is a Cauchy sequence with respect to the norm $\|\cdot\|_0$. By using (iii), we obtain

$$\|x_n - x_m\| \le c_0(\|x_n - x_m\|_0 + |x_n - x_m|) \le c_0(\|x_n - x_m\|_0 + |x_n| + |x_m|).$$

Using the fact that $|x_n| = |\hat{x}_n|_{E/F}$ for all representative x_n of $\hat{x}_n$ (indeed, if x_n and x'_n are two representatives of $\hat{x}_n$, then $x'_n - x_n \in F$ and $|x'_n| = |x_n + x'_n - x_n| \le |x_n| + |x'_n - x_n| = |x_n|$; in the same way we get $|x_n| \le |x'_n|$, so $|x'_n| = |x_n|$) and the fact that $|\hat{x}_n|_{E/F} \to 0$, we obtain that (x_n) is a Cauchy sequence with respect to the norm $\|\cdot\|$.

Since $(E, \|\cdot\|)$ is a Banach space, there exists $x \in E$ such that $\lim_{n\to\infty} \|x_n - x\| = 0$. But $\|\hat{x}_n - \hat{x}\|_{E/F} = \|\widehat{x_n - x}\| \le \|x_n - x\|$, so $\lim_{n\to\infty} \|\hat{x}_n - \hat{x}\|_{E/F} = 0$. Consequently, $\|\hat{x}_n\|_{E/F} \to \|\hat{x}\|_{E/F}$, so we have $\|\hat{x}\|_{E/F} = 1$.

On the other hand, we have

$$\big||\hat{x}_n|_{E/F} - |\hat{x}|_{E/F}\big| \le |\hat{x}_n - \hat{x}|_{E/F} = |x_n - x| \le c\|x_n - x\| \overset{n\to\infty}{\longrightarrow} 0. \tag{4.5}$$

But $|\cdot|_{E/F}$ is a norm on E/F. Indeed, if $|\hat{x}|_{E/F} = 0$, then there exists a sequence $(x + y_n)$ of representatives of $\hat{x}$, $x + y_n$, (where x is a fixed representative of $\hat{x}$ and $y_n \in F$) such that $|x + y_n| < \frac{1}{n}$, for all $n \in \mathbb{N}^*$. We have $|x| \le |x + y_n| + |y_n| = |x + y_n| < \frac{1}{n}$ for all $n \in \mathbb{N}^*$. So $|x| = 0$, which implies that $x \in F$ and finally $\hat{x} = \hat{0}$.

Now, using (4.5) and the fact that $|\hat{x}_n|_{E/F} \overset{n\to\infty}{\longrightarrow} 0$, we obtain $|\hat{x}|_{E/F} = 0$, so $\hat{x} = \hat{0}$. But this is in contradiction with the fact that $\|\hat{x}\|_{E/F} = 1$ and the proof is complete.

Now we can solve problem (4.4) by the following corollary:

Corollary 4.1. *Let $(E, \|\cdot\|)$ be a Banach space and let $B : E \times E \to \mathbb{R}$ be a symmetric continuous bilinear form (there exists $c > 0$ such that $|B(x, y)| \le c\|x\|\|y\|$ for all $x, y \in E$)*

which satisfies $B(x,x) \geq 0$ *for all* $x \in E$. *Let* $F := \{x \in E\,;\, B(x,x) = 0\}$ *and let* $L \in E'$ *be a linear form on* E *which satisfies* $L_{|F} = 0$. *Assume that there exists a larger space* $(\tilde{E}, \|\cdot\|_0)$ *such that*

(i) *The inclusion of* $(E, \|\cdot\|)$ *in* $(\tilde{E}, \|\cdot\|_0)$ *is compact,*

(ii) *There exists* $c_0 > 0$ *such that* $\|x\| \leq c_0\big(\|x\|_0 + \sqrt{B(x,x)}\big)$ *for all* $x \in E$.

Then there exists one and only one solution of the variational problem

$$\text{Find } \hat{x} \in E/F \text{ such that } B(\hat{x}, \hat{y}) = L(\hat{y}) \text{ for all } \hat{y} \in E/F.$$

Proof. We consider the seminorm $x \in E \mapsto |x| := \sqrt{B(x,x)}$ and the induced norm on E/F. We have seen in the proof of Theorem 4.1 that $|\cdot|_{E/F}$ is a norm on E/F.

The continuity of B implies that $B(x,x) \leq c\|x\|^2$ for all $x \in E$, so

$$|x| \leq \sqrt{c}\|x\| \text{ for all } x \in E. \tag{4.6}$$

Taking the *inf* with respect to $x \in \hat{x}$ in (4.6), we obtain that $|\hat{x}|_{E/F} \leq \sqrt{c}\|\hat{x}\|_{E/F}$ for all $\hat{x} \in E/F$. In addition, inequality (4.6) shows that assumption (i) of Theorem 4.1 is satisfied. The other two hypotheses are given in the statement of the corollary, so we can apply Theorem 4.1. Consequently, there exists $C > 0$ such that $\|\hat{x}\|_{E/F} \leq C|\hat{x}|_{E/F}$ for all $\hat{x} \in E/F$. Therefore, the norms $\|\cdot\|_{E/F}$ and $|\cdot|_{E/F}$ are equivalent.

We also see that $L \in (E/F, \|\cdot\|_{E/F})'$. Indeed, if $\|\hat{y}_n - \hat{y}\|_{E/F} \to 0$, then we can choose y_n and y as representatives for $\hat{y}_n$, respectively $\hat{y}$, such that $\|y_n - y\| \to 0$ (using the same technique as in Theorem 4.1). Since L is a linear form over E, we have $L(y_n) \to L(y)$. Since $L(y_n) = L(\hat{y}_n)$ and $L(y) = L(\hat{y})$, we obtain $L(\hat{y}_n) \to L(\hat{y})$.

We know from the general theory that $(E/F, \|\cdot\|_{E/F})$ is a Banach space. So, since the norms $\|\cdot\|_{E/F}$ and $|\cdot|_{E/F}$ are equivalent, we have that $(E/F, |\cdot|_{E/F})$ is a Banach space too. Moreover, the last one is a Hilbert space, because $(\hat{x}, \hat{y}) \mapsto B(\hat{x}, \hat{y})$ is a scalar product. From the equivalence of the norms, we also deduce the eqality $(E/F, |\cdot|_{E/F})' = (E/F, \|\cdot\|_{E/F})'$. So $L \in (E/F, |\cdot|_{E/F})'$. Now, we can conclude by applying Riesz's Theorem to the Hilbert space $\big(E/F, B(\cdot,\cdot)\big)$ and to the linear form L.

Let us come back to our particular case, where

$$\begin{aligned}
&E = \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S), \text{ which is a Banach space,}\\
&\tilde{E} = \boldsymbol{L}^2_\tau(S) \oplus \boldsymbol{H}^1_\nu(S),\\
&B : E \times E \to \mathbb{R}\ ,\ B(\boldsymbol{\zeta}, \boldsymbol{\eta}) = \int_S A(\boldsymbol{\zeta}, \boldsymbol{\eta})\, ds,\\
&L : E \to \mathbb{R}\ ,\ L(\boldsymbol{\eta}) = \int_S \boldsymbol{f} \cdot \boldsymbol{\eta}, ds,
\end{aligned}$$

where $\boldsymbol{f}$ is a vector field in $\boldsymbol{L}^2(S)$ and where

$$A(\boldsymbol{\zeta}, \boldsymbol{\eta})(p) := a^{\alpha\beta\sigma\tau}(p)\gamma_{\sigma\tau}(\boldsymbol{\zeta})(p)\gamma_{\alpha\beta}(\boldsymbol{\eta})(p) + a^{\alpha\beta\sigma\tau}(p)\rho_{\sigma\tau}(\boldsymbol{\zeta})(p)\rho_{\alpha\beta}(\boldsymbol{\eta})(p),$$

the expression in the right hand side being taken for a map (θ, ω) such that $p \in \theta(\omega)$. We have already seen that the value of this expression does not depend on the chosen map.

It is a classical result that $E \hookrightarrow \tilde{E}$ is a compact inclusion and it is not difficult to verify that L is a linear form over E and that B is a symmetric continuous bilinear form over the same space. We know from [4, Theorem 3.3-2] that $A(\boldsymbol{\eta}, \boldsymbol{\eta})(p) \geq 0$ on S, because $\big(\gamma_{\alpha\beta}(\boldsymbol{\eta})(p)\big)$ and $\big(\rho_{\alpha\beta}(\boldsymbol{\eta})(p)\big)$ are symmetric matrices. So $B(\boldsymbol{\eta}, \boldsymbol{\eta}) \geq 0$ for all $\eta \in E$.

In order to apply Theorem 4.1, it remains to verify hypothesis (*iii*), which can be viewed in this setting as a "weak" inequality of Korn's type in the entire space E. To this end, we shall use the inequality of Korn's type without boundary conditions for one-mapping surfaces that is proved in [4].

Since S is a compact regular surface of class C^3, we can apply Theorem 1.2 and find N maps (θ_t, ω_t) of class C^3, satisfying conditions $(1)-(3)$ of this theorem such that $S = \bigcup_{t=1}^{N} \theta_t(\omega_t)$. We apply the inequality of Korn's type without boundary conditions for each map. Accordingly, if $\boldsymbol{\eta}(p) = \tilde{\eta}_i(x)\boldsymbol{a}^i(p)$, we have for all $t \in \{1, \ldots, N\}$ that

$$\begin{aligned}&\sum_\alpha \|\tilde{\eta}_\alpha\|^2_{H^1(\omega_t)} + \|\tilde{\eta}_3\|^2_{H^2(\omega_t)}\\ &\le c_t\Big\{\sum_\alpha \|\tilde{\eta}_\alpha\|^2_{L^2(\omega_t)} + \|\tilde{\eta}_3\|^2_{H^1(\omega_t)} + \sum_{\alpha,\beta}\|\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\|^2_{L^2(\omega_t)} + \sum_{\alpha,\beta}\|\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})\|^2_{L^2(\omega_t)}\Big\}\\ &\qquad \text{for all } \ \tilde{\boldsymbol{\eta}} = (\tilde{\eta}_i) \in H^1(\omega_t) \times H^1(\omega_t) \times H^2(\omega_t).\end{aligned}$$

Here, we have applied Theorem 2.6-1 of [4]. The hypotheses of this theorem are satisfied, since $\eta \in \boldsymbol{H}^m_\tau(S) \oplus \boldsymbol{H}^n_\nu(S)$ (with $m, n \le 2$) implies that

$$\eta \in \boldsymbol{H}^m_\tau(\theta_t(\omega_t)) \oplus \boldsymbol{H}^n_\nu(\theta_t(\omega_t)) \quad \text{for all} \ \ t,$$

which is equivalent to $\tilde{\eta}_\alpha \in H^m(\omega_t)$, $\tilde{\eta}_3 \in H^n(\omega_t)$. Moreover, the norms $\|\boldsymbol{\eta}_\tau\|_{\boldsymbol{H}^m(\theta_t(\omega_t))}$ and $\|\boldsymbol{\eta}_\nu\|_{\boldsymbol{H}^n(\theta_t(\omega_t))}$ are equivalent to the norms $\|(\tilde{\eta}_\alpha)\|_{H^m(\omega_t)\times H^m(\omega_t)}$ and respectively $\|\tilde{\eta}_3\|_{H^n(\omega_t)}$.

Indeed, we have $\tilde{\eta}_i(x) = (\eta \cdot \boldsymbol{a}^i)(p) = (\boldsymbol{\eta}_\tau + \boldsymbol{\eta}_\nu)(p) \cdot \boldsymbol{a}^i(p)$, so $\tilde{\eta}_\alpha(x) = (\boldsymbol{\eta}_\tau \cdot \boldsymbol{a}^\alpha)(p)$ and $\tilde{\eta}_3(x) = (\boldsymbol{\eta}_\nu \cdot \boldsymbol{a}^3)(p)$. Conversely, we have $\boldsymbol{\eta}_\tau(p) = (\tilde{\eta}_\alpha \mathbf{a}^\alpha)(x)$ and $\boldsymbol{\eta}_\nu(p) = (\tilde{\eta}_3\mathbf{a}^3)(x)$. The desired equivalences come from the fact that $\boldsymbol{a}_i$ and $\boldsymbol{a}^i$ are $\boldsymbol{C}^2(\overline{\omega}_t)$-vector fields.

By using Theorem 3.3-2 of [4], we deduce the existence of a constant $\tilde{c}_t > 0$ such that

$$A(\boldsymbol{\eta}, \boldsymbol{\eta})(p) \ge \tilde{c}_t\Big(\sum_{\alpha,\beta}|\gamma_{\alpha\beta}(\tilde{\boldsymbol{\eta}})(x)|^2 + \sum_{\alpha,\beta}|\rho_{\alpha\beta}(\tilde{\boldsymbol{\eta}})(x)|^2\Big). \tag{4.8}$$

Therefore, we infer from (4.7) that

$$\begin{aligned}&\|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{H}^1(\theta_t(\omega_t))} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^2(\theta_t(\omega_t))}\\ &\le C_t\Big\{\|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{L}^2(\theta_t(\omega_t))} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^1(\theta_t(\omega_t))} + \int_{\theta_t(\omega_t)} A(\boldsymbol{\eta}, \boldsymbol{\eta})\, ds\Big\}\\ &\qquad \text{for all } \eta \in \boldsymbol{H}^1_\tau(\theta_t(\omega_t)) \oplus \boldsymbol{H}^2_\nu(\theta_t(\omega_t)).\end{aligned}$$

Since $A(\boldsymbol{\eta}, \boldsymbol{\eta}) \ge 0$ over S, we also have

$$\|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{H}^1(\theta_t(\omega_t))} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^2(\theta_t(\omega_t))} \le C_t\Big\{\|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{L}^2(\theta_t(\omega_t))} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^1(\theta_t(\omega_t))} + \int_S A(\boldsymbol{\eta}, \boldsymbol{\eta})\, ds\Big\}$$

for all $\eta \in \boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$. Taking the sum with respect to t, we obtain

$$\|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{H}^1(S)} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^2(S)} \le c\Big\{\|\boldsymbol{\eta}_\tau\|^2_{\boldsymbol{L}^2(S)} + \|\boldsymbol{\eta}_\nu\|^2_{\boldsymbol{H}^1(S)} + N\int_S A(\boldsymbol{\eta}, \boldsymbol{\eta})\, ds\Big\}.$$

Consequently, there exists a constant $c_0 > 0$ such that

$$\|\eta|_E \le c_0(\|\eta|_{\tilde{E}} + \sqrt{B(\boldsymbol{\eta}, \boldsymbol{\eta})})$$

for all $\eta \in E$, which is exactly hypothesis (iii) of Theorem 4.1, where we have denoted $|\boldsymbol{\eta}| := \sqrt{B(\boldsymbol{\eta}, \boldsymbol{\eta})}$.

For simplicity, let us denote $E/\ker B$ by $\dot{\boldsymbol{V}}(S)$. We recall that the space F that appears in Theorem 4.1 is in fact $\ker B$ (see Remark 4.1). Applying Theorem 4.1 to our case, we obtain the following theorem which gives the desired inequality of Korn's type on compact surfaces without boundary:

Theorem 4.2. *Let S be a regular compact surface of class C^3. Then there exists a constant $c > 0$ such that*

$$\|\hat{\boldsymbol{\eta}}\|_{\dot{\boldsymbol{V}}(S)} \le c\Big(\int_S A(\boldsymbol{\eta},\boldsymbol{\eta})\,ds\Big)^{\frac{1}{2}} \text{ for all } \hat{\eta} \in \dot{\boldsymbol{V}}(S),$$

where $\boldsymbol{\eta}$ is an arbitrary representative of $\hat{\boldsymbol{\eta}}$.

Of course, the same result still holds true if one replaces the bilinear form A with the following bilinear form

$$A^\varepsilon(\boldsymbol{\zeta},\boldsymbol{\eta}) := \varepsilon a^{\alpha\beta\sigma\tau,\varepsilon}\gamma_{\sigma\tau}(\boldsymbol{\zeta})\gamma_{\alpha\beta}(\boldsymbol{\eta}) + \frac{\varepsilon^3}{3}a^{\alpha\beta\sigma\tau,\varepsilon}\rho_{\sigma\tau}(\boldsymbol{\zeta})\rho_{\alpha\beta}(\boldsymbol{\eta}).$$

Indeed, with the notation $B^\varepsilon(\boldsymbol{\zeta},\boldsymbol{\eta}) := \int_S A^\varepsilon(\boldsymbol{\zeta},\boldsymbol{\eta})\,ds$, it is obvious that A^ε, respectively B^ε, satisfies the properties of A that we have used in our analysis respectively B. Moreover, we have $\ker B^\varepsilon = \ker B$.

The compatibility condition on L^ε (where $L^\varepsilon := \int_S \boldsymbol{f}^\varepsilon \cdot \eta, ds)$ becomes in our particular case $\int_S \boldsymbol{f}^\varepsilon \cdot \eta, ds = 0$ for all $\eta \in \ker B$. We recall that, if $\hat{\boldsymbol{\zeta}}, \hat{\eta} \in \dot{\boldsymbol{V}}(s)$ and $(\boldsymbol{\zeta},\boldsymbol{\zeta}')$, $(\boldsymbol{\eta},\boldsymbol{\eta}')$ are two pairs of representatives for $\hat{\boldsymbol{\zeta}}$ and respectively $\hat{\boldsymbol{\eta}}$, then $B^\varepsilon(\boldsymbol{\zeta}',\boldsymbol{\eta}') = B^\varepsilon(\boldsymbol{\zeta},\boldsymbol{\eta})$ (thanks to (4.2)) and $L(\boldsymbol{\eta}') = L(\boldsymbol{\eta})$ (thanks to the compatibility condition). By applying Corollary 4.1 to the spaces E, $\tilde{E}$, to the bilinear form B^ε and to the linear form L^ε appearing in our particular case, we establish the existence of a solution to the Koiter's model (4.9) for a linearly elastic shell with a compact middle surface (note that problem (4.9) is the analogue of the problem (3.1) in the case of compact regular surfaces):

Theorem 4.3. *Let S be a regular compact surface of class C^3 and let $\boldsymbol{f}^\varepsilon \in \boldsymbol{L}^2(S)$ be a vector field on S such that $\int_S \boldsymbol{f}^\varepsilon \cdot \eta, ds = 0$ for all $\eta \in \ker B^\varepsilon$. Then there exists one and only one solution to the problem*

$$\begin{cases} \text{Find } \hat{\boldsymbol{\zeta}}^\varepsilon \in \dot{\boldsymbol{V}}(S) \text{ such that} \\ \displaystyle\int_S A^\varepsilon({}^\varepsilon\boldsymbol{\zeta},\boldsymbol{\eta})\,ds = \int_S \boldsymbol{f}^\varepsilon \cdot \eta, ds \text{ for all } \hat{\eta} \in \dot{\boldsymbol{V}}(S), \end{cases} \tag{4.9}$$

where ${}^\varepsilon\boldsymbol{\zeta}$, $\boldsymbol{\eta}$ are arbitrary representatives of $\widehat{{}^\varepsilon\boldsymbol{\zeta}}$ and respectively $\hat{\boldsymbol{\eta}}$.

Now, we would like to better describe the space $\dot{\boldsymbol{V}}(S)$. For exemple, the first natural question is to find out if $\dot{\boldsymbol{V}}(S)$ is a proper quotient space of $\boldsymbol{H}^1_\tau(S) \oplus \boldsymbol{H}^2_\nu(S)$, i. e., to find out if $\ker B = \{\boldsymbol{0}\}$ or not. The answer is given by the following theorem, which describes the space $\ker B$ in the case of general regular surfaces (defined by (1.1)).

Theorem 4.4 (Infinitesimal Rigid Displacement Lemma on a General Regular Surface). *Let S be a regular surface of class C^3 and let $\boldsymbol{\eta}$ be a vector field in $\ker B$. Then there exists two vectors $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^3$ such that*

$$\boldsymbol{\eta}(p) = \mathbf{c} + \mathbf{d} \wedge \mathbf{p}, \text{ for all } p \in S,$$

where $\mathbf{p} := \overrightarrow{Op}$ is the position vector of p.

Proof. Let $p_0 \in S$ and let (θ,ω) be a local map at p_0 (i.e., $p_0 \in \theta(\omega) \subset S$) such that $\omega \subset \mathbb{R}^2$ is connected, $\theta : \omega \to \mathbb{R}^3$ is an injective application of class C^3, $\theta(\omega)$ is open in S, and the vectors $(\partial_\alpha\theta)$ are linearly independent in all points of ω. Notice that such a map exists, by the definition of a regular surface of class C^3 (see Section 1).

Since $\eta \in \ker B$, we have in particular $B(\boldsymbol{\eta},\boldsymbol{\eta}) = 0$, which is equivalent to $A(\boldsymbol{\eta},\boldsymbol{\eta}) = 0$. Consider the surface $S' := \theta(\omega)$ and notice that $A(\boldsymbol{\eta}_{|S'},\boldsymbol{\eta}_{|S'}) = A(\boldsymbol{\eta},\boldsymbol{\eta})_{|S'} = 0$ on S'. Therefore, it follows from (4.8) that $\gamma_{\alpha\beta}(\boldsymbol{\eta})$ and $\rho_{\alpha\beta}(\boldsymbol{\eta})$ vanish on S'. Applying Theorem 2.6-3 of [4] to the surface S' and to the vector field $\boldsymbol{\eta}_{S'}$ thus gives the existence of two vectors $\mathbf{c}(\theta)$ and $\mathbf{d}(\theta)$ such that $\boldsymbol{\eta}(p) = \mathbf{c}(\theta) + \mathbf{d}(\theta) \wedge \theta(x)$ for all $x \in \omega$, or equivalently, that

$$\boldsymbol{\eta}(p) = \mathbf{c}(\theta) + \mathbf{d}(\theta) \wedge \mathbf{p} \quad \text{for all} \quad p \in S'.$$

So "locally", the theorem is true. To prove the global result, it suffices to show that $\mathbf{c}(\theta)$ and $\mathbf{d}(\theta)$, which apparently depend on the local map, are in fact constants over the entire surface S. To this end, the decisive argument is the connectedness of S.

Since S is a regular surface of class C^3, there exists a collection of maps $(\theta_t,\omega_t)_{t\in A}$ such that ω_t are connected, $S = \bigcup_{t\in A} \theta_t(\omega_t) = \bigcup_{t\in A} S_t$, and θ_t, ω_t satisfy conditions (1)–(3) of (1.1). Notice that $S_t := \theta_t(\omega_t)$ are also regular surfaces of class C^3.

We have seen in the first part of the proof that, for all $t \in A$, there exist $\mathbf{c}_t, \mathbf{d}_t \in \mathbb{R}^3$ such that $\boldsymbol{\eta}(p) = \mathbf{c}_t + \mathbf{d}_t \wedge \mathbf{p}$, for all $p \in S_t$. Now, fix $t_0 \in A$ and define the sets

$$A_0 := \{t \in A\,;\, \mathbf{c}_t = \mathbf{c}_{t_0} \text{ and } \mathbf{d}_t = \mathbf{d}_{t_0}\} \text{ and } A_1 = A \setminus A_0.$$

Then $S_0 := \cup_{t\in A_0} S_t$ and $S_1 := \cup_{t\in A_1} S_t$ are open sets in S, since each S_t is open in S. Obviously, $S = S_0 \cup S_1$ and $t_0 \in A_0$, so $S_{t_0} \subset S_0$, which proves that $S_0 \neq \emptyset$.

Now, let us prove that $S_1 = \emptyset$. We argue by contradiction. Suppose that $S_1 \neq \emptyset$, which is equivalent to $A_1 \neq \emptyset$. Then $S_0 \cap S_1 \neq \emptyset$, because S is connected. So there exists $t_1 \in A_1$ such that $S_{t_1} \cap S_0 \neq \emptyset$. For all $p \in S_{t_1} \cap S_0$, we have

$$\boldsymbol{\eta}(p) = \mathbf{c}_{t_1} + \mathbf{d}_{t_1} \wedge \mathbf{p} = \mathbf{c}_{t_0} + \mathbf{d}_{t_0} \wedge \mathbf{p}. \tag{4.10}$$

Since $S_{t_1} \cap S_0$ is open in S, there exist three non-colinear points $p, q, r \in S_{t_1} \cap S_0$ (i.e. $\mathbf{p} - \mathbf{q} \neq \lambda(\mathbf{p} - \mathbf{r})$ for all $\lambda \in \mathbb{R}$). Otherwise, for any local map that describes a portion of $S_{t_1} \cap S_0$, the vectors $\mathbf{a}_\alpha$ cannot be linearly independent. We write (4.10) for $\mathbf{p}$, $\boldsymbol{q}$ and $\boldsymbol{r}$:

$$\begin{aligned}
\mathbf{c}_{t_1} + \mathbf{d}_{t_1} \wedge \mathbf{p} &= \mathbf{c}_{t_0} + \mathbf{d}_{t_0} \wedge \mathbf{p},\\
\mathbf{c}_{t_1} + \mathbf{d}_{t_1} \wedge \boldsymbol{q} &= \mathbf{c}_{t_0} + \mathbf{d}_{t_0} \wedge \boldsymbol{q},\\
\mathbf{c}_{t_1} + \mathbf{d}_{t_1} \wedge \boldsymbol{r} &= \mathbf{c}_{t_0} + \mathbf{d}_{t_0} \wedge \boldsymbol{r}.
\end{aligned}$$

Substracting the second equation from the first one, we obtain $\mathbf{d}_{t_1} \wedge (\mathbf{p} - \boldsymbol{q}) = \mathbf{d}_{t_0} \wedge (\mathbf{p} - \boldsymbol{q})$, so that $(\mathbf{d}_{t_1} - \mathbf{d}_{t_0}) \wedge (\mathbf{p} - \boldsymbol{q}) = 0$. Therefore $\mathbf{d}_{t_1} - \mathbf{d}_{t_0}$ is colinear with $\mathbf{p} - \boldsymbol{q}$. We make the same operations with the first and the last equation and we obtain that $\mathbf{d}_{t_1} - \mathbf{d}_{t_0}$ is also colinear with $\mathbf{p} - \boldsymbol{r}$. That is, there exist $\lambda, \mu \in \mathbb{R}$ such that

$$\mathbf{d}_{t_1} - \mathbf{d}_{t_0} = \lambda(\mathbf{p} - \mathbf{q}) = \mu(\mathbf{p} - \mathbf{r}).$$

But $\mathbf{p} - \mathbf{q}$ and $\mathbf{p} - \mathbf{r}$ are non-colinear, so we must have $\lambda = \mu = 0$. Consequently, $\mathbf{d}_{t_1} = \mathbf{d}_{t_0}$. Moreover, $\mathbf{c}_{t_1} = \mathbf{c}_{t_0}$, thanks to (4.10). But this proves that $t_1 \in A_0$, which contradicts the fact that $t_1 \in A_1$ (because A_0 and A_1 are disjoint sets, by definition).

So $S_1 = \emptyset$ and $S = S_0$. To conclude the proof, we take $\mathbf{c} = \mathbf{c}_{t_0}$ and $\mathbf{d} = \mathbf{d}_{t_0}$.

Remark 4.2. Using the same arguments as in the previous proof, where we have shown that $S_1 = \emptyset$, we can prove that $\mathbf{c}$ and $\mathbf{d}$ of Theorem 4.3 are unique for a given vector field $\eta \in \ker B$.

Theorem 4.4 shows not only that $\dot{\boldsymbol{V}}(S) \neq \boldsymbol{H}_\tau^1(S) \oplus \boldsymbol{H}_\nu^2(S)$, but more precisely, that $\dot{\boldsymbol{V}}(S)$ is isomorphic with a subspace of $\boldsymbol{H}_\tau^1(S) \oplus \boldsymbol{H}_\nu^2(S)$ of codimension 6.

Remarks 4.3. (1) Theorems 4.3 and 4.4 also holds true for general bounded surfaces with boundary. Indeed, we have used the compactedness of the surface only in the proof of Theorem 1.2, or this theorem also holds true for bounded surfaces with boundary.

(2) Let S be a general bounded surface with boundary and let Γ be a relative open subset of the boundary of S. if we require that the solution of Koiter's problem (3.2) satisfy in addition some boundary conditions on Γ, then the quotient space $\dot{\boldsymbol{V}}(S) = E/\ker B$ appearig in Theorem 4.2 coincides with the entire space E. More specifically, problem (3.2) is posed in this case over the space $E = V_K(S)$, where

$$\boldsymbol{V}_K(S) := \{\eta \in \boldsymbol{H}_\tau^1(S) \oplus \boldsymbol{H}_\nu^2(S)\,;\, \boldsymbol{\eta} = \partial_\nu \boldsymbol{\eta}_\nu = 0 \text{ on } \Gamma\},$$

where the normal derivative of any vector field $\boldsymbol{\xi} = (\xi_i) \in \boldsymbol{H}^2(S)$ is defined by $\partial_\nu \boldsymbol{\xi} := (\nabla \xi_i \cdot \boldsymbol{\nu})$ over the boundary of S. Here $\nabla \xi_i := [\xi_i]_{|\alpha} \boldsymbol{a}^\alpha$, where $[\xi_i]_{|\alpha}$ are the covariant derivatives of the function ξ_i. The equality $E/\ker B = E$ comes from the fact that $\ker B = \{0\}$ over $V_K(S)$. One can prove this by using Theorem 4.4 together with a connectedness argument. Note that in the case of one-mapping surfaces, Theorems 4.2 and 4.3 have already been proved in [4] (see Theorem 2.6-4 and the begining of Chapter 7).

(3) Even in the case of compact surfaces without boundary, we can avoid considering quotient spaces. It suffices to consider the space $\boldsymbol{V}_\perp(S) := \{\eta \in \boldsymbol{H}_\tau^1(S) \oplus \boldsymbol{H}_\nu^2(S)\,;\, \int_S \eta \cdot \boldsymbol{\zeta} \text{ for all } \boldsymbol{\zeta} \in \ker B\}$ instead of the space $\dot{\boldsymbol{V}}(S)$. Note that the space $\boldsymbol{V}_\perp(S)$ is in fact the subspace $(\ker B)^\perp$ of $\boldsymbol{H}_\tau^1(S) \oplus \boldsymbol{H}_\nu^2(S)$, where the orthogonal of $\ker B$ is taken with respect to the scalar product $(\boldsymbol{\eta}, \boldsymbol{\zeta}) \mapsto \int_S \eta \cdot \boldsymbol{\zeta}\, ds$. This idea has already been used by Şlicaru in his doctoral dissertation [10]. However, fixing in this manner a representative of $\hat{\eta} \in \dot{\boldsymbol{V}}(S)$ (since this is what we do eventually) does not correspond to any physical requirement or principle. This is why we have preferred to solve the problem over the quotient space $\dot{\boldsymbol{V}}(S)$.

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