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THE SUM OF DEFICIENCIES OF ENTIRE FUNCTION ON $\mathbf{C}^n$***

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Abstract

This paper proves that: Let f be an entire function of finite order λ on $\mathbf{C}^n$. Then
(1) $\sum_{a\in\mathbf{C}}\delta(a,f)\le 1-k(\lambda)$, where $k(\lambda)$ is a nonnegative constant depending only on λ;
(2) If $\sum_{a\in\mathbf{C}}\delta(a,f)=1$, then λ is a positive integer and equals the lower order of f.

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§1. Introduction

It is well known that the defect relation

$$\sum_a \delta(a,f)\le 2$$

is valid for meromorphic functions on the complex plane, and this relation was generalized successfully to meromorphic maps from $\mathbf{C}^n$ into projective space and more general case. Since the upper bound 2 is not reached by all meromorphic functions on the complex plane, ones start to find the more accurate expression of upper bound of sum of deficiencies, for examples, the works in [1, 6, 9]. In [6] A. Pfluger proved that if f is an entire function of finite order λ on the complex plane, then

(1) $\sum_{a\in\mathbf{C}}\delta(a,f)\le 1-\tilde{k}(\lambda)$, where $\tilde{k}(\lambda)$ is a nonnegative constant depending only on λ;

(2) If $\sum_{a\in\mathbf{C}}\delta(a,f)=1$, then λ is a positive integer.

For meromorphic functions on $\mathbf{C}^n$, few results of this kind are known. In this paper we study this problem and prove the following:

Theorem 1.1. *Let f be a transcendental entire function of finite order λ on $\mathbf{C}^n$, then*

$$\sum_{a\in\mathbf{C}}\delta(a,f)\le 1-k(\lambda),$$

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where $k(\lambda)=\frac{2\Gamma^4(3/4)|\sin\lambda\pi|}{\pi^2\lambda+\Gamma^4(3/4)|\sin\lambda\pi|}$.

Theorem 1.2. *Let f be a transcendental entire function of finite order λ on $\mathbf{C}^n$. If*

$$\sum_{a\in\mathbf{C}}\delta(a,f)=1,$$

then λ is a positive integer and equals the lower order of f.

§2. Notations and Lemmas

For $z=(z_1,\cdots,z_n)\in\mathbf{C}^n$, define $|z|=(|z_1|^2+\cdots+|z_n|^2)^{\frac{1}{2}}$. Denote

$$S_n(r)=\{z\in\mathbf{C}^n;|z|=r\},\qquad \bar{B}_n(r)=\{z\in\mathbf{C}^n;|z|\le r\}.$$

Let $d=\partial+\bar\partial$ and $d^c=(\partial-\bar\partial)/4\pi i$, denote

$$\omega_n(z)=dd^c\log|z|^2,\quad \sigma_n(z)=d^c\log|z|^2\wedge\omega_n^{n-1}(z),\quad \nu_n(z)=dd^c|z|^2.$$

Then $\sigma_n(z)$ defines a positive measure on $S_n(r)$ with total measure one.

Let $a\in\mathbf{P}^1$. If $f^{-1}(a)\ne\mathbf{C}^n$, we denote by Z_a^f the $a-$divisor of f, write $Z_a^f(r)=\bar{B}_n(r)\cap Z_a^f$ and define

$$n_f(r,a)=r^{2-2n}\int_{Z_a^f(r)}\nu_n^{n-1}(z).$$

Then the counting function is defined as

$$N_f(r,a)=\int_0^r[n_f(t,a)-n_f(0,a)]\frac{dt}{t}+n_f(0,a)\log r,$$

where $n_f(0,a)$ is the Lelong number of Z_a^f at the origin.

Let $a\in\mathbf{P}^1$ and $f^{-1}(a)\ne\mathbf{C}^n$. Then we define the proximity function as

$$\begin{aligned}m_f(r,a)&=\int_{S_n(r)}\log^+\frac{1}{|f(z)-a|}\sigma_n(z),\ a\ne\infty;\\&=\int_{S_n(r)}\log^+|f(z)|\sigma_n(z),\qquad a=\infty,\end{aligned}$$

and define the characteristic function as $T_f(r)=m_f(r,\infty)+N_f(r,\infty)$. The first main theorem states that[7, Chapter 4, A5.1]

$$T_f(r)=m_f(r,a)+N_f(r,a)+O(1).$$

Define

$$\delta(a,f)=\liminf_{r\to\infty}\frac{m_f(r,a)}{T_f(r)}=1-\limsup_{r\to\infty}\frac{N_f(r,a)}{T_f(r)}.$$

Denote by f_{z_j} the partial derivative of f with respect to z_j $(j=1,2,\cdots,n)$, then we have

Lemma 2.1.[10, Lemma 6] *Let f be a non-constant meromorphic function on $\mathbf{C}^n$. Then*

$$m_{f_{z_j}/f}(r,\infty)=\int_{S_n(r)}\log^+\left|\frac{f_{z_j}}{f}(z)\right|\sigma_n(z)=O(\log rT_f(r)),\quad j=1,2,\cdots,n$$

hold for all large r outside a set with finite measure. Furthermore, if f is of finite order, then $m_{f_{z_j}/f}(r,\infty)=O(\log r)$, $\quad j=1,2,\cdots,n$ hold for all large r.

We call f to be transcendental if $\lim\limits_{r\to\infty}\dfrac{T_f(r)}{\log r}=\infty$.

Lemma 2.2. *Let f be a transcendental entire function on $\mathbf{C}^n$. Then $D_f(z)=\sum\limits_{j=1}^n z_jf_{z_j}(z)$ is a transcendental entire function on $\mathbf{C}^n$ too, and $m_{D_f/f}(r,\infty)=O(\log rT_f(r))$ holds for* *all large r outside a set with finite measure. Furthermore, if f is of finite order, then $m_{D_f/f}(r,\infty)=O(\log r)$ holds for all large r.*

Proof. Since f is an entire function on $\mathbf{C}^n$, then we can expand it as a convergent series $f(z)=\sum\limits_{m=0}^{\infty}P^m(z)$, where $P^m(z)$ is either identically zero or a homogeneous polynomial of degree m $(m=0,1,2,\cdots)$. Since f is transcendental, there are infinitely many terms of $\{P^m(z)\}$ which are not identically zero. By the homogeneity of $P^m(z)$ we have

$$\sum_{j=1}^{n} z_j P^m_{z_j}(z)=mP^m(z),\qquad m=1,2,\cdots,$$

hence

$$D_f(z)=\sum_{j=1}^{n} z_j f_{z_j}(z)=\sum_{m=0}^{\infty} mP^m(z),$$

and there are infinitely many terms of $\{mP^m(z)\}$ which are not identically zero. So D_f is a transcendental entire function on $\mathbf{C}^n$ too. Since for any rational function $R(z)$, we have $m_R(r,\infty)=O(\log r)$, then

$$\begin{aligned}
m_{D_f/f}(r,\infty)&=\int_{S_n(r)}\log^+\Big|\sum_{j=1}^{n} z_j\frac{f_{z_j}}{f}(z)\Big|\sigma_n(z)\\
&\le\sum_{j=1}^{n}\int_{S_n(r)}\log^+\Big|\frac{f_{z_j}}{f}(z)\Big|\sigma_n(z)+\sum_{j=1}^{n}\int_{S_n(r)}\log^+|z_j|\sigma_n(z)+O(1)\\
&=\sum_{j=1}^{n} m_{f_{z_j}/f}(r,\infty)+\sum_{j=1}^{n} m_{z_j}(r,\infty)+O(1)=\sum_{j=1}^{n} m_{f_{z_j}/f}(r,\infty)+O(\log r).
\end{aligned}$$

Hence from Lemma 2.1 we conclude the proof.

Let f be an entire function on $\mathbf{C}^n$, and set $M(r,f)=\max\limits_{|z|=r}|f(z)|$.

Lemma 2.3.[5, Lemma 1] *Let f be an entire function on $\mathbf{C}^n$. Then for any $0<r<R$,*

$$T_f(r)+O(1)\le\log M(r,f)\le\frac{1-(r/R)^2}{(1-r/R)^{2n}}T_f(R)+O(1).$$

Lemma 2.4. *Let f be a transcendental entire function on $\mathbf{C}^n$. Then $f(z)$ and $D_f(z)=\sum\limits_{j=1}^{n} z_j f_{z_j}(z)$ are of the same order and lower order.*

Proof. Since f is a transcendental entire function on $\mathbf{C}^n$. Then

$$\begin{aligned}
T_{D_f}(r)=m_{D_f}(r,\infty)&=\int_{S_n(r)}\log^+|D_f(z)|\sigma_n(z)\\
&\le\int_{S_n(r)}\log^+\Big|\frac{D_f}{f}(z)\Big|\sigma_n(z)+\int_{S_n(r)}\log^+|f(z)|\sigma_n(z)\\
&=\int_{S_n(r)}\log^+\Big|\frac{D_f}{f}(z)\Big|\sigma_n(z)+m_f(r,\infty)\\
&=\int_{S_n(r)}\log^+\Big|\frac{D_f}{f}(z)\Big|\sigma_n(z)+T_f(r).
\end{aligned}\tag{2.1}$$

Hence from Lemma 2.2 we deduce that

$$T_{D_f}(r)\le O(\log rT_f(r))+T_f(r)\tag{2.2}$$

holds for all large r outside a set with finite measure.

Let z_r be a point on $|z| = r$ such that $|f(z_r)| = \max_{|z|=r} |f(z)| (= M(r,f))$. We write $z_r = r\xi$, $\xi = (\xi_1, \cdots, \xi_n) \in S_n(1)$. Consider the function $\tilde{f}(t) = f(t\xi)$, $t \in \mathbf{C}$, then $\tilde{f}'(t) = \sum_{j=1}^{n} \xi_j f_{z_j}(t\xi)$. Therefore

$$|f(z_r) - f(0)| = |\tilde{f}(r) - \tilde{f}(0)| \le \int_0^r |\tilde{f}'(t)| dt$$
$$= \int_0^r \Big| \sum_{j=1}^{n} \xi_j f_{z_j}(t\xi) \Big| dt = \int_0^{\frac{1}{r}} \Big| \sum_{j=1}^{n} \xi_j f_{z_j}(t\xi) \Big| dt + \int_{\frac{1}{r}}^{r} \Big| \sum_{j=1}^{n} \xi_j f_{z_j}(t\xi) \Big| dt. \tag{2.3}$$

Since $|t\xi| \le 1/r$ when $0 < t < 1/r$, if $r \ge 1$, we have $|f_{z_j}(t\xi)| \le M(1, f_{z_j})$, hence

$$\int_0^{\frac{1}{r}} \Big| \sum_{j=1}^{n} \xi_j f_{z_j}(t\xi) \Big| dt \le \frac{1}{r} \sum_{j=1}^{n} M(1, f_{z_j}).$$

Since $|t\xi| \le r$ when $0 < t < r$,

$$\int_{\frac{1}{r}}^{r} \Big| \sum_{j=1}^{n} \xi_j f_{z_j}(t\xi) \Big| dt = \int_{\frac{1}{r}}^{r} \frac{1}{t} \Big| \sum_{j=1}^{n} t\xi_j f_{z_j}(t\xi) \Big| dt$$
$$\le r \int_{\frac{1}{r}}^{r} \Big| \sum_{j=1}^{n} t\xi_j f_{z_j}(t\xi) \Big| dt \le r \int_{\frac{1}{r}}^{r} M(r, D_f) dt = M(r, D_f)(r^2 - 1).$$

Hence from (2.3) we have

$$\log M(r,f) = \log |f(z_r)|$$
$$\le \log^+ |f(0)| + \log^+ \sum_{j=1}^{n} M(1, f_{z_j}) + \log^+ M(r, D_f) + O(\log r).$$

Therefore from above inequality and taking $R = 2r$ in Lemma 2.3 we deduce that when r is large enough,

$$T_f(r) \le \log M(r,f) + O(1) \le \log M(r, D_f) + O(\log r) \le 3 \cdot 2^{2n-2} T_{D_f}(2r) + O(\log r). \tag{2.4}$$

Since f is a transcendental entire function, from Lemma 2.2 D_f is a transcendental entire function too. By a standard way one can deduce from (2.2) and (2.4) that f and D_f are of same order and lower order.

Let g be a meromorphic map from $\mathbf{C}^n$ into $\mathbf{P}^1$, and (g_0, g_1) be a reduced representation of g. If $|g_0(0)|^2 + |g_1(0)|^2 \ne 0$, define the characteristic function of g as

$$\tilde{T}_g(r) = \int_{S_n(r)} \log(|g_0(z)|^2 + |g_1(z)|^2)^{\frac{1}{2}} \sigma_n(z) - \log(|g_0(0)|^2 + |g_1(0)|^2)^{\frac{1}{2}}.$$

Let $a \in \mathbf{P}^1$, we can write $a = [a_0, a_1]$ ($|a_0|^2 + |a_1|^2 = 1$) by the homogeneous coordinate. Denote by $\tilde{Z}_a^g$ the $a-$divisor of g, i.e., the $0-$divisor of $a_0 g_0 + a_1 g_1$, then by the same way as above we can define the counting function $\tilde{N}_g(r,a)$ with respect to $\tilde{Z}_a^g$ and the deficiency $\tilde{\delta}(a,g)$. Now we can state the results of S.Mori and J.Noguchi as following:

Lemma 2.5.[5, Theorem 1]; [2, Corollary 3] *Let $g : \mathbf{C}^n \to \mathbf{P}^1$ be a non-constant meromorphic map of finite order λ. Then*

(1) *for any $H_1, H_2 \in \mathbf{P}^1$,*

$$\limsup_{r\to\infty} \frac{\tilde{N}_g(r, H_1) + \tilde{N}_g(r, H_2)}{\tilde{T}_g(r)} \ge k(\lambda),$$

where $k(\lambda)$ is defined in Theorem 1.1;

(2) *If $H_1, H_2 \in \mathbf{P}^1$ satisfy $\tilde{\delta}(H_1, g) = \tilde{\delta}(H_2, g) = 1$, then λ is a positive integer and equals the lower order of g.*

Let f be an entire function on $\mathbf{C}^n$, define a holomorphic map $g : \mathbf{C}^n \to \mathbf{P}^1$ as following:

$$g : \ z \to \ [1, f(z)].$$

Take $H_1 = [0, 1], H_2 = [1, 0] \in \mathbf{P}^1$, then $\tilde{Z}^g_{H_2} = 0$, $\tilde{Z}^g_{H_1} = Z^f_0$, hence $\tilde{N}_g(r, H_1) = N_f(r, 0)$,$\tilde{N}_g(r, H_2) = N_f(r, \infty) = 0$.

Since

$$|\log^+ |f(z)| - \log(1 + |f(z)|^2)^{\frac{1}{2}}| \le \log 2,$$

integrating above inequality on $S_n(r)$ and noticing $\int_{S_n(r)} \sigma_n(z) = 1$, by the definition of characteristic function we have

$$T_f(r) = \tilde{T}_g(r) + O(1).$$

Hence f and g are of the same order and lower order, and

$$\limsup_{r\to\infty} \frac{N_f(r, 0)}{T_f(r)} = \limsup_{r\to\infty} \frac{\tilde{N}_g(r, H_1)}{\tilde{T}_g(r)}.$$

Obviously $\tilde{\delta}(H_2, g) = \delta(\infty, f) = 1$, hence from Lemma 2.5 we obtain

Lemma 2.6. *Let f be a non-constant entire function of finite order λ on $\mathbf{C}^n$. Then*

(1)

$$\limsup_{r\to\infty} \frac{N_f(r, 0)}{T_f(r)} \ge k(\lambda);$$

(2) *If $\delta(0, f) = 1$, then λ is a positive integer and equals the lower order of f.*

§2. Proof of Theorems

The proof of Theorem 1.1. For any q distinct points $a_1, \cdots, a_q$ in $\mathbf{C}$, set

$$F(z) = \sum_{j=1}^{q} \frac{1}{f(z) - a_j}, \quad \text{and} \quad \delta = \frac{1}{3} \min_{j<k} |a_j - a_k|.$$

Then as in [4, p.239] we can deduce that

$$\log^+ |F(z)| \ge \sum_{j=1}^{q} \log^+ \frac{1}{|f(z) - a_j|} - q \log^+ \frac{3q}{\delta} - \log 3.$$

Since $\int_{S_n(r)} \sigma_n(z) = 1$, by integrating above inequality on $S_n(r)$ we have

$$m_F(r, \infty) \ge \sum_{j=1}^{q} m_f(r, a_j) - q \log^+ \frac{3q}{\delta} - \log 3. \tag{3.1}$$

Noticing $D_{f-a_j} = D_f$ $(j = 1, \cdots, q)$ we have

$$\log^+ F(z) \le \sum_{j=1}^{q} \log^+ \Big|\frac{D_{f-a_j}(z)}{f(z) - a_j}\Big| + \log^+ \frac{1}{|D_f(z)|} + \log q.$$

Since f and $f - a_j$ are of the same order, from Lemma 2.2 and above inequality we deduce that

$$m_F(r, \infty) \le \sum_{j=1}^{q} m_{\frac{D_{f-a_j}}{f-a_j}}(r, \infty) + m_{D_f}(r, 0) + \log q = m_{D_f}(r, 0) + O(\log r). \tag{3.2}$$

Therefore from (3.1), (3.2) and the first main theorem we have

$$\sum_{j=1}^{q} m_f(r, a_j) \leq T_{D_f}(r) - N_{D_f}(r, 0) + O(\log r). \tag{3.3}$$

From (2.1) and Lemma 2.2 we have

$$T_{D_f}(r) \leq T_f(r) + O(\log r).$$

Hence from (3.3) we obtain

$$\sum_{j=1}^{q} \liminf_{r\to\infty} \frac{m_f(r, a_j)}{T_f(r)} \leq \sum_{j=1}^{q} \liminf_{r\to\infty} \frac{m_f(r, a_j)}{T_{D_f}(r)} \leq 1 - \limsup_{r\to\infty} \frac{N_{D_f}(r, 0)}{T_{D_f}(r)},$$

i.e., $\sum\limits_{j=1}^{q} \delta(a_j, f) \leq \delta(0, D_f)$. Hence by a well known discussion we can derive that

$$\sum_{a\in\mathbf{C}} \delta(a, f) \leq \delta(0, D_f). \tag{3.4}$$

From Lemma 2.4 we know that the order of D_f is λ. Furthermore, from Lemma 2.6 we have

$$\delta(0, D_f) = 1 - \limsup_{r\to\infty} \frac{N_{D_f}(r, 0)}{T_{D_f}(r)} \leq 1 - k(\lambda).$$

Combining (3.4) with above inequality we can obtain the conclusion of Theorem 1.1.

The Proof of Theorem 1.2. Since $\sum\limits_{a\in\mathbf{C}} \delta(a, f) = 1$, from (3.4) we have $\delta(0, D_f) = 1$. Hence from Lemma 2.6 we know that the order of D_f equals the lower order of D_f and is a positive integer. Hence from Lemma 2.4 we deduce that the order of f equals the lower order of f and is a positive integer. The proof is completed.

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