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SOME FUNCTIONAL LIMIT THEOREMS FOR THE INFINITE SERIES OF OU PROCESSES

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Abstract

This paper obtains functional modulus of continuity and Strassen's functional LIL of the infinite series of independent Ornstein-Uhlenbeck processes, which also imply the Lévy's exact modulus of continuity and LIL of this process respectively.

Keywords Ornstein-Uhlenbeck processes, Stationary Gaussian processes, Modulus of continuity, Law of the iterated logarithm

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§1. Introduction

Let $\{Y(t), -\infty < t < \infty\} = \{X_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ be a sequence of independent Ornstein-Uhlenbeck (OU) processes with coefficients γ_k and λ_k, i.e., $\{X_k(t), -\infty < t < \infty\}$ are stationary, mean zero Gaussian processes with

$$E\{X_k(s)X_k(t)\} = \frac{\gamma_k}{\lambda_k}\exp\{-\lambda_k|t-s|\}, \quad k = 1, 2, \cdots,$$

where $\gamma_k \geq 0$, $\lambda_k > 0$.

The process $Y(\cdot)$ was first introduced by Dawson[5] as the stationary solution of the infinite array of stochastic differential equations

$$dX_i(t) = -\lambda_i X_i(t)dt + (2\gamma_i)^{1/2}dW_i(t), \quad i = 1, 2, \cdots,$$

where $\{W_i(t), -\infty < t < \infty\}$ are independent Wiener processes (see also [6, 13]). Since then the properties of $Y(\cdot)$ have been extensively studied in the literature. The continuity properties of $Y(\cdot)$ were studied by, for example, Iscoe et al. (see [10] and the references therein). The moduli of continuity for ℓ^p-valued OU processes as well as for the ℓ^2-norm squared process of these were investigated by, for example, Csáki et al., Csörgő and Shao (see

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[2,3] and the references thereins). Csáki et al.[1] studied the infinite series of the independent OU coordinate processes of $Y(\cdot)$, namely the process $X(\cdot)$ defined by

$$\{X(t), -\infty < t < \infty\} = \Big\{ \sum_{k=1}^{\infty} X_k(t), -\infty < t < \infty \Big\}, \tag{1.1}$$

and obtained the Lévy's exact modulus of continuity for $\{X(t), -\infty < t < \infty\}$.

Let $\log x$ denote $\log_e(\max\{x, 1\})$, and $\log\log x$ denote $\log_e(\log_e(\max\{x, e\}))$. In the sequel of the present paper we assume that for some $\delta > 0$,

$$0 < \Gamma_0 = \sum_{k=1}^{\infty} \gamma_k (\log(\lambda_k \vee e))^{1+\delta}/\lambda_k < \infty,$$

which, in turn, implies that $X(\cdot)$ is a stationary and almost surely (a.s.) continuous Gaussian process, and that

$$\sigma^2(h) = E(X(t+h) - X(t))^2 = 2\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k}(1 - e^{-\lambda_k h}), \quad h \ge 0, t \ge 0. \tag{1.2}$$

The modulus of continuity of $X(\cdot)$ which was established by Csáki et al. (see [1, Theorem 4.1]) is the following

Theorem 1.1. *Assume that $\sigma(\cdot)$ is a regularly varying function at zero with positive exponent, namely, $\sigma(s) = s^{\alpha} L(s)$ with $\alpha > 0$, where $L(\cdot)$ is slowly varying at zero, i.e. it is measurable, positive and satisfies*

$$\lim_{s\downarrow 0} \frac{L(\lambda s)}{L(s)} = 1 \quad \textit{for all} \ \ \lambda > 0.$$

Then

$$\lim_{h\downarrow 0} \sup_{0\le t\le 1-h} \sup_{0\le s\le h} \frac{|X(t+s) - X(t)|}{\sigma(h)(2\log h^{-1})^{1/2}} \overset{a.s.}{=} 1. \tag{1.3}$$

The LIL of $X(\cdot)$ is the following (see, e.g. [12]).

Theorem 1.2. *Assume that there exist positive constants α and C_0 such that*

$$(1/C_0)s^{\alpha} \le \sigma(s) \le C_0 s^{\alpha} \quad \textit{for all} \quad 0 < s \le 1.$$

Then

$$\limsup_{h\downarrow 0} \sup_{0\le s\le h} \frac{|X(s) - X(0)|}{\sigma(h)(2\log\log h^{-1})^{1/2}} \overset{a.s.}{=} 1. \tag{1.4}$$

The moduli of non-differentiability of $X(\cdot)$ were studied by Csörgő and Shao[4] and Zhang[15].

The purpose of this paper is to study functional modulus of continuity and Strassen's functional LIL for $X(\cdot)$, namely to establish Theorems 3.1, 3.2 and 4.1 below (see §3 and §4), which also imply (1.3) and (1.4) respectively.

For use later on, we introduce some notations: Let $C_0[0, 1]$ denote the set of all continuous functions f on $[0, 1]$ with $f(0) = 0$, and $\|\cdot\|_{\infty}$ denote the sup-norm in $C_0[0, 1]$, i.e.,

$$\|f\|_{\infty} = \sup_{0\le t\le 1} |f(t)| \quad \text{for} \quad f \in C_0[0, 1].$$

Let $H_\mu \subset C_0[0,1]$ denote the reproducing kernel Hilbert space corresponding to the Gaussian measure μ on the seperable Banach space $C_0[0,1]$ induced by $\{X(t)-X(0), 0 \le t \le 1\}$. (The construction of H_μ can be found in, e.g., [11]). If $f \in H_\mu$, then $\|f\|_\mu$ denotes the H_μ-norm of f. Set $\mathcal{U} = \{f \in H_\mu : \|f\|_\mu \le 1\}$. For any $h \in (0,1)$ and $t \in [0, 1-h]$, put

$$\begin{aligned} M_{t,h}(x) &= \frac{X(t+hx) - X(t)}{\sigma(h)(2\log h^{-1})^{1/2}}, \quad 0 \le x \le 1, \\ S_h(x) &= \frac{X(hx) - X(0)}{\sigma(h)(2\log\log h^{-1})^{1/2}}, \quad 0 \le x \le 1, \\ V_h &= \{M_{t,h}(\cdot) \in C_0[0,1], 0 \le t \le 1-h\}. \end{aligned}$$

§2. Several Lemmas

In this section we give some lemmas which will be used in the following two sections, some of which are of independent interst.

Lemma 2.1. *For any $\varepsilon > 0$, there exists a positive number $\lambda_0 = \lambda_0(\varepsilon)$ such that for any $\lambda \ge \lambda_0$,*

$$P\Big(\inf_{f\in\mathcal{U}} \Big\| \frac{X(\cdot) - X(0)}{\lambda} - f(\cdot) \Big\|_\infty \ge 2\varepsilon\Big) \le \exp\Big\{ -\frac{(\lambda(1+\varepsilon))^2}{2} \Big\}. \tag{2.1}$$

Proof. This inequality appears in Lemma 2.1 of [14], or see Theorem 2.1 of [19]. Hence the lemma holds.

The following is an extension of Lemma 2.1.

Lemma 2.2. *Assume that there exist positive constants α and θ such that*

$$\sigma(s) = \theta s^\alpha \quad \textit{for all} \ \ 0 < s \le 1. \tag{2.2}$$

Then, for any $\varepsilon > 0$, there exist positive numbers $\lambda_0 = \lambda_0(\varepsilon)$ such that

$$\begin{aligned} &P\Big(\sup_{0\le t\le 1-h} \inf_{f\in\mathcal{U}} \Big\| \frac{\theta(X(t+h\cdot) - X(t))}{\lambda\sigma(h)} - f(\cdot) \Big\|_\infty \ge 6\varepsilon\Big) \\ &\le \frac{C}{h} \exp\Big\{ -\frac{(\lambda(1+\varepsilon))^2}{2} \Big\} \end{aligned} \tag{2.3}$$

for any $\lambda \ge \lambda_0$ and every $0 < h \le h_0$ with some $h_0 < 1$. Here, and in the sequel, C stands a positive constant, whose value is irrelavant.

Proof. Let $t_j = h\big[t \cdot \frac{2^j}{h}\big]/2^j$ for real number $t \in [0,1)$, where j is a positive integer which will be specified later on. Notice that

$$\begin{aligned} &P\Big(\sup_{0\le t\le 1-h} \inf_{f\in\mathcal{U}} \Big\| \frac{\theta(X(t+h\cdot) - X(t))}{\lambda\sigma(h)} - f(\cdot) \Big\|_\infty \ge 6\varepsilon\Big) \\ &\le P\Big(\sup_{0\le t\le 1-h} \inf_{f\in\mathcal{U}} \Big\| \frac{\theta(X(t_j+h\cdot) - X(t_j))}{\lambda\sigma(h)} - f(\cdot) \Big\|_\infty \ge 4\varepsilon\Big) \\ &\quad + 2P\left(\sup_{0\le t\le 1-h} \frac{\theta\|X(t+h\cdot) - X(t_j+h\cdot)\|_\infty}{\lambda\sigma(h)} \ge \varepsilon\right) \\ &=: I_1 + I_2. \end{aligned} \tag{2.4}$$

Put $t_j(l) = lh/2^j$, $l \ge 1$. Noting that $X(\cdot)$ is stationary, and that for any $h > 0$, $\Big\{\frac{\theta(X(hx)-X(0))}{\sigma(h)}, 0 \le x \le 1\Big\}$ and $\{X(x) - X(0), 0 \le x \le 1\}$ have the same distribution by

(2.2), we get

$$\begin{aligned} I_1 &\le P\Big(\max_{1\le l\le [2^j h^{-1}]+1} \sup_{t_j(l-1)\le t< t_j(l)} \inf_{f\in\mathcal{U}} \Big\| \frac{\theta(X(t_j+h\cdot)-X(t_j))}{\lambda\sigma(h)} - f(\cdot)\Big\|_\infty \ge 4\varepsilon\Big) \\ &\le \sum_{l=1}^{[2^j h^{-1}]+1} P\Big(\inf_{f\in\mathcal{U}} \Big\| \frac{\theta(X(t_j(l-1)+h\cdot)-X(t_j(l-1)))}{\lambda\sigma(h)} - f(\cdot)\Big\|_\infty \ge 4\varepsilon\Big) \\ &\le \frac{2^j+1}{h} P\Big(\inf_{f\in\mathcal{U}} \Big\| \frac{X(\cdot)-X(0)}{\lambda} - f(\cdot)\Big\|_\infty \ge 4\varepsilon\Big) \\ &\le \frac{2^j+1}{h} \exp\Big\{-\frac{(\lambda(1+2\varepsilon))^2}{2}\Big\}, \end{aligned} \tag{2.5}$$

where the last inequality follows from (2.1).

It holds that

$$\begin{aligned} I_2 &\le \sum_{l=1}^{[2^j h^{-1}]+1} P\Big(\sup_{t_j(l-1)\le t<t_j(l)} \frac{\theta\|X(t+h\cdot)-X(t_j(l-1)+h\cdot)\|_\infty}{\lambda\sigma(h)} \ge \varepsilon\Big) \\ &\le \frac{2^j+1}{h} P\Big(\sup_{0\le t<2^{-j}} \|X(t+\cdot)-X(\cdot)\|_\infty \ge \varepsilon\lambda\Big) \\ &\le \frac{4^j+2^j}{h} \exp\Big\{-\frac{\varepsilon^2\lambda^2}{3\sigma^2(2^{-j})}\Big\}, \end{aligned} \tag{2.6}$$

where for obtaining the last inequality we have used the following inequality (see Lemma 3.2 of [1]): For any $\mu>0$, there exists $0<h_0'=h_0'(\mu)<1$ such that for every $0<h'\le h_0'$ and $u>0$,

$$P\Big(\sup_{0\le t\le 1-h'} \sup_{0\le s\le h'} |X(t+s)-X(t)| \ge u\sigma(h')\Big) \le \frac{C}{h'} \exp\Big\{-\frac{u^2}{2+\mu}\Big\} \tag{2.7}$$

if $\sigma(\cdot)$ satisfies (2.2).

Choose j such that $2^j = (\frac{3(1+2\varepsilon)^2\theta^2}{\varepsilon^2})^{1/2\alpha}$. Then by combining (2.4)–(2.6), we have

$$I_1+I_2 \le \frac{C}{h}\exp\Big\{-\frac{(\lambda(1+2\varepsilon))^2}{2} - \frac{2}{\alpha}\log\varepsilon\Big\} \le \frac{C}{h}\exp\Big\{-\frac{(\lambda(1+\varepsilon))^2}{2}\Big\}$$

for $\lambda>\lambda_0$ with $\lambda_0=\lambda_0(\varepsilon)=-2\log\varepsilon/\alpha\varepsilon$. So, (2.3) is proved.

The following lemma follows from the well-known inequalities about the shift of symmetric convex sets (see, for example, [7]).

Lemma 2.3. *For any $f\in H_\mu$ and $r>0$,*

$$\begin{aligned} \exp\{-\|f\|_\mu^2/2\} P(\|X(\cdot)-X(0)\|_\infty \le r) &\le P(\|(X(\cdot)-X(0))-f(\cdot)\|_\infty \le r) \\ &\le P(\|X(\cdot)-X(0)\|_\infty \le r), \end{aligned}$$

where H_μ is defined as in Section 1.

The following lemma is due to Fernique[8].

Lemma 2.4. *Let $\{\xi(t), 0\le t\le T\}$ be a Gaussian process with mean zero and*

$$E(\xi(t)-\xi(s))^2 \le \Lambda^2(|t-s|) \quad \text{for } 0\le s,t\le T,$$

where Λ is continuous, nondecreasing and satisfies $\int_1^\infty \Lambda(e^{-y^2}T)dy<\infty$ and also $E\xi^2(t)\le$

Γ^2 *for* $0 \le t \le T$. *Then, for each* $x > 0$,

$$P\Big(\sup_{0\le t\le T}|\xi(t)| > x\Big(\Gamma + \int_1^\infty \Lambda(e^{-y^2}T)dy\Big)\Big) \le Ce^{-x^2/2}.$$

§3. Functional Modulus of Continuity of $X(\cdot)$

In this section, we prove the following

Theorem 3.1. *Assume that there exist positive constants* α *and* θ *such that* (2.2) *is satisfied. Then for any* $\varepsilon > 0$, *with probability one, there exists* $h_0 = h_0(\varepsilon) > 0$ *such that*

$$\theta V_h \subset \mathcal{U}^\varepsilon \tag{3.1$'$}$$

and

$$\mathcal{U} \subset (\theta V_h)^\varepsilon \tag{3.2$'$}$$

if $h \le h_0$, *where the following notations are used*

$$E^\varepsilon =: \Big\{g \in C_0[0,1], \inf_{f\in E}\|g-f\|_\infty < \varepsilon\Big\},$$

$$\theta V_h = \{\theta M_{t,h}(\cdot) \in C_0[0,1], 0 \le t \le 1-h\}.$$

An equivalent result is the following theorem.

Theorem 3.2. *Assume that there exist positive constants* α *and* θ *such that* (2.2) *is satisfied. Then*

$$\lim_{h\downarrow 0}\sup_{0\le t\le 1-h}\inf_{f\in\mathcal{U}}\|\theta M_{t,h} - f\|_\infty \overset{a.s.}{=} 0, \tag{3.1}$$

and, for each $f \in \mathcal{U}$,

$$\lim_{h\downarrow 0}\inf_{0\le t\le 1-h}\|\theta M_{t,h} - f\|_\infty \overset{a.s.}{=} 0. \tag{3.2}$$

Proof. At first we prove (3.1). For any $\vartheta \in (1,2)$ and integer $n \ge 1$, let $h_n = \vartheta^{-n}$. For all $h \in (0,1)$, there is n such that $h_{n+1} \le h \le h_n$. Then we have

$$\begin{aligned}
&\sup_{0\le t\le 1-h}\inf_{f\in\mathcal{U}}\|\theta M_{t,h} - f\|_\infty \\
\le& \sup_{0\le t\le 1-h_{n+1}}\inf_{f\in\mathcal{U}}\|\theta M_{t,h_{n+1}} - f\|_\infty \\
&+ \sup_{h_{n+1}\le h\le h_n}\sup_{0\le t\le 1-h_{n+1}}\theta|\Gamma_{h_{n+1}}^{-1}\Gamma_h - 1|\|M_{t,h_{n+1}}\|_\infty \\
&+ \sup_{h_{n+1}\le h\le h_n}\sup_{0\le t\le 1-h_{n+1}}\theta\Gamma_{h_{n+1}}^{-1}\Gamma_h\|M_{t,h_{n+1}}(h_{n+1}^{-1}h\cdot) - M_{t,h_{n+1}}(\cdot)\|_\infty \\
=:& J_1^{(n)} + J_2^{(n)} + J_3^{(n)}.
\end{aligned} \tag{3.3}$$

Here, and in the sequel,

$$\Gamma_h =: (2h\log h^{-1})^{-1/2}.$$

By Lemma 2.2, we get that for any $\varepsilon > 0$ and n large enough,

$$P(J_1^{(n)} \ge 6\varepsilon) \le \frac{C}{h_{n+1}}\exp\{-(1+\varepsilon)^2\log h_{n+1}^{-1}\} \le Ch_{n+1}^\varepsilon.$$

Therefore, we have

$$\sum_n P(J_1^{(n)} \geq 6\varepsilon) < \infty,$$

which implies

$$\lim_{n\to\infty} J_1^{(n)} \overset{\text{a.s.}}{=} 0 \tag{3.4}$$

by the Borel-Cantelli lemma. By (2.7), the definition of h_n, and the Borel-Cantelli lemma, it is easy to show that

$$\limsup_{n\to\infty}\{J_2^{(n)} + J_3^{(n)}\} \overset{\text{a.s.}}{\leq} 4\sqrt{\vartheta - 1},$$

which implies that

$$\lim_{n\to\infty}\{J_2^{(n)} + J_3^{(n)}\} \overset{\text{a.s.}}{=} 0 \tag{3.5}$$

if we let $\vartheta \downarrow 1$. From (3.3)–(3.5) we obtain (3.1).

Next, we prove (3.2) holds for each $f \in \mathcal{U}$. For any $\vartheta \in (0,1)$, let $h_n = \vartheta^{-n}, n \geq 1$. Let

$$t_i = 2dih_n, \quad i = 0, 1, \cdots, [(2dh_n)^{-1}] =: \rho_{h_n},$$

where $d > 1$ is a sufficiently large constant which will be specified later on.

We first show

$$\lim_{n\to\infty} \min_{0\leq i\leq \rho_{h_n}} \|\theta M_{t_i,h_n} - f\|_\infty \overset{\text{a.s.}}{=} 0. \tag{3.6}$$

Let $\{W_k(t), -\infty < t < \infty\}_{k=1}^{\infty}$ be a sequence of independent standard Wiener processes. It is easy to see that

$$\Big\{\Big(\frac{\gamma_k}{\lambda_k}\Big)^{1/2}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}}, -\infty < t < \infty\Big\} \quad \text{and} \quad \{X_k(t), -\infty < t < \infty\}$$

have the same distribution. Hence we can rewrite $\{X(t), -\infty < t < \infty\}$ as

$$X(t) = \sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^{1/2}\frac{W_k(e^{2\lambda_k t})}{e^{\lambda_k t}} \tag{3.7}$$

and remain the almost sure path properties of $X(\cdot)$ without change.

By (3.7), we have

$$\begin{aligned}
&X(t_i + h_n x) - X(t_i) \\
&= \sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^{1/2}\Big(\frac{W_k(e^{2\lambda_k(2di+x)h_n})}{e^{\lambda_k(2di+x)h_n}} - \frac{W_k(e^{2\lambda_k(2di)h_n})}{e^{\lambda_k(2di)h_n}}\Big) \\
&= \sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^{1/2}\frac{W_k(e^{2\lambda_k(2di+x)h_n}) - W_k(e^{2\lambda_k(2i-1)dh_n})}{e^{\lambda_k(2di+x)h_n}} - \\
&\quad - \sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^{1/2}\frac{W_k(e^{2\lambda_k(2di)h_n}) - W_k(e^{2\lambda_k(2i-1)dh_n})}{e^{\lambda_k(2di)h_n}} \\
&\quad + \sum_{k=1}^{\infty}\Big(\frac{\gamma_k}{\lambda_k}\Big)^{1/2}\frac{W_k(e^{2\lambda_k(2i-1)dh_n})}{e^{\lambda_k(2di)h_n}}(e^{-\lambda_k h_n x} - 1) \\
&=: \xi_{i,d}^{(n)}(x) - \xi_{i,d}^{(n)}(0) + \eta_{i,d}^{(n)}(x).
\end{aligned} \tag{3.8}$$

Note that $\{\xi_{i,d}^{(n)}(x)-\xi_{i,d}^{(n)}(0), 0 \le x \le 1\}_{i=1}^{\infty}$ is a sequence of independent Gaussian processes. Therefore, $\forall 0 < \varepsilon < 1$,

$$\begin{aligned}
&P\Big(\min_{0\le i\le \rho_{h_n}} \|\theta M_{t_i,h_n} - f\|_\infty \ge 4\varepsilon\Big)\\
&\le P\Big(\min_{0\le i\le \rho_{h_n}} \Big\|\frac{\theta(\xi_{i,d}^{(n)}(\cdot)-\xi_{i,d}^{(n)}(0))}{\sigma(h_n)\sqrt{2\log h_n^{-1}}} - f(\cdot)\Big\|_\infty \ge 3\varepsilon\Big) + P\Big(\max_{0\le i\le \rho_{h_n}} \frac{\theta\|\eta_{i,d}^{(n)}(\cdot)\|_\infty}{\sigma(h_n)\sqrt{2\log h_n^{-1}}} \ge \varepsilon\Big)\\
&\le \prod_{i=0}^{\rho_{h_n}} P\Big(\Big\|\frac{\theta(\xi_{i,d}^{(n)}(\cdot)-\xi_{i,d}^{(n)}(0))}{\sigma(h_n)\sqrt{2\log h_n^{-1}}} - f(\cdot)\Big\|_\infty \ge 3\varepsilon\Big) + \sum_{i=0}^{\rho_{h_n}} P\Big(\frac{\theta\|\eta_{i,d}^{(n)}(\cdot)\|_\infty}{\sigma(h_n)\sqrt{2\log h_n^{-1}}} \ge \varepsilon\Big) \qquad (3.9)\\
&\le \prod_{i=0}^{\rho_{h_n}} \Big\{P(\|\theta M_{t_i,h_n}(\cdot) - f(\cdot)\|_\infty \ge 2\varepsilon) + P\Big(\frac{\theta\|\eta_{i,d}^{(n)}(\cdot)\|_\infty}{\sigma(h_n)\sqrt{2\log h_n^{-1}}} \ge \varepsilon\Big)\Big\}\\
&\quad + \sum_{i=0}^{\rho_{h_n}} P\Big(\frac{\theta\|\eta_{i,d}^{(n)}(\cdot)\|_\infty}{\sigma(h_n)\sqrt{2\log h_n^{-1}}} \ge \varepsilon\Big).
\end{aligned}$$

For any $f \in \mathcal{U}$ and $0 < \varepsilon < 1$, let $f_\varepsilon = (1-\varepsilon)f$. Then $f_\varepsilon \in \mathcal{U}$ and $\|f - f_\varepsilon\|_\infty \le \varepsilon$. By Lemma 2.3, we get that $\forall i \ge 0$,

$$\begin{aligned}
&P(\|\theta M_{t_i,h_n}(\cdot) - f(\cdot)\|_\infty \ge 2\varepsilon)\\
&\le P(\|(X(\cdot)-X(0)) - \sqrt{2\log h_n^{-1}} f_\varepsilon(\cdot)\|_\infty \ge \varepsilon\sqrt{2\log h_n^{-1}})\\
&\le 1 - P(\|X(\cdot)-X(0)\|_\infty \le \varepsilon\sqrt{2\log h_n^{-1}})\exp\{-(1-\varepsilon)^2\|f\|_\mu^2 \log h_n^{-1}\}\\
&\le 1 - C\exp\{-(1-\varepsilon)^2 \log h_n^{-1}\}\\
&\le \exp\{-C\vartheta^{-(1-\varepsilon)^2 n}\}
\end{aligned}$$

for n large enough since $f \in \mathcal{U}$. Thus, if we can show that for any $i \ge 0$ and $\varepsilon > 0$,

$$P\Big(\sup_{0\le t\le 1} |\eta_{i,d}^{(n)}(t)| \ge \frac{\varepsilon}{\theta}\sigma(h_n)\sqrt{2\log h_n^{-1}}\Big) \le C\vartheta^{-C\varepsilon^2 d^{1/2} n} \qquad (3.10)$$

for large n, then by (3.9) we get

$$\begin{aligned}
&P\Big(\min_{0\le i\le \rho_{h_n}} \|\theta M_{t_i,h_n} - f\|_\infty \ge 4\varepsilon\Big)\\
&\le \exp\{-C\rho_{h_n}\vartheta^{-(1-\varepsilon)^2 n}\} + C\rho_{h_n}\vartheta^{-C\varepsilon^2 d^{1/2} n} \qquad (3.11)\\
&\le \exp\{-C\vartheta^{\varepsilon n}\} + C\vartheta^{-(C\varepsilon^2 d^{1/2}-1)n}.
\end{aligned}$$

Choosing $d > 1$ sufficiently large such that $C\varepsilon^2 d^{1/2} > 1$, we have the sum (3.11) is finite. Therefore we obtain (3.6) by the Borel-Cantelli lemma.

We prove (3.10). For $0 \le s \le s+t \le 1$ we have $\forall i \ge 0$,

$$\begin{aligned}
&E(\eta_{i,d}^{(n)}(s+t) - \eta_{i,d}^{(n)}(s))^2\\
&\le \sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} e^{-2d\lambda_k h_n}(1-e^{-\lambda_k h_n t})^2\\
&\le \Big(\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_n t})\Big)^{1/2} \Big(\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} e^{-4d\lambda_k h_n}(1-e^{-\lambda_k h_n t})^3\Big)^{1/2}\\
&\le \sigma(h_n t)\Big(\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} e^{-4d\lambda_k h_n}(1-e^{-\lambda_k h_n})^3\Big)^{1/2}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k} e^{-4d\lambda_k h_n}(1-e^{-\lambda_k h_n})^3\\
&\le \sum_{\lambda_k h_n \le d^{-1/2}} \frac{\gamma_k}{\lambda_k} e^{-4d\lambda_k h_n}(1-e^{-\lambda_k h_n})^3\\
&\quad + \sum_{\lambda_k h_n > d^{-1/2}} \frac{\gamma_k}{\lambda_k} e^{-4d\lambda_k h_n}(1-e^{-\lambda_k h_n})^3\\
&\le d^{-1} \sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_n}) + e^{-d^{1/2}} \sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda_k}(1-e^{-\lambda_k h_n})\\
&\le 4d^{-1}\sigma^2(h_n).
\end{aligned}$$

We have also

$$E(\eta_{i,d}^{(n)}(s+t) - \eta_{i,d}^{(n)}(s))^2 \le 2d^{-1/2}\sigma(h_n t)\sigma(h_n)$$

and

$$E(\eta_{i,d}^{(n)}(t))^2 = E(\eta_{i,d}^{(n)}(t) - \eta_{i,d}^{(n)}(0))^2 \le 2d^{-1/2}\sigma(h_n t)\sigma(h_n).$$

By the condition (2.2) of $\sigma(\cdot)$,

$$\int_1^{\infty} \Lambda(e^{-y^2})dy = (2d^{-1/2}\sigma^2(h_n))^{1/2} \int_1^{\infty} e^{-\alpha y^2} dy \le 2(2d^{-1/2}\sigma^2(h_n))^{1/2}/\sqrt{2\alpha}.$$

Appying Lemma 2.4 yields for each $x > 0$,

$$P\Big(\sup_{0 \le t \le 1} |\eta_{i,d}^{(n)}(t)| > x(1+2/\sqrt{2\alpha})(2d^{-1/2})^{1/2}\sigma(h_n)\Big) \le Ce^{-x^2/2},$$

which implies easily that (3.10) holds. So (3.6) is proved.

For all $h \in (0,1)$, there is n such that $h_{n+1} \le h \le h_n$. Then, for each $f \in \mathcal{U}$ we have

$$\begin{aligned}
&\inf_{0 \le t \le 1-h} \|\theta M_{t,h} - f\|_\infty\\
&\le \min_{0 \le i \le \rho_{h_n}} \|\theta M_{t_i,h_n} - f\|_\infty\\
&\quad + \sup_{h_{n+1} \le h \le h_n} \sup_{0 \le t \le 1-h_n} \theta\left\{|\Gamma_{h_n}^{-1}\Gamma_h - 1|\|M_{t,h_n}\|_\infty + \Gamma_{h_n}^{-1}\Gamma_h\|M_{t,h_n}(h_n^{-1}h\cdot) - M_{t,h_n}(\cdot)\|_\infty\right\}\\
&=: J_4^{(n)} + J_5^{(n)}.
\end{aligned}$$

It follows from (3.6) that

$$\lim_{n\to\infty} J_4^{(n)} \overset{\text{a.s.}}{=} 0.$$

Similar to the arguments for $J_2^{(n)}$ and $J_3^{(n)}$ in (3.3), it is easy to show that

$$\lim_{n\to\infty} J_5^{(n)} \overset{\text{a.s.}}{=} 0.$$

From these estimates we obtain (3.2). The proof is completed.

Corollary 3.1. *Assume that there exist positive constants α and θ such that* (2.2) *is satisfied. Then*

$$\lim_{h\downarrow 0} \sup_{0\le t\le 1-h} \Phi(\theta M_{t,h}) \overset{a.s.}{=} \sup_{f\in\mathcal{U}} \Phi(f)$$

for any continuous function $\Phi : C_0[0,1] \to \mathbf{R}$. In particular, (1.3) *holds.*

§4. Strassen's Functional LIL of $X(\,\cdot\,)$

In this section, we prove the following

Theorem 4.1. *Assume that there exist positive constants α and θ such that* (2.2) *is satisfied. Then*

$$\lim_{h\downarrow 0} \inf_{f\in\mathcal{U}} \|\theta S_h - f\|_\infty \overset{a.s.}{=} 0. \tag{4.1}$$

If we also assume that one of the following conditions holds:

(i) $\alpha > 1$;

(ii) $0 < \max_{k\ge 1} \lambda_k < \infty$;

(iii) $\sum_{k=1}^{\infty} \gamma_k < \infty$,

then, for each $f \in \mathcal{U}$,

$$\liminf_{h\downarrow 0} \|\theta S_h - f\|_\infty \overset{a.s.}{=} 0. \tag{4.2}$$

Proof. The proof of (4.1), which is completely similar to that of (3.1), is omitted.

Now we show that (4.2) holds for each $f \in \mathcal{U}$ if also one of the conditions (i)$\sim$(iii) is satisfied. Let $h_n = e^{-n^p}$, $n \ge 1, p > 1$. It is enough to show that for each $f \in \mathcal{U}$,

$$\liminf_{n\to\infty} \|\theta S_{h_n} - f\|_\infty \overset{\text{a.s.}}{=} 0. \tag{4.3}$$

Notice that

$$\begin{aligned}
&\liminf_{n\to\infty} \|\theta S_{h_n} - f\|_\infty \\
&\le \liminf_{n\to\infty} \sup_{0\le x\le 1} \Big| \frac{\theta(X(h_n x + r h_n) - X(r h_n))}{\sigma(h_n)(2\log\log h_n^{-1})^{1/2}} - f(x) \Big| \\
&\quad + 2 \limsup_{n\to\infty} \sup_{0\le x\le 1} \frac{\theta |X(h_n x + r h_n) - X(h_n x)|}{\sigma(h_n)(2\log\log h_n^{-1})^{1/2}} \\
&=: L_1 + L_2,
\end{aligned}$$

where $r > 0$, which will be specified later on. It is enough to show that the following two equalities are true:

$$L_1 \overset{\text{a.s.}}{=} 0, \tag{4.4}$$

$$L_2 \overset{\text{a.s.}}{=} 0. \tag{4.5}$$

We proceed with the proofs of (4.4) and (4.5) by considering the case of (i)–(iii), separately.

Case 1. $\alpha > 1$. Choose $r > 0$ sufficiently large. Notice that

$$\begin{aligned}
&\limsup_{n\to\infty} \sup_{0\le x\le 1} \frac{\theta|X(h_n x + rh_n) - X(h_n x)|}{\sigma(h_n)(2\log\log h_n^{-1})^{1/2}} \\
&\le \limsup_{n\to\infty} \sum_{i=1}^{[r/\delta]} \sup_{0\le x\le 1} \frac{\theta|X(h_n x + i\delta h_n) - X(h_n x + (i-1)\delta h_n)|}{\sigma(h_n)(2\log\log h_n^{-1})^{1/2}} \\
&\quad + \limsup_{n\to\infty} \sup_{0\le x\le 1} \sup_{0\le s\le \delta h_n} \frac{\theta|X(h_n x + [r/\delta]\delta h_n + s) - X(h_n x + [r/\delta]\delta h_n)|}{\sigma(h_n)(2\log\log h_n^{-1})^{1/2}} \\
&=: L_2' + L_2'',
\end{aligned}$$

where $0 < \delta \le r$, which will be specified later on. By (2.7), it holds that $\forall \varepsilon > 0$,

$$\begin{aligned}
&P\Big(\sum_{i=1}^{[r/\delta]} \sup_{0\le x\le 1} |X(h_n x + i\delta h_n) - X(h_n x + (i-1)\delta h_n)| \ge \frac{\varepsilon}{\theta}\sigma(h_n)(2\log\log h_n^{-1})^{1/2} \Big) \\
&\le \frac{r}{\delta} P\Big(\sup_{0\le x\le 1} |X(x+\delta) - X(x)| \ge \varepsilon\delta(2\log\log h_n^{-1})^{1/2}/r \Big) \qquad (4.6)\\
&\le \frac{r}{\delta}\exp\Big\{ -\frac{\varepsilon^2\delta^2}{r^2\sigma^2(\delta)}\log\log h_n^{-1} \Big\} \le Cn^{-\frac{\varepsilon^2\delta^2 p}{r^2\sigma^2(\delta)}},
\end{aligned}$$

where $0 < \delta < r$. If we choose δ small enough such that

$$\frac{\varepsilon^2 p}{\theta^2 r^2 \delta^{2\alpha-2}} > 1,$$

then the sum (4.6) is finite. Hence by the Borel-Cantelli lemma we obtain

$$L_2' \overset{\text{a.s.}}{=} 0.$$

Similarly, by (2.7) and the Borel-Cantelli lemma, we have $L_2'' \overset{\text{a.s.}}{=} 0$ when $\delta \downarrow 0$. Therefore (4.5) is proved.

Put

$$\Upsilon_n(x) = \xi_{1,r/2}^{(n)}(x) \quad \text{and} \quad \zeta_n(x) = \eta_{1,r/2}^{(n)}(x),$$

where $\xi_{1,r/2}^{(n)}(x)$ and $\eta_{1,r/2}^{(n)}(x)$ are defined as in (3.8) (with $i = 1$ and $d = r/2$ there). Then

$$X(h_n x + rh_n) - X(rh_n) = \Upsilon_n(x) - \Upsilon_n(0) + \zeta_n(x). \tag{4.7}$$

Along the same lines as that of the proof of (3.10), we have $\forall \varepsilon > 0$,

$$P\Big(\sup_{0\le x\le 1} \frac{\theta|\zeta_n(x)|}{\sigma(h_n)(2\log\log h_n^{-1})^{1/2}} \ge \varepsilon \Big) \le Cn^{-2} \quad \text{for large } n, \tag{4.8}$$

which implies

$$\limsup_{n\to\infty} \sup_{0\le x\le 1} \frac{\theta|\zeta_n(x)|}{\sigma(h_n)(2\log\log h_n^{-1})^{1/2}} \overset{\text{a.s.}}{=} 0 \tag{4.9}$$

by the Borel-Cantelli lemma.

By (4.7), (4.8) and Lemma 2.3, we get that $\forall 0<\varepsilon<1$,

$$\begin{aligned}
&P\Big(\sup_{0\le x\le 1}\Big|\frac{\theta(\Upsilon_n(x)-\Upsilon_n(0))}{\sigma(h_n)(2\log\log h_n^{-1})^{1/2}}-f(x)\Big|\le 3\varepsilon\Big)\\
&\ge P\Big(\sup_{0\le x\le 1}\Big|\frac{\theta(X(h_nx+rh_n)-X(rh_n))}{\sigma(h_n)(2\log\log h_n^{-1})^{1/2}}-f(x)\Big|\le 2\varepsilon\Big)\\
&\quad -P\Big(\sup_{0\le x\le 1}\frac{\theta|\zeta_n(x)|}{\sigma(h_n)(2\log\log h_n^{-1})^{1/2}}\ge\varepsilon\Big)\\
&\ge P\Big(\sup_{0\le x\le 1}\Big|\frac{X(x)-X(0)}{(2\log\log h_n^{-1})^{1/2}}-f_\varepsilon(x)\Big|\le\varepsilon\Big)-Cn^{-2}\\
&\ge P(\|X(\cdot)-X(0)\|_\infty\le\varepsilon(2\log\log h_n^{-1})^{1/2})\\
&\quad\circ\exp\{-\|f_\varepsilon\|_\mu^2\log\log h_n^{-1}\}-Cn^{-2}\\
&\ge Cn^{-(1-\varepsilon)^2p}-Cn^{-2},
\end{aligned}\tag{4.10}$$

where f_ε is defined as in §3. Choosing $p>1$ small enough such that $(1-\varepsilon)^2p\le 1$, we have the sum (4.10) is infinite. Since $\{\Upsilon_n(x)-\Upsilon_n(0), 0\le x\le 1\}_{n=1}^\infty$ is a sequence of independent Gaussian processes, we get

$$\liminf_{n\to\infty}\sup_{0\le x\le 1}\Big|\frac{\theta(\Upsilon_n(x)-\Upsilon_n(0))}{\sigma(h_n)(2\log\log h_n^{-1})^{1/2}}-f(x)\Big|\overset{\text{a.s.}}{=}0 \tag{4.11}$$

by the Borel-Cantelli lemma. Putting (4.7), (4.9) and (4.11) together implies (4.4).

Case 2. $0<\max\limits_{k\ge1}\lambda_k<\infty$. Choose $r=\delta$ small enough in Case 1. Note the following fact: for $0\le s\le s+t\le 1$,

$$E(\zeta_n(s+t)-\zeta_n(s))^2\le\sum_{k=1}^\infty\frac{\gamma_k}{\lambda_k}e^{-2\delta\lambda_kh_n}(1-e^{-\lambda_kh_n})^2\le\delta\sigma^2(h_n).$$

Then, along the same lines as that of Case 1, we have that (4.4) and (4.5) are true.

Case 3. $\sum\limits_{k=1}^\infty\gamma_k<\infty$. Choose $r=\delta$ small enough in Case 1. If

$$\Gamma_1=:\sum_{k=1}^\infty\gamma_k<\infty,$$

then

$$\lim_{h\downarrow0}\frac{\sigma^2(h)}{h}=(\sigma^2(h))'\big|_{h=0}=2\sum_{k=1}^\infty\gamma_k(1-e^{-\lambda_kh})\Big|_{h=0}=2\Gamma_1.$$

In this case, by (2.2), we have $\sigma^2(s)=2\Gamma_1 s$ for all $0<s\le1$. It follows easily that

$$\{X(h_nx+\delta h_n)-X(\delta h_n); 0\le x\le 1\}_{n=1}^\infty$$

are independent. By Lemma 2.3, $\forall 0<\varepsilon<1$, we have

$$P\Big(\sup_{0\le x\le 1}\Big|\frac{\theta(X(h_nx+\delta h_n)-X(\delta h_n))}{\sigma(h_n)(2\log\log h_n^{-1})^{1/2}}-f(x)\Big|\le 2\varepsilon\Big)\ge Cn^{-(1-\varepsilon)^2p}. \tag{4.12}$$

Choosing $p>1$ small enough such that $(1-\varepsilon)^2p\le1$, the sum (4.12) is infinite. Hence by the Borel-Cantelli lemma, (4.4) follows in this case as well. The proof of (4.5) is the same as that in Case 1. The proof is completed.

Corollary 4.1. *Assume that there exist positive constants* α *and* θ *such that* (2.2) *is satisfied and that one of the conditions* (i)–(iii) *is satisfied. Then, with probability one, the process* $\{\theta S_h(x), 0 \le x \le 1, 0 < h < 1/3\}$ *is relatively compact in* $C_0[0,1]$, *and the set of its limit points (as* $h \to 0$*) is* $\mathcal{U}$.

Corollary 4.2. *Assume that there exist positive constants* α *and* θ *such that* (2.2) *is satisfied and that one of the conditions* (i)–(iii) *is satisfied. Then*

$$\limsup_{h\downarrow 0} \Phi(\theta S_h) \overset{\text{a.s.}}{=} \sup_{f\in\mathcal{U}} \Phi(f)$$

for any continuous function $\Phi : C_0[0,1] \to \mathbf{R}$. *In particular,* (1.4) *holds.*

Remark. The assumption (2.2) used in Theorems 3.1, 3.2 and 4.1 is weak and standard, since for studying the functional limit law of $X(\cdot)$ we need this condition to ensure that for any $h > 0$,

$$\Big\{\frac{\theta(X(hx) - X(0))}{\sigma(h)}, 0 \le x \le 1\Big\} \quad \text{and} \quad \{X(x) - X(0), 0 \le x \le 1\}$$

have the same distribution.

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