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# HYERS-ULAM-RASSIAS STABILITY OF QUADRATIC FUNCTIONAL EQUATIONS IN BANACH MODULES OVER A $C^*$-ALGEBRA****

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**Abstract**

The authors prove the Hyers-Ulam-Rassias stability of the quadratic mapping in Banach modules over a unital $C^*$-algebra, and prove the Hyers-Ulam-Rassias stability of the quadratic mapping in Banach modules over a unital Banach algebra.

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## §1. Introduction

In 1940, S.M. Ulam[14] raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping?

Let $E_1$ and $E_2$ be Banach spaces. Consider $f : E_1 \to E_2$ to be a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E_1$. Assume that there exist constants $\epsilon \geq 0$ and $p \in [0, 1)$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon(||x||^p + ||y||^p)$$

for all $x, y \in E_1$. Th.M. Rassias[11] showed that there exists a unique $\mathbb{R}$-linear mapping $T : E_1 \to E_2$ such that

$$\|f(x) - T(x)\| \leq \frac{2\epsilon}{2-2^p}||x||^p$$

for all $x \in E_1$.

A mapping $f : E_1 \to E_2$ is called quadratic if $f$ satisfies the quadratic functional equation

$$f(x+y) + f(x-y) = 2f(x) + 2f(y)$$

for all $x, y \in E_1$. F. Skof[13] was the first author to treat the Hyers-Ulam stability of a quadratic functional equation. S. Czerwik[6] generalized the Skof's result.

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Let $f : E_1 \to E_2$ be a mapping with $f(0) = 0$ satisfying the inequality

$$\|f(x+y) + f(x-y) - 2f(x) - 2f(y)\| \le \varphi(x,y)$$

for all $x, y \in E_1$. Assume that one of the series

$$\sum_{n=1}^{\infty} 2^{-2n}\varphi(2^{n-1}x, 2^{n-1}x) \quad \text{and} \quad \sum_{n=1}^{\infty} 2^{2n-2}\varphi(2^{-n}x, 2^{-n}x)$$

converges for every $x \in E_1$ and denote by $\widetilde{\varphi}(x)$ its sum. If, for every $x, y \in E_1$, as $n \to \infty$, (i) $2^{-2n}\varphi(2^{n-1}x, 2^{n-1}y) \to 0$ or (ii) $2^{2n-2}\varphi(2^{-n}x, 2^{-n}y) \to 0$, respectively, then there exists a unique quadratic mapping $Q : E_1 \to E_2$ such that

$$\|f(x) - Q(x)\| \le \widetilde{\varphi}(x)$$

for all $x \in E_1$ (see [4] for details).

Throughout this paper, let $B$ be a unital Banach algebra with norm $|\cdot|$, $B_1 = \{a \in B \mid |a| = 1\}$, $B_{in}$ the set of invertible elements in $B$, $\mathbb{R}^+$ the set of nonnegative real numbers, and let ${}_BM_1$ and ${}_BM_2$ be left Banach $B$-modules with norms $||\cdot||$ and $\|\cdot\|$, respectively. Assume that $F(2^n x) = 4^n F(x)$ for all $x \in {}_BM_1$ if $\varphi$ satisfies (i), and that $F\left(\frac{x}{2^n}\right) = \frac{1}{4^{n-1}}F(x)$ for all $x \in {}_BM_1$ if $\varphi$ satisfies (ii).

In this paper, we are going to prove the Hyers-Ulam-Rassias stability of the quadratic mapping in Banach modules over a unital Banach algebra.

## §2. Stability of the Quadratic Mapping in Banach Modules over a $C^*$-Algebra

In this section, let $B$ be a unital $C^*$-algebra of stable rank 1, which implies that $B_{in}$ is dense in $B$ (see [3, 10]). Let $F : {}_BM_1 \to {}_BM_2$ be a mapping such that (iii) $F(ax)$ is continuous in $a \in B$ for each fixed $x \in {}_BM_1$.

A quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ is called $B$-quadratic if $Q(ax) = a^2 Q(x)$ for all $a \in B$ and all $x \in {}_BM_1$.

**Theorem 2.1.** *Let $F : {}_BM_1 \to {}_BM_2$ be a mapping with $F(0) = 0$ for which there exists a function $\varphi : {}_BM_1 \times {}_BM_1 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x) < \infty,$$

$$\|F(ax+ay) + F(ax-ay) - 2a^2F(x) - 2a^2F(y)\| \le \varphi(x,y)$$

*for all $a \in B_1 \cap B_{in}$ and all $x, y \in {}_BM_1$, where $\widetilde{\varphi}(x)$ is as defined in the introduction. Then the mapping $F : {}_BM_1 \to {}_BM_2$ is a $B$-quadratic mapping.*

**Proof.** Put $a = 1 \in B_1 \cap B_{in}$. By [4, Theorem 2], there exists a unique quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ such that

(iv) $\|F(x) - Q(x)\| \le \widetilde{\varphi}(x)$

for all $x \in {}_BM_1$. The mapping $Q : {}_BM_1 \to {}_BM_2$ was given by $Q(x) = \lim\limits_{n\to\infty} \frac{F(2^n x)}{4^n}$ for all $x \in {}_BM_1$ if $\varphi$ satisfies (i), and $Q(x) = \lim\limits_{n\to\infty} 4^{n-1}F(\frac{x}{2^n})$ for all $x \in {}_BM_1$ if $\varphi$ satisfies (ii). The mapping $Q : {}_BM_1 \to {}_BM_2$ is similar to the additive mapping $T$ given in the proof of [11, Theorem]. Under the assumption that $F(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_BM_1$, by the same reasoning as the proof of [11, Theorem], the quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ is $\mathbb{R}$-quadratic.

Let us prove the theorem for the case that $\varphi$ satisfies (i). By the assumption, for each $a \in B_1 \cap B_{in}$,

$$\|F(2^n ax) - 4a^2F(2^{n-1}x)\| \le \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $x \in {}_BM_1$. Using the fact that for each $a \in B$ and each $z \in {}_BM_2$, $\|az\| \le K|a| \cdot \|z\|$ for some $K > 0$, one can show that

$$\|a^2F(2^nx) - 4a^2F(2^{n-1}x)\| \le K|a|^2 \cdot \|F(2^nx) - 4F(2^{n-1}x)\| \le K\varphi(2^{n-1}x, 2^{n-1}x)$$

for all $a \in B_1 \cap B_{in}$ and all $x \in {}_BM_1$. So

$$\begin{aligned}\|F(2^nax) - a^2F(2^nx)\| &\le \|F(2^nax) - 4a^2F(2^{n-1}x)\| \\ &\quad + \|4a^2F(2^{n-1}x) - a^2F(2^nx)\| \\ &\le \varphi(2^{n-1}x, 2^{n-1}x) + K\varphi(2^{n-1}x, 2^{n-1}x)\end{aligned}$$

for all $a \in B_1 \cap B_{in}$ and all $x \in {}_BM_1$. Thus $2^{-2n}\|F(2^nax) - a^2F(2^nx)\| \to 0$ as $n \to \infty$ for all $a \in B_1 \cap B_{in}$ and all $x \in {}_BM_1$. Hence

$$Q(ax) = \lim_{n\to\infty} \frac{F(2^nax)}{2^{2n}} = \lim_{n\to\infty} \frac{a^2F(2^nx)}{2^{2n}} = a^2Q(x) \tag{2.1}$$

for all $a \in B_1 \cap B_{in}$ and all $x \in {}_BM_1$.

Let $b \in B_1 \setminus B_{in}$. Since $B_{in}$ is dense in $B$, there exists a sequence $\{b_m\}$ in $B_{in}$ such that $b_m \to b$ as $m \to \infty$. Put $a_m = \frac{1}{|b_m|}b_m$, then $a_m \to \frac{1}{|b|}b = b$ as $m \to \infty$ and $a_m \in B_1 \cap B_{in}$. Thus there exists a sequence $\{a_m\}$ in $B_1 \cap B_{in}$ such that $a_m \to b$ as $m \to \infty$, and so

$$\begin{aligned}\lim_{m\to\infty} Q(a_mx) &= \lim_{m\to\infty}\lim_{n\to\infty} 4^{-n}F(2^na_mx) \\ &= \lim_{m\to\infty} F(a_mx) = F(\lim_{m\to\infty} a_mx)) \quad \text{by (iii)} \\ &= \lim_{n\to\infty} 4^{-n}F(2^nbx) = Q(bx)\end{aligned} \tag{2.2}$$

for all $x \in {}_BM_1$. By (2.1),

$$\|Q(a_mx) - b^2Q(x)\| = \|a_m^2Q(x) - b^2Q(x)\| \to \|b^2Q(x) - b^2Q(x)\| = 0 \tag{2.3}$$

as $m \to \infty$. By (2.2),

$$\|4^{-n}F(2^na_mx) - Q(a_mx)\| \to \|4^{-n}F(2^nbx) - Q(bx)\| \tag{2.4}$$

as $m \to \infty$. By (2.3) and (2.4),

$$\begin{aligned}\|Q(bx) - b^2Q(x)\| &\le \|Q(bx) - 4^{-n}F(2^nbx)\| + \|4^{-n}F(2^nbx) - 4^{-n}F(2^na_mx)\| \\ &\quad + \|4^{-n}F(2^na_mx) - Q(a_mx)\| + \|Q(a_mx) - b^2Q(x)\| \\ &\to \|Q(bx) - 4^{-n}F(2^nbx)\| + \|4^{-n}F(2^nbx) - Q(bx)\| \quad \text{as} \quad m \to \infty \\ &\to 0 \quad \text{as} \quad n \to \infty\end{aligned}$$

for all $x \in {}_BM_1$. So

$$Q(bx) = b^2Q(x) \tag{2.5}$$

for all $b \in B_1 \setminus B_{in}$ and all $x \in {}_BM_1$. By (2.1) and (2.5), $Q(ax) = a^2Q(x)$ for all $a \in B_1$ and all $x \in {}_BM_1$.

Similarly, for the case that $\varphi$ satisfies (ii), one can obtain that

$$Q(ax) = \lim_{n\to\infty} \frac{F(2^{-n}ax)}{4^{1-n}} = \lim_{n\to\infty} \frac{a^2F(2^{-n}x)}{4^{1-n}} = a^2Q(x)$$

for all $a \in B_1$ and all $x \in {}_BM_1$.

Since $Q$ is $\mathbb{R}$-quadratic and $Q(ax) = a^2Q(x)$ for each $a \in B_1$,

$$Q(ax) = Q\Big(|a| \cdot \frac{a}{|a|}x\Big) = |a|^2 \cdot Q\Big(\frac{a}{|a|}x\Big) = |a|^2 \cdot \frac{a^2}{|a|^2} \cdot Q(x) = a^2Q(x)$$

for all $a \in B (a \neq 0)$ and all $x \in {}_BM_1$. And $Q(0x) = 0^2Q(x)$ for all $x \in {}_BM_1$. So the unique quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ is a $B$-quadratic mapping satisfying (iv).

But $Q(x) = \lim\limits_{n\to\infty} \frac{F(2^n x)}{4^n} = F(x)$ for all $x \in {}_BM_1$ if $\varphi$ satisfies (i), and

$$Q(x) = \lim_{n\to\infty} 4^{n-1} F\Big(\frac{x}{2^n}\Big) = F(x)$$

for all $x \in {}_BM_1$ if $\varphi$ satisfies (ii). So the $B$-quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ is the mapping $F$, as desired.

Now we prove the Hyers-Ulam-Rassias stability of another quadratic mapping in Banach modules over a unital $C^*$-algebra.

**Theorem 2.2.** *Let $F : {}_BM_1 \to {}_BM_2$ be a mapping with $F(0) = 0$ for which there exists a function $\varphi : {}_BM_1 \times {}_BM_1 \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x) < \infty,$$

$$\|a^2F(x+y) + a^2F(x-y) - 2F(ax) - 2F(ay)\| \le \varphi(x,y)$$

*for all $a \in B_1 \cap B_{in}$ and all $x, y \in {}_BM_1$, where $\widetilde{\varphi}(x)$ is as defined in the introduction. Then the mapping $F : {}_BM_1 \to {}_BM_2$ is a $B$-quadratic mapping.*

**Proof.** By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{R}$-quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ satisfying (iv).

Let us prove the theorem for the case that $\varphi$ satisfies (i). By the assumption, for each $a \in B_1 \cap B_{in}$,

$$\|a^2F(2^n x) - 4F(2^{n-1}ax)\| \le \varphi(2^{n-1}x, 2^{n-1}x)$$

for all $x \in {}_BM_1$. So

$$\begin{aligned}\|a^2F(2^n x) - F(2^n ax)\| &\le \|a^2F(2^n x) - 4F(2^{n-1}ax)\| + \|4F(2^{n-1}ax) - F(2^n ax)\| \\ &\le \varphi(2^{n-1}x, 2^{n-1}x) + \varphi(2^{n-1}ax, 2^{n-1}ax)\end{aligned}$$

for all $a \in B_1 \cap B_{in}$ and all $x \in {}_BM_1$. Thus $2^{-2n}\|a^2F(2^n x) - F(2^n ax)\| \to 0$ as $n \to \infty$ for all $a \in B_1 \cap B_{in}$ and all $x \in {}_BM_1$. Hence

$$a^2Q(x) = \lim_{n\to\infty} \frac{a^2F(2^n x)}{2^{2n}} = \lim_{n\to\infty} \frac{F(2^n ax)}{2^{2n}} = Q(ax)$$

for all $a \in B_1 \cap B_{in}$ and all $x \in {}_BM_1$.

Similarly, for the case that $\varphi$ satisfies (ii), one can obtain that

$$a^2Q(x) = \lim_{n\to\infty} \frac{a^2F(2^{-n} x)}{2^{2-2n}} = \lim_{n\to\infty} \frac{F(2^{-n} ax)}{2^{2-2n}} = Q(ax)$$

for all $a \in B_1 \cap B_{in}$ and all $x \in {}_BM_1$.

The rest of the proof is the same as the proof of Theorem 2.1. So the unique quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ is a $B$-quadratic mapping, which is the mapping $F$.

**Theorem 2.3.** *Let $F : {}_BM_1 \to {}_BM_2$ be a mapping with $F(0) = 0$ for which there exists a function $\varphi : {}_BM_1 \times {}_BM_1 \to [0,\infty)$ such that*

$$\widetilde{\varphi}(x) < \infty,$$

$$\|F(x+y) + F(x-y) - 2F(x) - 2F(y)\| \le \varphi(x,y),$$

$$\|F(ax) - a^2F(x)\| \le \varphi(x,x)$$

*for all $a \in B_1 \cap B_{in}$ and all $x, y \in {}_BM_1$, where $\widetilde{\varphi}(x)$ is as defined in the introduction. Then the mapping $F : {}_BM_1 \to {}_BM_2$ is a $B$-quadratic mapping.*

**Proof.** By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{R}$-quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ satisfying (iv).

Assume that $\varphi$ satisfies (i). Since $2^{-2n}\|F(2^n ax) - a^2F(2^n x)\| \to 0$ as $n \to \infty$ for all

$a \in B_1 \cap B_{in}$ and all $x \in {}_BM_1$,

$$Q(ax) = \lim_{n\to\infty} \frac{F(2^n ax)}{4^n} = \lim_{n\to\infty} a^2 \frac{F(2^n x)}{4^n} = a^2 Q(x)$$

for all $a \in B_1 \cap B_{in}$ and all $x \in {}_BM_1$.

Similarly, for the case that $\varphi$ satisfies (ii), one can obtain that

$$Q(ax) = \lim_{n\to\infty} \frac{F(2^{-n} ax)}{2^{2-2n}} = \lim_{n\to\infty} \frac{a^2 F(2^{-n} x)}{2^{2-2n}} = a^2 Q(x)$$

for all $a \in B_1 \cap B_{in}$ and all $x \in {}_BM_1$.

The rest of the proof is the same as the proof of Theorem 2.1. So the unique $\mathbb{R}$-quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ is a $B$-quadratic mapping, which is the mapping $F$.

## §3. Stability of the Quadratic Functional Equation in Banach Modules a over Banach Algebra

In this section, we prove the Hyers-Ulam-Rassias stability of a quadratic mapping in Banach modules over a unital Banach algebra.

**Theorem 3.1.** *Let $F : {}_BM_1 \to {}_BM_2$ be a mapping with $F(0) = 0$ for which there exists a function $\varphi : {}_BM_1 \times {}_BM_1 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x) < \infty,$$

$$\|F(ax + ay) + F(ax - ay) - 2a^2 F(x) - 2a^2 F(y)\| \le \varphi(x, y)$$

*for all $a \in B_1$ and all $x, y \in {}_BM_1$, where $\widetilde{\varphi}(x)$ is as defined in the introduction. If $F(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in {}_BM_1$, then the mapping $F : {}_BM_1 \to {}_BM_2$ is a $B$-quadratic mapping.*

**Proof.** By the same reasoning as the proof of Theorem 2.1, there exists a unique $\mathbb{R}$-quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ satisfying (iv).

Let us prove the theorem for the case that $\varphi$ satisfies (i). By the same method as the proof of Theorem 2.1, one can obtain that

$$Q(ax) = \lim_{n\to\infty} \frac{F(2^n ax)}{2^{2n}} = \lim_{n\to\infty} \frac{a^2 F(2^n x)}{2^{2n}} = a^2 Q(x)$$

for all $a \in B_1$ and all $x \in {}_BM_1$.

Similarly, for the case that $\varphi$ satisfies (ii), one can obtain that

$$Q(ax) = \lim_{n\to\infty} \frac{F(2^{-n} ax)}{2^{2-2n}} = \lim_{n\to\infty} \frac{a^2 F(2^{-n} x)}{2^{2-2n}} = a^2 Q(x)$$

for all $a \in B_1$ and all $x \in {}_BM_1$.

The rest of the proof is the same as the proof of Theorem 2.1. So the unique $\mathbb{R}$-quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ is a $B$-quadratic mapping, which is the mapping $F$. □

Similarly, one can obtain similar results to Theorem 2.2 and Theorem 2.3.

**Theorem 3.2.** *Let $F : {}_BM_1 \to {}_BM_2$ be a mapping with $F(0) = 0$ for which there exists a function $\varphi : {}_BM_1 \times {}_BM_1 \to [0, \infty)$ such that*

$$\widetilde{\varphi}(x) < \infty,$$

$$\|F(ax + ay) + F(ax - ay) - 2a^2 F(x) - 2a^2 F(y)\| \le \varphi(x, y)$$

*for all $a \in B_1 \cup \mathbb{R}^+$ and all $x, y \in {}_BM_1$, where $\widetilde{\varphi}(x)$ is as defined in the introduction. Then the mapping $F : {}_BM_1 \to {}_BM_2$ is a $B$-quadratic mapping.*

**Proof.** Put $a = 1 \in B_1 \cup \mathbb{R}^+$. By [4, Theorem 2], there exists a unique quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ satisfying (iv).

Let us prove the theorem for the case that $\varphi$ satisfies (i). By the same method as the proof of Theorem 2.1, one can obtain that

$$Q(ax) = \lim_{n\to\infty} \frac{F(2^n ax)}{2^{2n}} = \lim_{n\to\infty} \frac{a^2 F(2^n x)}{2^{2n}} = a^2 Q(x)$$

for all $a \in B_1 \cup \mathbb{R}^+$ and all $x \in {}_BM_1$.

Similarly, for the case that $\varphi$ satisfies (ii), one can obtain that

$$Q(ax) = \lim_{n\to\infty} \frac{F(2^{-n} ax)}{2^{2-2n}} = \lim_{n\to\infty} \frac{a^2 F(2^{-n} x)}{2^{2-2n}} = a^2 Q(x)$$

for all $a \in B_1 \cup \mathbb{R}^+$ and all $x \in {}_BM_1$.

The rest of the proof is similar to the proof of Theorem 2.1. So the unique quadratic mapping $Q : {}_BM_1 \to {}_BM_2$ is a $B$-quadratic mapping, which is the mapping $F$.

**Remark 3.1.** If the second inequality in the statement of Theorem 2.1 is replaced by

$$\|F(ax+y) + F(ax-y) - 2a^2F(x) - 2F(y)\| \le \varphi(x,y),$$

then

$$\|F(ax+x) + F(ax-x) - 2a^2F(x) - 2F(x)\| \le \varphi(x,x),$$
$$\|F(ax+x) + F(ax-x) - 2F(ax) - 2F(x)\| \le \varphi(ax,x).$$

So

$$\|2F(ax) - 2a^2F(x)\| \le \varphi(x,x) + \varphi(ax,x),$$

hence the result does also hold as a corollary of Theorem 2.3.

Similarly, one can prove the stability of the other quadratic mappings in Banach modules over a unital $C^*$-algebra or a unital Banach algebra, and obtain similar results to Theorem 2.2 and Theorem 2.3.

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