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DIMENSIONS OF SELF-AFFINE SETS WITH OVERLAPS**

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Abstract

The authors develop an algorithm to show that a class of self-affine sets with overlaps can be viewed as sofic affine-invariant sets without overlaps, thus by using the results of [11] and [10], the Hausdorff and Minkowski dimensions are determined.

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§1. Introduction

1.1. McMullen's Self-Affine Carpets

McMullen[11] and Bedford[1] independently studied plane sets constructed as follows. Let $1 < m \le n$ be integers. By drawing $n-1$ vertical lines and $m-1$ horizontal lines, partition the unit square into nm congruent rectangles. Let S be a subcollection of these rectangles; erase all other rectangles and partition the remaining ones into nm congruent subrectangles, again keeping only those which correspond to the pattern S. Repeating this procedure and infinitum, a compact set K is obtained, which is called McMullen's self-affine carpet.

Let $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ be the 2-torus. The McMullen's self-affine carpets can be viewed as a subsets of $\mathbb{T}^2$. They are invariant under the toral endomorphism

$$T = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}.$$

Denote by $D_0 = \{0, 1, \cdots, n-1\} \times \{0, 1, \cdots, m-1\}$. Let $D \subset D_0$. Then the corresponding self-affine carpets are

$$K(T, D) = \Big\{ \sum_{k=1}^{\infty} T^{-k} d_k : d_k \in D \Big\}.$$

The set $K(T, D)$ can be also viewed as the attractor for the family of $|D|$ affine contractions

$$f_j \begin{pmatrix} x \\ y \end{pmatrix} = T^{-1} \left(\begin{pmatrix} x \\ y \end{pmatrix} + d_j \right), \qquad d_j \in D,$$

i.e., $K(T, D)$ is the unique compact invariant set of the above iterated function system[8].

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Though these sets are the simplest self-affine sets, when $n > m$ the different expansion coefficients in the horizontal and vertical directions make the quantitative analysis of these sets quite delicate; McMullen and Bedford succeeded in calculating their Hausdorff dimension and showing it is usually strictly smaller than the Box dimension. For instance, the set determined by the pattern in Fig. 1 has Hausdorff dimension $\log_2(1+2^{\log_3 2})$; for a general pattern S, the dimension is $\log_m\Big(\sum\limits_{i=0}^{m-1} b_i^{\log_n m}\Big)$, where b_i is the number of rectangles in row i of S.

Fig. 1. Typical McMullen's example

1.2. Sofic Affine-Invariant Sets

Let $G=\langle V,E\rangle$ be a finite directed graph in which loops and multiple edges are allowed. Let D be the set of symbols, $|D|=l$. Suppose the edges of G are labeled in symbols in D in a right resolving fashion: no two edge emanating from the same vertex marked the same symbol. Then the symbol sequences which arise from infinite paths in G form a sofic system on l symbols. That is

$$\Omega=\{\{d_k\}_{k\geq 1}:\ \{d_k\}_{k\geq 1} \text{ is an infinite path in } G, \text{ where } d_k\in D\}.$$

Definition 1.1. *Let $G=\langle V,E\rangle$ be a directed graph and $D=\{0,1,\cdots,n-1\}\times\{0,1,\cdots,m-1\}$ be the set of symbols. Suppose $\Omega\subset D^{\mathbb{N}}$ is the resulting sofic system. We call*

$$R_T(\Omega)=\Big\{\sum_{k=1}^{\infty}T^{-k}d_k:\ \{d_k\}_{k\geq 1}\in\Omega\Big\}$$

a T-invariant sofic set.

In some papers such sets are called recurrent sets[2, 3] or graph-directed sets[12].

The edges in G are conveniently represented by the adjacency matrix A where for any two vertices v,w in G, $A(v,w)$ is the number of edges in E from v to w. Now we construct m matrices $A_0,A_1,\cdots,A_{m-1}$ as follows: A_j is a $|V|\times|V|$ matrix such that for vertices v,v' in G, $A_j(v,v')$ denotes the number of edges from v to v' in G such that the second coordinate of their label is j. R. Kenyon and Y. Peres[10] studied the dimensions of the sofic affine-invariant sets and showed that

$$\dim_H R_T(\Omega)=\lim_{k\to\infty}\frac{1}{k}\log_m\sum_{0\leq i_1,\cdots,i_k\leq m-1}\|A_{i_k}\cdot A_{i_{k-1}}\cdots A_{i_1}\|^{\alpha},$$

where $\alpha=\log m/\log n\leq 1$ (the choice of norm is clearly immaterial).

Example 1.1. Here we give an example of sofic invariant set[10, Example 4.2]. Let

$$G=\langle V,E\rangle \quad\text{and}\quad V=\{a,b\}$$

and the label set

$$D=\{0,1,2\}\times\{0,1\}.$$

The graph is shown by Fig. 2a.

Fig. 2a

A sofic system can also be illustrated by a group patterns. For this example the sofic system is illustrated by Fig. 2b.

Fig. 2b

The adjacency matrix of this sofic system is

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A_1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

[10] has calculated that $\dim_H K = 0.883127\cdots$ and $\dim_B K = 0.8875138\cdots$.

1.3. The Main Result

Example 1.2. Kenyon and Peres[10] have considered the following one parameter family of sets. Let $0 \le u \le 1/2$ and let K_u be the attractor for the affine maps

$$f_j \begin{pmatrix} x \\ y \end{pmatrix} = T^{-1}\left(\begin{pmatrix} x \\ y \end{pmatrix} + d_j\right), \qquad d_j \in D,$$

where $T = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and $D = \left\{\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ u \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$. See Fig. 3a.

Fig. 3a

By a general result of Deliu et al[4] and Falconer[7], for all $0 \le u \le 1/2$,

$$\dim_B K_u = 2 - \log_3 2 = 1.36907\cdots.$$

For the Hausdorff dimension, they observed that $2K_{1/2}$ is a sofic set determined by the substitution in Fig. 3b and clearly

$$\dim_H K_{1/2} = \dim_H 2K_{1/2} = 1.36629\cdots.$$

Fig. 3b

They claimed that the method by involving more work might apply to calculating $\dim_H K_u$ for any rational value of u.

In this paper we introduce a method which not only may apply to the above case, but also is possible to deal with the self-affine sets with overlaps.

Let $T = \begin{pmatrix} n & 0 \\ 0 & m \end{pmatrix}$ with $1 \le m \le n$, $D = \{d_1, d_2, \cdots, d_N\} \subset \mathbb{R}^2$. Let $d_j = (d_j(x), d_j(y))$. Without loss of generality we can assume that

$$0 \le d_j(x) \le n-1, \quad 0 \le d_j(y) \le m-1.$$

Let $\{f_j\}_{j=1}^N$ be a family of contraction maps defined by

$$f_j \begin{pmatrix} x \\ y \end{pmatrix} = T^{-1}\left(\begin{pmatrix} x \\ y \end{pmatrix} + d_j \right), \qquad d_j \in D, \tag{1.1}$$

and $K(T, D)$ by the associated invariant set such that

$$K(T, D) = \bigcup_{j=1}^{N} f_j(K(T, D)).$$

The typical pattern of these sets is shown in Fig. 4. Usually they are self-affine sets with overlaps. Generally, it is very difficult to calculate the dimensions of self-similar sets with overlaps , and it is awfully hard for self-affine sets with overlaps.

Fig. 4

In this paper we also require that the digits set D is rational, that is, the coordinates of digits of D are rational (the case that D is irrational, that is, some coordinates of digits of D may be irrational, seems difficult). Our main result is

Theorem 1.1. *Let $D = \{d_1, \cdots, d_N\}$ be rational and $\{f_j\}_{j=1}^N$ be a function system defined as in* (1.1). *Then there is a sofic affine invariant set $R(\Omega)$ such that $R(\Omega) = K(T, D)$.*

In fact we develop an algorithm to construct a sofic system Ω (that is, the graph directed set G with labeled edge) such that $K(T, D) = R(\Omega)$ and the Hausdorff dimension follows. [9] has used this idea to study the self-similar sets with overlaps, but indeed it also works for the setting of self-affine sets.

Example 1.3. Let $T = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and

$$D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

See Fig. 5a.

Fig. 5a

We will show that the equivalent sofic system is

Fig. 5b

The adjacency matrix is

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

and

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The details are in Subsection 2.3.

Example 1.4. Let $T = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ and

$$D = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{2}{3} \\ \frac{1}{2} \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}.$$

See Fig. 6a.

Fig. 6a

We will show that the equivalent sofic system is

Fig. 6b

The adjacency matrix is

$$A=\begin{pmatrix}1&1&1&1&1\\0&2&1&1&1\\0&0&1&0&0\\1&0&0&1&0\\0&0&0&0&1\end{pmatrix}$$

and

$$A_0=\begin{pmatrix}0&1&0&0&1\\0&1&0&0&1\\0&0&1&0&0\\1&0&0&1&0\\0&0&0&0&0\end{pmatrix},$$

$$A_1=\begin{pmatrix}1&0&1&1&0\\0&1&1&1&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{pmatrix}.$$

§2. Algorithm to Construct Sofic Invariant Sets

2.1. Label of Cylinders

Let $T=\begin{pmatrix}n&0\\0&m\end{pmatrix}$ and $D=\{d_1,\cdots,d_N\}$ be rational. $\{f_i\}_{i=1}^N$ is the function system defined in (1.1) and denote by $K(T,D)$ its invariant sets. $K(T,D)$ most likely is a self-affine set with overlaps. Even if $K(T,D)$ has not overlaps, usually it is not McMullen's affine carpets (see Fig. 4) and thus very hard to calculate its dimensions.

In this section we develop an algorithm to construct a sofic invariant set $R(\Omega)$ such that

$$K(T,D)=R(\Omega).$$

Let $D_0=\{0,1,\cdots,n-1\}\times\{0,1,\cdots,m-1\}$. Let $\{g_j\}_{j=1}^{nm}$ be a function system defined by

$$g_j\begin{pmatrix}x\\y\end{pmatrix}=T^{-1}\left(\begin{pmatrix}x\\y\end{pmatrix}+e_j\right),\qquad e_j\in D_0.$$

Then the associated invariant set $K(T, D_0)$ is exactly the unit square $[0, 1] \times [0, 1]$, denoted by K_0. We call

$$g_{i_1 \cdots i_k}(K_0) = g_{i_1} \circ g_{i_2} \circ \cdots \circ g_{i_k}(K_0), \qquad 1 \le i_j \le nm,$$

a k-cylinder of $K(T, D_0)$. Denote the collection of all k-cylinder of $K(T, D_0)$ by $\mathcal{C}_{D_0,k}$. Similarly we call

$$f_{i_1 \cdots i_k}(K_0) = f_{i_1} \circ f_{i_2} \cdots \circ f_{i_k}(K_0), \qquad 1 \le i_j \le N,$$

a k-cylinder of $K(T, D)$. Denote the collection of all k-cylinders of $K(T, D)$ by $\mathcal{C}_{D,k}$. By convention, K_0 is defined as a 0-cylinder. A k-cylinder of $K(T, D)$ or $K(T, D_0)$ is a rectangle of size $n^{-k} \times m^{-k}$ and sides parallel the coordinate axis.

Now we concentrate on the system $\{g_i\}_{i=1}^{nm}$. We study how the k-cylinders of $\{g_i\}_{i=1}^{nm}$ intersect the k-cylinders of $\{f_i\}_{i=1}^{N}$.

Definition 2.1. *Suppose $\tau \in \mathcal{C}_{D_0,k}$. The neighborhood of τ with respect to $\mathcal{C}_{D,k}$ is defined as*

$$\mathcal{N}(\tau) = \{\delta :\ \delta \in \mathcal{C}_{D,k} \text{ and the interior of } \delta \cap \tau \text{ not empty}\}.$$

Notice that a k-cylinder of $\mathcal{C}_{D,k}$, say δ, is a rectangle of size $n^{-k} \times m^{-k}$ and its sides parallel to the coordinates axis, so it is completely determined by its left-lower corner, denoted this point by $\bar{\delta}$. Now we give each cylinder of $\mathcal{C}_{D_0,k}$ a label.

Definition 2.2. *The label of $\tau \in \mathcal{C}_{D_0,k}$ is defined as*

$$L(\tau) = \{T^k(\bar{\delta} - \bar{\tau}) :\ \delta \in \mathcal{N}_k(\tau)\}.$$

Denote

$$\mathcal{L} = \bigcup_{k \ge 0} \{L(\tau) :\ \tau \in \mathcal{C}_{D_0,k}\}.$$

Proposition 2.1. *$\mathcal{L}$ is a finite set.*

Proof. For $\tau \in \mathcal{C}_{D_0,k}$, $\delta \in \mathcal{C}_{D,k}$,

$$\bar{\tau} = \sum_{j=1}^{k} T^{-j} e_{i_j}, \qquad e_{i_j} \in D_0;$$

$$\bar{\delta} = \sum_{j=1}^{k} T^{-j} d_{i_j}, \qquad d_{i_j} \in D.$$

So

$$T^k(\bar{\delta} - \bar{\tau}) = \sum_{j=1}^{k} T^{k-j}(d_{i_j} - e_{i_j}).$$

Suppose that M is an integer such that Md_j is integer for each $d_j \in D$. Then the coordinates of $MT^k(\bar{\delta} - \bar{\tau})$ are integers. $\tau \cap \delta \ne \emptyset$ implies

$$T^k(\bar{\delta} - \bar{\tau}) \in [-(n-1), n-1] \times [-(m-1), m-1],$$

so

$$MT^k(\bar{\delta} - \bar{\tau}) \in [-M(n-1, M(n-1)] \times [-M(m-1), M(m-1)].$$

Then for any $\tau \in \mathcal{C}_{D_0,k}$,

$$L(\tau) \subset \frac{([-M(n-1), M(n-1)] \times [-M(m-1), M(m-1)]) \cap \mathbb{Z}^2}{M}.$$

Notice the right side of the above formula is a finite set, so $\mathcal{L}$ is also a finite set.

2.2. Sofic Invariant Set

It is possible that for some cylinder τ, it does not meet any cylinder of $K(T,D)$ and it has no contribution. This means that $\emptyset$ is an element of $\mathcal{L}$. Delete empty from $\mathcal{L}$ we get $\mathcal{L}^*$, that is, $\mathcal{L}^* = \mathcal{L} \setminus \{\emptyset\}$.

A k-cylinder of $K(T, D_0)$ will be partitioned into $n \times m$ $(k+1)$-cylinders. Marking each sub-cylinder by its label and deleting the sub-cylinders with label $\emptyset$, we get a pattern.

Proposition 2.2. *Suppose that τ and τ' are two cylinders of $K(T, D_0)$. If $L(\tau) = L(\tau')$, then the pattern of τ coincides with the pattern of τ'.*

Proof. Suppose that τ is a k-cylinder and its label is $L(\tau)$. τ can be partitioned into $n \times m$ $(k+1)$-cylinders. The neighbourhood of τ is

$$\mathcal{N}(\tau) = \{T^{-k}e + \bar{\tau} : \ e \in L(\tau)\}.$$

For a vector v of the plane, we denote v_x the x-coordinate of v and v_y the y-coordinate of v. Let $f = (f_x, f_y)$ be one of the sub-cylinders, $0 \le f_x \le m-1$, $0 \le f_y \le n-1$. Denote

$$\mathcal{N} = \{\bar{e} + T^{-(k+1)}d : \ e \in \mathcal{N}(\tau), \ d \in D\}.$$

Then the neighbour of f can only come from $\mathcal{N}$, thus the label of f is

$$\begin{aligned}
L(f) &= \{T^{k+1}(\bar{e} - (\bar{\tau} + T^{-(k+1)}f)) : e \in \mathcal{C}_{D_0,k+1}, \ e \cap f \ne \emptyset\} \\
&= \{T^{k+1}(e - \bar{\tau}) - f) : e \in \mathcal{N}, \ e \cap f \ne \emptyset\} \\
&= \{T^{k+1}(\bar{e} + T^{-(k+1)}d - \bar{\tau}) - f : e \in \mathcal{N}(\tau), \ d \in D, \\
&\quad \text{As a (k+1)-cylinder } (e + T^{-(k+1)}d) \cap f \ne \emptyset\} \\
&= \{T^{k+1}(\bar{e} - \bar{\tau}) + d - f : e \in \mathcal{N}(\tau), \ d \in D, \\
&\quad |(T^{k+1}(\bar{e} - \bar{\tau}) + d - f)_x| \le 1, \ |(T^{k+1}(\bar{e} - \bar{\tau}) + d - f)_y| \le 1\}.
\end{aligned}$$

Therefore we get

$$\begin{aligned}
L(f) = \{&Te + d - f : e \in L(\tau), \ d \in D, \\
&|(Te + d - f)_x| \le 1, \ |(Te + d - f)_y| \le 1\}.
\end{aligned} \tag{2.1}$$

The labels of the sub-cylinders are completely determined by $L(\tau)$, so the pattern of τ is completely determined by $L(\tau)$.

Suppose that $\mathcal{L}^*$ has l elements. By the above proposition we will obtain l patterns. These patterns determine a sofic system Ω and a sofic invariant set $R(\Omega)$. Our last step is to show that $K(T,D)$ and $R(\Omega)$ coincide.

Theorem 2.1. *With the above notation we have*

$$K(T,D) = R(\Omega).$$

Proof. We abbreviate $K(T,D)$ as K and $R(\Omega)$ as R. Let $K(k)$ be the union of the k-cylinders of K and $R(k)$ be the k-th iteration of the sofic system. Then by [5],

$$K(k) \to K \quad \text{and} \quad R(k) \to R \qquad \text{in } \ d_H,$$

where d_H denotes the Hausdorff metric. By the construction of $R(\Omega)$, we have

$$d_H(K(k), R(k)) \le m^{-k}.$$

So for any $k \in \mathbb{N}$,

$$d_H(K,R) \le d_H(K, K(k)) + d_H(R, R(k)) + d_H(K(k), R(k)) \le 3m^{-k},$$

thus $d_H(K,R) = 0$ which yields $K = R$.

2.3. Algorithm

Now we make the algorithm more practical.

(1) We begin with the 0-cylinder whose label is $\left\{\begin{pmatrix}0\\0\end{pmatrix}\right\}$. We put it as the initial vertex into V.

(2) We work out the patterns of all vertices in V by the fomula (2.1). We may get some new types of label. Regard them as new vertices and add them to V.

(3) Repeat the procedure 2 until there is no new type of label occurring. By Proposition 2.1 the procedure will stop at finite steps.

Now we get the vertex set V and the patterns, and thus get the desired sofic system.

Example 2.1. Using the above algorithm we calculate that $V = \{A, B, C\}$, where

$$A = \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\},$$
$$B = \left\{\begin{pmatrix}0\\-\frac{1}{2}\end{pmatrix}\right\},$$
$$C = \left\{\begin{pmatrix}0\\\frac{1}{2}\end{pmatrix}\right\}.$$

The corresponding patterns are exactly the same as Fig. 3b.

Example 2.2. Using the above algorithm we calculate that $V = \{A, B, C\}$, where

$$A = \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\},$$
$$B = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\frac{2}{3}\\0\end{pmatrix}\right\},$$
$$C = \left\{\begin{pmatrix}-\frac{1}{3}\\0\end{pmatrix}\right\}.$$

The corresponding patterns are depicted by Fig. 5b.

Example 2.3. We calculate that

$$V = \{A, B, C, D, E\},$$

where

$$A = \left\{\begin{pmatrix}0\\0\end{pmatrix}\right\},$$
$$B = \left\{\begin{pmatrix}0\\0\end{pmatrix}, \begin{pmatrix}\frac{2}{3}\\\frac{1}{2}\end{pmatrix}\right\},$$
$$C = \left\{\begin{pmatrix}\frac{2}{3}\\-\frac{1}{2}\end{pmatrix}\right\},$$
$$D = \left\{\begin{pmatrix}-\frac{1}{3}\\-\frac{1}{2}\end{pmatrix}\right\},$$
$$E = \left\{\begin{pmatrix}-\frac{1}{3}\\\frac{1}{2}\end{pmatrix}\right\}.$$

The corresponding patterns are depicted by Fig. 6b.

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