Chin. Ann. Math.
24B:3(2003),285–292.

THEOREMS OF BARTH-LEFSCHETZ TYPE ON KÄHLER MANIFOLDS WITH PARTIALLY POSITIVE CURVATURE**

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Abstract

The author obtains the theorems of Barth-Lefschetz type on Kähler manifolds with partially positive bisectional curvature without the assumption of nonnegative bisectional curvature. Some applications of the results to holomorphic mappings are given.

Keywords Kähler manifold, Bisectional curvature, Holomorphic mapping
2000 MR Subject Classification 58E10, 32Q15, 57R70, 58E05
Chinese Library Classification O176.3 **Document Code** A
Article ID 0252-9599(2003)03-0285-08

§1. Introduction

Let V be a v-dimensional compact Kähler manifold. In the case of $V = CP^v$, Lefschetz[9] stated now known as the Lefschetz theorem on hyperplane sections. This celebrated result has been generalized by many authors (see, for example, [1, 11, 5] and references therein). In particular they proved a "connectedness" theorem for closed local complete intersections $M, N \subset CP^v$ of complex dimensions m, n respectively. It is proved that the relative homotopy groups satisfy

$$\pi_j(N, N \cap M) = 0, \quad j \leq \min\{n + m - v, 2m - v + 1\}.$$

On the other hand using the second variation formula of arc length Frankel[4] proved a "connectedness" theorem for complex submanifolds of a Kähler manifold of positive sectional curvature. Kenmotsu and Xia[10] generalized the intersection result of Frankel by using partially positive bisectional curvature. The relevance of Frankel's work to the theorem of Barth and Larsen was noted by Fulton[5]. In 1998, Schoen and Wolfson[15] obtained an elegant Morse-theoretic proof and some generalizations of the theorem of Barth and Larsen. Also in 1984 in the case of codimension 1, Wu[17] proved that in another way one could be led to stronger topological conclusions. He showed that a partial positivity condition on the bisectional curvature can be utilized to prove q-completeness ($q > 1$) for the complex manifolds (for the definition of q-positive bisectional curvature, see §2). For example, let M

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**Project supported by the CNPq of Brazil.

be a compact Kähler manifold and let N be a nonsingular complex hypersurface in M. If the bisectional curvature of M is q-positive in a neighborhood of N, then $M-N$ is strongly q-pseudoconvex. If, in addition, the bisectional curvature is everywhere q-nonnegative on M, then $M-N$ is q-complete.

The notion of q-positive bisectional curvature was studied independently in several papers (e.g., [7, 17]) and the determination of this integer q for Hermitian symmetric spaces was carried out in [7].

The aim of this paper is to obtain some estimates for the index of geodesics normally connecting two complex manifolds under the condition of partially positive bisectional curvatures (without the assumption of nonnegative bisectional curvature) and combine some abstract results obtained in [15] to give some topological applications. We present in §3 a setup for the calculation of the Morse index of geodesics with respect to submanifolds. This approach is different from that of [15] and has some advantages for the proof and applications. (In fact it can be seen from our proofs that the proofs of Theorems 2.1 and 2.5 in [15] are not complete). For more applications of our index estimate we introduce the notion of q-positive normal bisectional curvature with respect to a submanifold which is naturally appeared in applications. For example, as special cases of Theorem 3.4 and Theorem 4.2 we have

Theorem 1.1. *Let V be a v-dimensional compact Kähler manifold and let M, N be complex submanifolds in V. Suppose that V has q-positive bisectional curvature. Then the homomorphism induced by the inclusion*

$$i_* : \pi_j(N, N\cap M) \to \pi_j(V, M)$$

is an isomorphism for $j \le m+n-v+1-q$ and is a surjection for $j = m+n-v+2-q$.

Theorem 1.2. *Let V be a v-dimensional compact Kähler manifold of positive bisectional curvature and let M be complex submanifolds in V and Δ is the diagonal of $V\times V$. Then the homomorphism induced by the inclusion*

$$i_* : \pi_j(\Delta, \Delta\cap M) \to \pi_j(V\times V, M)$$

is an isomorphism for $j \le m-v$ and is a surjection for $j = m-v+1$.

The manifolds with partially positive curvature cover many known examples and a refined case-by-case study of the positivity of the curvature of irreducible symmetric spaces of the compact type has been carried out by Lee[12]. Combining these results we actually generalize the theorems of Barth-Lefschetz type for CP^v to compact Hermitian symmetric spaces. Also as a special case of another application we obtain that any holomorphic map from V to itself has a fixed point provided V has positive bisectional curvature.

It should be remarked that one can also use the method here to discuss the corresponding problem in [6]. One can reprove and extend the connectedness theorem and intersections in a geometric way.

§2. Notations and Preliminaries

Let V be a v-dimensional compact Kähler manifold and TV be its holomorphic tangent bundle. The letters J, G, R will denote, respectively, the complex structure tensor, the (complex-valued) Kähler metric tensor, and the curvature tensor of the Riemannian metric which is the real part of G; in turn this Riemannian metric will always be denoted by $\langle\, , \rangle$

and its norm by $|\cdot|$. If X, Y are vectors in T_xV, as in [8] denote by $H(X,Y)$ the bisectional curvature determined by X and Y, i.e.,

$$H(X,Y) = \begin{cases} 0 & \text{if } X \text{ or } Y = 0, \\ \frac{\langle R(X,JX)Y,JY\rangle}{|X|^2|Y|^2} & \text{if } X \neq 0, Y \neq 0. \end{cases} \tag{2.1}$$

Recall that $H(X,Y)$ depends only on the planes spanned by X, JX and Y, JY, furthermore, by Bianchi's first identity,

$$H(X,Y) = \frac{1}{|X|^2|Y|^2}\{\langle R(X,Y)X,Y\rangle + \langle R(X,JY)X,JY\rangle\}. \tag{2.2}$$

Definition 2.1. *V is said to have q-positive bisectional curvature in a subset W $(1 \leq q \leq v)$ if for every $x \in W$ and for every orthonormal basis $\{e_1, Je_1, \cdots, e_v, Je_v\}$ of T_xV, $\sum\limits_{i=1}^{q} H(X,e_i) > 0$ for all unit vectors $X \in T_xV$. We can similarly define q-nonnegative bisectional curvature by replacing the last strict inequality with "$\geq$".*

Hermitian symmetric spaces of the compact type furnish many examples of Kähler manifolds with q-positive bisectional curvatures. Let V be an irreducible Hermitian symmetric space of compact type. From the calculations in [2] and [3], one can read off the smallest integer q such that $1 \leq q \leq \dim_C V - 1$ and the bisectional curvature of V is q-positive. The general situation where V may not be irreducible is handled by noting that if V_1 and V_2 have q_1-positive and q_2-positive bisectional curvatures respectively, then $V_1 \times V_2$ in the product metric has q-positive bisectional curvature, where

$$q = \max\{q_1 + \dim_C V_2, \quad q_2 + \dim_C V_1\}.$$

§3. The Index Estimate and Homotopy Results

In this section we will get some estimates of Morse index for the nontrivial critical point of the energy function.

Let M, N be two complex submanifolds in V. We denote, by $\Omega(V; M, N)$ (Ω, for short), the space of paths $\gamma : [0,1] \to V$ with $\gamma(0) \in M, \gamma(1) \in N$. The energy function E of the path defines a function on Ω given by

$$E(\gamma) = \frac{1}{2}\int_0^1 |\dot\gamma(t)|^2 dt. \tag{3.1}$$

It is shown in [15] that $\gamma : [0,1] \to V$ is a critical point of E if and only if γ is a smooth geodesic which is normal to M and N at $\gamma(0)$, and $\gamma(1)$, respectively.

If γ is a critical point of E, then the second variation of E in the directions $W_1, W_2 \in T_\gamma\Omega$, denoted $E_{**}(W_1, W_2)$, is given by

$$E_{**}(W_1, W_2) = -\int_0^1 \Big\langle W_2, \frac{D^2W_1}{dt^2} + R(\dot\gamma, W_1)\dot\gamma\Big\rangle dt. \tag{3.2}$$

Recall that the Morse index of a critical point γ is defined as the dimension of the maximal subspace of $T_\gamma\Omega$ on which the restriction of the symmetric bilinear form E_{**} is negative definite.

As seen above, if $\gamma : [0,1] \to V$ is a critical point of the energy function E defined in (3.1), then γ is a smooth geodesic normal to M and N at $\gamma(0)$ and $\gamma(1)$ respectively. Let $W_1, W_2 \in T_\gamma\Omega$. If γ is a critical point of the energy function E, the second variation formula

(3.2) can be rewritten as

$$E_{**}(W_1,W_2)=\langle\nabla_{W_1}W_2,\dot\gamma\rangle|_0^1+\int_0^1\langle\nabla_{\dot\gamma}W_1,\nabla_{\dot\gamma}W_2\rangle-\int_0^1 R(\dot\gamma,W_1)\dot\gamma,W_2\rangle dt. \tag{3.3}$$

Define the subspace $S\subset T_\gamma\Omega$ by $S:=S_1\cap S_2$, where

$$S_1:=\{W(t):W(t) \text{ is a parallel transport along } \gamma \text{ of a vector } v\in T_{\gamma(0)}M\},$$
$$S_2:=\{W(t):W(t) \text{ is a parallel transport along } \gamma \text{ of a vector } v\in T_{\gamma(1)}N\}.$$

Clearly both the subspaces S_1 and S_2 are J-invariant and therefore they are complex subspaces. So S is also a complex subspace and its complex dimension $s:=\dim S=\dim S_1+\dim S_2-\dim(S_1+S_2)\geq m+n-v+1$. Since for all $X\in S$, $\nabla_{\dot\gamma}X=0$. Then the restriction of the index form of E_{**} to S is

$$B(X,Y):=E_{**}(X,Y)=\langle\nabla_XY,\dot\gamma\rangle|_0^1-\int_0^1\langle R(\dot\gamma,X)\dot\gamma,Y\rangle dt. \tag{3.4}$$

Since the second fundamental forms $\mathrm{II}_{\dot\gamma(0)}(X,Y):=\langle\nabla_XY,\dot\gamma(0)\rangle$, $\mathrm{II}_{\dot\gamma(1)}(X,Y)$ and the curvature operator $\langle R(\dot\gamma,X)\dot\gamma,Y\rangle$ are symmetric with respect to X,Y, B is a symmetric bilinear form on the Euclidean space S with the induced inner product from $T_{\gamma(0)}M$. So there exist $2s$ orthonormal eigenvectors $e_1,e_2,\cdots,e_{2s}$ and $2s$ corresponding eigenvalues $\lambda_1\leq\lambda_2\leq\cdots\leq\lambda_{2s}$, namely,

$$B(e_i,X)=\lambda_i(e_i,X)\quad\text{for all}\quad X\in S, \tag{3.5}$$

where $i=1,2,\cdots,2s$. Thus we have

$$\operatorname{index}(\gamma)\geq\ \text{the number of negative } \lambda'\text{s}. \tag{3.6}$$

For any $X\in S$, we have $JX\in S$. Thus

$$B(X,X)=\langle\nabla_XX,\dot\gamma\rangle|_0^1-\int_0^1\langle R(\dot\gamma,X)\dot\gamma,X\rangle dt, \tag{3.7}$$

$$B(JX,JX)=\langle\nabla_{JX}JX,\dot\gamma\rangle|_0^1-\int_0^1\langle R(\dot\gamma,JX)\dot\gamma,JX\rangle dt. \tag{3.8}$$

Since $J\dot\gamma(0)$ is normal to M, $\langle\nabla_XY,J\dot\gamma(0)\rangle=\langle\nabla_YX,J\dot\gamma(0)\rangle$ for any $X,Y\in T_{\gamma(0)}M$, and

$$\begin{aligned}&\langle\nabla_{JX}JX,\dot\gamma(0)\rangle=\langle J\nabla_{JX}X,\dot\gamma(0)\rangle=-\langle\nabla_{JX}X,J\dot\gamma(0)\rangle\\&=-\langle\nabla_XJX,J\dot\gamma(0)\rangle=-\langle J\nabla_XX,J\dot\gamma(0)\rangle=-\langle\nabla_XX,\dot\gamma(0)\rangle.\end{aligned} \tag{3.9}$$

Similarly we have

$$\langle\nabla_{JX}JX,\dot\gamma(1)\rangle=-\langle\nabla_XX,\dot\gamma(1)\rangle. \tag{3.10}$$

Summing up (3.7) and (3.8) and using (3.9) and (3.10), we have,

$$\begin{aligned}B(X,X)+B(JX,JX)&=-\int_0^1\{\langle R(\dot\gamma,JX)\dot\gamma,JX\rangle+\langle R(\dot\gamma,JX)\dot\gamma,JX\rangle\}dt\\&=-\int_0^1\langle R(\dot\gamma,J\dot\gamma)X,JX\rangle\,dt.\end{aligned} \tag{3.11}$$

To continue our discussion we need the following definition.

Definition 3.1. *V is said to have q-positive normal bisectional curvature with respect to a submanifold M $(1\leq q\leq m)$ if for every $x\in V$, all geodesic $\gamma:[0,d]\to V$ with $\gamma(0)\in M,\gamma(d)=x$ and $\dot\gamma(0)\perp T_{\gamma(0)}M$, and for every orthonormal basis $\{e_1,Je_1,\cdots,e_m,Je_m\}$*

of $T_{\gamma(0)}M$, $\sum\limits_{i=1}^{q} H(\gamma(t), e_i(t)) > 0$, *where* $e_i(t)$ *is the parallel transport of* e_i *along* γ. *We can similarly define q-nonnegative normal bisectional curvature by replacing the last strict inequality with "*$\geq$*".*

We are now ready to state and prove our index estimate theorem.

Theorem 3.1. *Let* V *be a v-dimensional compact Kähler manifold and let* M, N *be complex submanifolds in* V. *Suppose that* V *has q-positive normal bisectional curvature with respect to* M *(or* N*). Let* γ *be a smooth geodesic which is normal to* M *and* N *at* $\gamma(0)$, *and* $\gamma(1)$, *respectively. Then*

$$\operatorname{index}(\gamma) \geq s - q + 1 \geq m + n - v + 2 - q. \tag{3.12}$$

Proof. Denote $p := s - q + 1$. Let $\lambda_1 \leq \cdots \leq \lambda_{2s}$ be the eigenvalves of the symmetric bilinear form E_{**} on the subspace S. It suffices to show that $\lambda_1 \leq \cdots \leq \lambda_p < 0$. Otherwise we assume for the sake of contradiction that $\lambda_p \geq 0$. Let $e_1, \cdots, e_{2s}$ be the eigenvectors corresponding to $\lambda_1, \cdots, \lambda_{2s}$. The assumption that $\lambda_p \geq 0$ implies that E_{**} is nonnegative definite on subspace $V_1 := \operatorname{span}\{e_p, \cdots, e_{2s}\}$. Since S is J-invariant, $JV_1 \subset JS = S$, and

$$\begin{aligned}\dim_R(V_1 \cap JV_1) &= \dim_R(V_1) + \dim_R(JV_1) - \dim_R(V_1 + JV_1)\\ &\geq 2s - (p-1) + 2s - (p-1) - 2s\\ &= 2s - 2p + 2 = 2q,\end{aligned}$$

where $\dim_R$ denotes real dimension. It is easy to see that $V_1 \cap JV_1$ is a complex subspace with complex dimension at least q. Denote by t the dimension of $V_1 \cap JV_1$. We can choose an orthonormal basis of $V_1 \cap JV_1$, $\{W_1, JW_1, \cdots, W_q, JW_q, \cdots, W_t, JW_t\}$.

Since B is nonnegative definite on V_1, from (3.11),

$$0 \leq B(W_i, W_i) + B(JW_i, JW_i) = -\int_0^1 \langle R(\dot\gamma, J\dot\gamma)W_i, JW_i\rangle\, dt = -\int_0^1 H(\dot\gamma, W_i)\, dt. \tag{3.13}$$

Thus

$$0 \leq \sum_{i=1}^{q}[B(W_i, W_i) + B(JW_i, JW_i)] = -\int_0^1 \sum_{i=1}^{q} H(\dot\gamma, W_i)\, dt < 0. \tag{3.14}$$

The last inequality is due to the condition that V has q-positive normal bisectional curvature with respect to M(or N). We arrive at a contradiction which shows our conclusion.

Denote $\Omega_c := E^{-1}([0, c])$ and $\Omega_c^\circ := E^{-1}([0, c))$. The following theorem was proved by Schoen and Wolfson[15, Theorem 1.5].

Theorem 3.2.[15] *Let* V *be a v-dimensional compact Kähler manifold and let* M, N *be complex submanifolds in* V. *Suppose that every nontrivial critical point of* E *on* Ω *has index* $\mu > \mu_0 \geq 0$. *Then the relative homotopy groups* $\pi_j(\Omega, \Omega_0)$ *are zero for* $0 \leq j \leq \mu_0$.

Combining this with the usual argument with the homotopy exact sequence and the relative Hurewicz isomorphism, we get the following theorem. We will not include here the proof because the proof of Theorem 3.2 in [15] works here without change since they used only homotopy exact sequence.

Theorem 3.3. *Let* V *be a v-dimensional compact Kähler manifold and let* M, N *be complex submanifolds in* V. *Suppose that every nontrivial smooth geodesic which is normal to* M *and* N *at* $\gamma(0)$, $\gamma(1)$ *has index* $\mu > \mu_0 \geq 0$. *Then the homomorphism induced by the inclusion* $i_* : \pi_j(N, N \cap M) \to \pi_j(V, M)$ *is an isomorphism for* $j \leq \mu_0$ *and is a surjection for* $j = \mu_0 + 1$.

Combining this theorem and Theorem 3.1, we have

Theorem 3.4. *Let V be a v-dimensional compact Kähler manifold and let M, N be complex submanifolds in V. Suppose that V has q-positive normal curvature with respect to M (or N). Then the homomorphism induced by the inclusion*

$$i_* : \pi_j(N, N \cap M) \to \pi_j(V, M)$$

is an isomorphism for $j \le m + n - v + 1 - q$ and is a surjection for $j = m + n - v + 2 - q$.

Remark 3.1. The conclusions of Theorem 3.1 and Theorem 3.4 hold if we replace the curvature conditions by the condition that V has q-nonnegative normal curvature with respect to M (or N) and has q-positive normal curvature with respect to M (or N) at all points in M (or N). This can be seen from the proof of Theorem 3.1. This observation is important in applications and will be seen in §4.

Remark 3.2. In fact we can consider the corresponding relative homotopy group induced by immersion maps. Since we care mainly about the image of submanifolds (as many people considered, see e.g. [4, 15]) we prefer to concentrate on the present case.

§4. Applications

To give some applications of our theorems we need to calculate curvatures in the normal directions.

Let (V, g, J) be a Kähler manifold. It can be verified that $(V \times V, g \times g, J \times J)$ is a Hermitian manifold. The almost complex structure $J \times J$, denoted by $\bar{J}$, and the product metric $g \times g$, are given as follows: for $\overline{X}, \overline{Y} \in T_{(p,q)}(V \times V)$, $(p, q) \in V \times V$,

$$\begin{aligned} \bar{J}\,\overline{X} &= (JX_1, JX_2), \\ (g \times g)(\overline{X}, \overline{Y}) &= g(X_1, Y_1) + g(X_2, Y_2), \end{aligned} \tag{4.1}$$

where $\overline{X} = (X_1, X_2), \overline{Y} = (Y_1, Y_2)$, $X_1, Y_1 \in T_pM$ and $X_2, Y_2 \in T_qM$.

In fact, $(V \times V, g \times g, J \times J)$ is a Kähler manifold. It is sufficient to verify that $\overline{\nabla}\bar{J} = 0$, where $\overline{\nabla}$ is the Levi-Civita connection of $(V \times V, g \times g)$.

It is well known that the connection $\overline{\nabla}$ satisfies, for $\overline{X}, \overline{Y}$ above,

$$\overline{\nabla}_{\overline{X}}\overline{Y} = (\nabla_{X_1} Y_1, \nabla_{X_2} Y_2), \tag{4.2}$$

where ∇ is the Levi-Civita connection of (M, g). Hence

$$\begin{aligned} (\overline{\nabla}\bar{J})(\overline{Y}, \overline{X}) &= \overline{\nabla}_{\overline{X}}(\bar{J}\,\overline{Y}) - \bar{J}(\overline{\nabla}_{\overline{X}}\overline{Y}) \\ &= \overline{\nabla}_{(X_1, X_2)}(JY_1, JY_2) - \bar{J}(\nabla_{X_1} Y_1, \nabla_{X_2} Y_2) \\ &= (\nabla_{X_1}(JY_1), \nabla_{X_2}(JY_2)) - (J\nabla_{X_1} Y_1, J\nabla_{X_2} Y_2) = 0. \end{aligned}$$

The third equality above is due to (4.2).

Theorem 4.1. *Let (V, g, J, ω) be a compact connected Kähler manifold of positive bisectional curvature. Assume that f and g are holomorphic maps from V to itself and at least one of them is a local diffeomorphism. Then there exists a point $x \in V$ such that $f(x) = g(x)$.*

Proof. We have that $(V \times V, g \times g, J \times J)$ is a Kähler manifold. We first claim that $(V \times V, g \times g, J \times J)$ has nonnegative bisectional curvature when V has a nonnegative one.

For any unit vectors $\overline{X}, \overline{Y} \in T_{(p,q)}(V \times V)$ as in (4.1), we denote $(X_1, 0)$, $(Y_1, 0)$ and $(0, X_2)$, $(0, Y_2)$ by $\overline{X}_1, \overline{Y}_1$ and $\overline{X}_2, \overline{Y}_2$ respectively. The bisectional curvature $\overline{H}(\overline{X}, \overline{Y})$

satisfies

$$\begin{aligned}\overline{H}(\overline{X},\overline{Y}) &= \langle \overline{R}(\overline{X}_1+\overline{X}_2, \bar{J}\,\overline{X}_1+\bar{J}\,\overline{X}_2)(\overline{Y}_1+\overline{Y}_2), \bar{J}\,\overline{Y}_1+\bar{J}\,\overline{Y}_2\rangle \\ &= \langle \overline{R}(\overline{X}_1, \bar{J}\,\overline{X}_1)\overline{Y}_1, \bar{J}\,\overline{Y}_1\rangle + \langle \overline{R}(\overline{X}_2, \bar{J}\,\overline{X}_2)\overline{Y}_2, \bar{J}\,\overline{Y}_2\rangle \\ &= \langle R(X_1, JX_1)Y_1, JY_1\rangle + \langle R(X_2, JX_2)Y_2, JY_2\rangle.\end{aligned} \tag{4.3}$$

The second equality in (4.3) holds since $\overline{\nabla}_{\overline{X}_1}\overline{X}_2 = \overline{\nabla}_{\overline{X}_2}\overline{X}_1 = 0$. It is easy to see that both terms of the right hand side of (4.3) are nonnegative and the claim holds.

Assume that g is a local diffeomorphism. We now consider the graphs of f and g, denoted by $G(f)$ and $G(g)$ respectively.

We see that both graphs are compact complex analytic submanifolds of $(V\times V, \omega\times\omega)$. It remains to show that $G(f)\cap G(g)\neq\emptyset$.

It suffices to show that any nontrivial geodesic connecting $G(f)$ and $G(g)$ has positive index since no intersection implies the existence of a stable nontrivial normal minimal geodesic connecting them.

From Theorem 3.1 and Remark 3.1, we only need to verify that $G(g)$ has normal positive bisectional curvature for any point $(x, g(x))\in\Delta$. For any two unit vectors $\overline{X},\overline{Y}\in T_{(x,g(x))}(V\times V)$ such that $\overline{X}\in T_{(x,g(x))}G(g)$ and $\overline{Y}\perp T_{(x,g(x))}G(g)$, there exists a vector $X_1\in T_xV$ such that $\overline{X}=(X_1, dg(X_1))$, and therefore since dg is a isomorphism, we have $\overline{X}\notin T_{(x,g(x))}(\{x\}\times V)$. We also have $\overline{Y}\notin T_{(x,g(x))}(\{x\}\times V)$, otherwise $\overline{Y}=(0,Y_2)$, we could choose a vector $(dg^{-1}Y_2, Y_2)\in T_{(x,g(x))}G(g)$ which is not orthonormal to $\overline{Y}$.

Then $X_1\neq 0$ and $Y_1\neq 0$ and positivity of bisectional curvature of V implies

$$\langle R(X_1, JX_1)Y_1, JY_1\rangle > 0.$$

It follows immediately from (4.3) that $H(\overline{X},\overline{Y})>0$ for any two unit vectors $\overline{X},\overline{Y}\in T_{(x,x)}(V\times V)$ such that $\overline{X}\in T_{(x,x)}G(g)$ and $\overline{Y}\perp T_{(x,x)}G(g)$. Hence by Theorem 3.1, the index of any nontrivial geodesic is greater than or equal to $v+v-2v+2-1=1$. Thus the graph $G(f)$ and the graph of $G(g)$ must intersect. That is, $f(x)=g(x)$ has a solution.

Remark 4.1. In [4], Frankel proved that every holomorphic map of a connected compact Kähler manifold with positive sectional curvature has a fixed point by using his intersection theorem for compact complex submanifolds of a Kähler manifold. In [16], Weinstein proved that every isometry of a compact oriented even-dimensional Riemannian manifold with positive sectional curvature has a fixed point if it preserves orientation. Sakai[14] proved a theorem for holomorphic isometric map which is considered as a holomorphic analogue of Weinstein's theorem. In our theorem we do not assume that f is isometric.

As a special case of Theorem 4.1, taking g as the identity map, we have the following fixed point theorem.

Corollary 4.1. *Let (V, g, J, ω) be a compact connected Kähler manifold of positive bisectional curvature. Assume that f is a holomorphic map from V to itself. Then f has a fixed point.*

Furthermore, as a special case of Theorem 3.4, we have

Theorem 4.2. *Let V be a v-dimensional compact Kähler manifold of positive bisectional curvature and let M be complex submanifolds in V and f is a holomorphic local diffeomorphism. Then the homomorphism induced by the inclusion*

$$i_*:\pi_j(G(f), G(f)\cap M)\to\pi_j(V\times V, M)$$

is an isomorphism for $j\leq m-v$ and is a surjection for $j=m-v+1$.

Proof. Take N in Theorem 3.4 as $G(f)$. From the proof of Theorem 4.1 we know that $V \times V$ has nonnegative normal curvature with respect to Δ and has positive normal curvature with respect to $G(f)$ at all points in $G(f)$. So in the Theorem 3.4 we replace v by $2v$, n by v and q by 1, then the homomorphism induced by the inclusion

$$i_* : \pi_j(G(f), G(f) \cap M) \to \pi_j(V \times V, M)$$

is an isomorphism for $j \leq m - v$ and is a surjection for $j = m - v + 1$. The proof is complete.

We conclude this paper by the following remark.

Remark 4.2. Theorem 1.2 is a special case of Theorem 4.2 by replacing f by the identity map because $\Delta = G(id)$.

Acknowledgements. Part of this work was carried out while the author was visiting IHES. The author wishes to express his thanks to IHES for the financial support and hospitality. He also wishes to thank J. Wolfson for some discussions about the proof of Theorem 3.1.

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