Chin. Ann. Math.
24B:3(2003),293–302.

ON THE EQUATION $\Box\phi = |\nabla\phi|^2$ IN FOUR SPACE DIMENSIONS**

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Abstract

This paper considers the following Cauchy problem for semilinear wave equations in n space dimensions

$$\Box\phi = F(\partial\phi),$$
$$\phi(0,x) = f(x), \quad \partial_t\phi(0,x) = g(x),$$

where $\Box = \partial_t^2 - \triangle$ is the wave operator, F is quadratic in $\partial\phi$ with $\partial = (\partial_t, \partial_{x_1}, \cdots, \partial_{x_n})$.

The minimal value of s is determined such that the above Cauchy problem is locally well-posed in H^s. It turns out that for the general equation s must satisfy

$$s > \max\Big(\frac{n}{2}, \frac{n+5}{4}\Big).$$

This is due to Ponce and Sideris (when $n = 3$) and Tataru (when $n \geq 5$). The purpose of this paper is to supplement with a proof in the case $n = 2, 4$.

Keywords Semilinear wave equation, Cauchy problem, Low regularity solution
2000 MR Subject Classification 35L
Chinese Library Classification O175.27 **Document Code** A
Article ID 0252-9599(2003)03-0293-10

§1. Introduction

In this paper, we consider the following Cauchy problem for semilinear wave equations in n space dimensions

$$\Box\phi = F(\partial\phi), \tag{1.1}$$
$$\phi(0,x) = f(x), \quad \partial_t\phi(0,x) = g(x), \tag{1.2}$$

where $\Box = \partial_t^2 - \triangle$ is the wave operator, F is quadratic in $\partial\phi$ with $\partial = (\partial_t, \partial_{x_1}, \cdots, \partial_{x_n})$.

We want to determine the minimal value of s such that the Cauchy problem (1.1),(1.2) is locally well-posed for

$$f \in H^s, \quad g \in H^{s-1}. \tag{1.3}$$

Manuscript received April 26, 2002.

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**Project supported by the 973 Project of the National Natural Science Foundation of China, the Key Teachers Program and the Doctoral Program Foundation of the Ministry of Education of China.

The classical local existence theorem requires $s > \frac{n}{2} + 1$, while the scaling limit is $s > \frac{n}{2}$. The correct one turns out to be

$$s > \max\Big(\frac{n}{2}, \frac{n+5}{4}\Big). \tag{1.4}$$

The counter-example which shows that (1.4) can not be improved in general is due to Lindblad[8]. The positive result is due to Ponce and Sideris[10] (when $n = 3$) and Tataru[11] (when $n \geq 5$). To our knowledge, the proofs have not been written for the case $n = 2, 4$. The purpose of this paper is to supplement with such a proof. The proof in the case $n = 2$ uses Strichartz inequality just as in the three space dimensional case. The proof in the case $n = 4$ uses so called Wave-Sobolev spaces (cf. [1, 3, 5–7, 12]). In the latter case, the proof is quite similar to that of our previous paper[12] when one deals with two space dimensional case and the nonlinearity satisfying a null condition. This is the main motivation for us to write down the present proof.

§2. Four Space Dimensional Case

For simplicity of the exposition, we consider the equation

$$\Box\phi = |\nabla\phi|^2, \tag{2.1}$$

where $\nabla = (\partial_{x_1}, \cdots, \partial_{x_n})$. We start our investigation by studying the regularity properties of the first iterate of the equation (2.1). For simplicity, we consider the following iterate:

$$\Box\phi = 0, \tag{2.2}$$

$$\phi(0,x) = f(x), \quad \partial_t\phi(0,x) = 0, \tag{2.3}$$

$$\Box\psi = |\nabla\phi|^2, \tag{2.4}$$

$$\psi(0,x) = 0, \quad \partial_t\psi(0,x) = 0. \tag{2.5}$$

We shall prove

Proposition 2.1. *Consider in R^{4+1} the Cauchy problem (2.4),(2.5) with ϕ satisfying (2.2),(2.3). If $f \in H^s$ with $\frac{9}{4} < s < \frac{5}{2}$, then the first iterate ψ belongs to H^s, and moreover, it holds that*

$$|\partial\psi(t,\cdot)|_{H^{s-1}} \leq C_\gamma t^\gamma |f|^2_{H^s} \tag{2.6}$$

for any γ with $\frac{1}{4} < \gamma < s - 2$. Here C_γ is a positive constant depending only on γ.

It turns out that only the knowledge of local regularity of ψ is not enough for our purposes. We shall also investigate the microlocal regularity properties of ψ.

Proposition 2.2. *Under the assumption of Proposition 2.1. Let $\tilde{\psi}(\tau,\xi)$ be the space time Fourier transform of ψ. Then microlocally at noncharacteristic point $\tau^2 - \xi^2 \neq 0$, ψ belongs to $H^{2s-\frac{3}{2}}$. More precisely, the following estimate holds*

$$\iint (|\tau| + |\xi|)^{2s-2} ||\tau| - |\xi||^{-(5-2s)} (\widetilde{|\nabla\phi|^2}(\tau,\xi))^2 d\tau d\xi \leq C|f|^4_{H^s}. \tag{2.7}$$

We are now ready to study the Cauchy problem (2.1),(1.2). It is suggested by Proposition 2.2 that we introduce the Wave-Sobolev norms (cf. [7])

$$N_{b,\delta}(\phi) = \Big(\iint w_+^{2b}(\tau,\xi) w_-^{2\delta}(\tau,\xi) \tilde{\phi}^2(\tau,\xi) d\tau d\xi\Big)^{\frac{1}{2}}, \tag{2.8}$$

where $\tilde{\phi}$ denotes the space-time Fourier transform of ϕ and

$$w_\pm(\tau,\xi) = 1 + ||\tau| \pm |\xi||. \tag{2.9}$$

After some cut off, we shall seek a solution with finite $N_{s,s-\frac{3}{2}}$ norm. Thus, we shall establish the following

Theorem 2.1. *Consider in R^{4+1} the space-time norms* (2.8) *and function ϕ defined on R^{4+1}. Then the estimate*

$$N_{s-1,-(\frac{5}{2}-s)}(|\nabla\phi|^2) \le CN^2_{s,s-\frac{3}{2}}(\phi) \tag{2.10}$$

holds for any $\frac{9}{4} < s < \frac{5}{2}$.

By applying Theorem 2.1, we can get

Theorem 2.2. *The initial value problem* (2.1),(1.2) *in* 4 *space dimensions is locally well-posed for $f \in H^s$, $g \in H^{s-1}$ with $\frac{9}{4} < s < \frac{5}{2}$.*

We first prove Proposition 2.1. It is easy to see that the space Fourier transform of ψ is

$$\hat{\psi}(t,\xi) = \frac{1}{|\xi|}\int_0^t \sin|\xi|(t-t')\hat{F}(t',\xi)dt', \tag{2.11}$$

where $F = |\nabla\phi|^2$. Noting that

$$\hat{\phi}(t',\xi) = \cos|\xi|t'\hat{f}(\xi), \tag{2.12}$$

we get

$$\hat{\psi}(t,\xi) = \frac{1}{|\xi|}\int_0^t\int_{R^4}(\xi-\eta)\cdot\eta\sin|\xi|(t-t')\cos|\xi-\eta|t'\cos|\eta|t'\hat{f}(\xi-\eta)\hat{f}(\eta)d\eta dt'. \tag{2.13}$$

By duality, we only have to prove

$$\begin{aligned}&\int_0^t\iint(\xi-\eta)\cdot\eta(1+|\xi|)^{s-1}\sin|\xi|(t-t')\cos|\xi-\eta|t'\cos|\eta|t'\\&\cdot\hat{f}(\xi-\eta)\hat{f}(\eta)H(\xi)d\xi d\eta dt'\\&\le Ct^\gamma|f|^2_{H^s}\|H\|,\end{aligned} \tag{2.14}$$

where $\|H\|$ denotes the L^2 norm of H. Let

$$F(\xi) = (1+|\xi|)^s\hat{f}(\xi). \tag{2.15}$$

Then (2.14) is equivalent to

$$\begin{aligned}&\iint\frac{(\xi-\eta)\cdot\eta(1+|\xi|)^{s-1}}{(1+|\xi-\eta|)^s(1+|\eta|)^s}\Big(\int_0^t\sin|\xi|(t-t')\cos|\xi-\eta|t'\cos|\eta|t'dt'\Big)\\&\cdot F(\xi-\eta)F(\eta)H(\xi)d\xi d\eta\\&\le Ct^\gamma\|F\|^2\|H\|.\end{aligned} \tag{2.16}$$

By making a change of variables from ξ to $\xi+\eta$, this is equivalent to

$$\iint\frac{\xi\cdot\eta(1+|\xi+\eta|)^{s-1}}{(1+|\xi|)^s(1+|\eta|)^s}P(t,\xi,\eta)F(\xi)F(\eta)H(\xi+\eta)d\xi d\eta \le Ct^\gamma\|F\|^2\|H\|, \tag{2.17}$$

where

$$P(t,\xi,\eta) = \int_0^t\sin|\xi+\eta|(t-t')\cos|\xi|t'\cos|\eta|t'dt'. \tag{2.18}$$

A simple calculation shows that

$$P(t,\xi,\eta) \le \sum_{i=1}^4 P_i(t,\xi,\eta), \tag{2.19}$$

where

$$P_1(t,\xi,\eta)=\frac{|\sin(|\xi+\eta|+|\xi|+|\eta|)t|}{|\xi+\eta|+|\xi|+|\eta|}\le\frac{t^\gamma}{(|\xi+\eta|+|\xi|+|\eta|)^{1-\gamma}},\tag{2.20}$$

$$P_2(t,\xi,\eta)=\frac{|\sin(-|\xi+\eta|+|\xi|+|\eta|)t|}{-|\xi+\eta|+|\xi|+|\eta|}\le\frac{t^\gamma}{(-|\xi+\eta|+|\xi|+|\eta|)^{1-\gamma}},\tag{2.21}$$

$$P_3(t,\xi,\eta)=\frac{|\sin(|\xi+\eta|-|\xi|+|\eta|)t|}{|\xi+\eta|-|\xi|+|\eta|}\le\frac{t^\gamma}{(|\xi+\eta|-|\xi|+|\eta|)^{1-\gamma}},\tag{2.22}$$

$$P_4(t,\xi,\eta)=\frac{|\sin(|\xi+\eta|+|\xi|-|\eta|)t|}{|\xi+\eta|+|\xi|-|\eta|}\le\frac{t^\gamma}{(|\xi+\eta|+|\xi|-|\eta|)^{1-\gamma}}.\tag{2.23}$$

Thus, we need to estimate

$$I_i=\iint\frac{\xi\cdot\eta(1+|\xi+\eta|)^{s-1}}{(1+|\xi|)^s(1+|\eta|)^s}P_i(t,\xi,\eta)F(\xi)F(\eta)H(\xi+\eta)d\xi d\eta.\tag{2.24}$$

Noting that

$$\frac{t^\gamma}{(|\xi+\eta|+|\xi|+|\eta|)^{1-\gamma}}\le\frac{t^\gamma}{(-|\xi+\eta|+|\xi|+|\eta|)^{1-\gamma}},\tag{2.25}$$

the estimate of I_1 can be reduced to that of I_2. We shall first estimate I_2. We have

$$I_2\le t^\gamma\iint\frac{|\xi\cdot\eta|(1+|\xi+\eta|)^{s-1}F(\xi)F(\eta)H(\xi+\eta)d\xi d\eta}{(1+|\xi|)^s(1+|\eta|)^s(|\xi|+|\eta|-|\xi+\eta|)^{1-\gamma}}.\tag{2.26}$$

Without loss of generality, we assume $|\eta|\le|\xi|$. Then it follows from Schwartz inequality that

$$I_2\le\Big(\iint\frac{F^2(\xi)d\xi d\eta}{(1+|\eta|)^{2(s-1)}(|\xi|+|\eta|-|\xi+\eta|)^{2-2\gamma}}\Big)^{\frac12}\|F\|\|H\|.\tag{2.27}$$

So it remains to prove

$$J=\iint\frac{d\eta}{(1+|\eta|)^{2(s-1)}(|\xi|+|\eta|-|\xi+\eta|)^{2-2\gamma}}\le C.\tag{2.28}$$

Without loss of generality, we assume $\xi=(r,0,0,0)$. Then by using polar coordinates, we get

$$J\le C\int\frac{dr}{(1+r)^{2s-2\gamma-3}}\int\frac{\sin^2\theta d\theta}{(1-\cos\theta)^{2-2\gamma}}<+\infty\tag{2.29}$$

provided that

$$\frac14<\gamma<s-2.$$

We now estimate I_3. I_4 is, in fact, equal to I_3. By making a change of variables from ξ to $\xi-\eta$ then η to $-\eta$, we get

$$I_3\le t^\gamma\iint\frac{(|\xi+\eta)\cdot\eta|(1+|\xi|)^{s-1}F(\xi+\eta)F(\eta)H(\xi)d\xi d\eta}{(1+|\xi+\eta|)^s(1+|\eta|)^s(|\xi|+|\eta|-|\xi+\eta|)^{1-\gamma}}.\tag{2.30}$$

If

$$\xi\cdot\eta\ge0,$$

then

$$|\xi+\eta|^2=\xi^2+\eta^2+2\xi\cdot\eta\ge|\xi|^2.$$

Thus

$$I_3\le Ct^\gamma\iint\frac{F(\xi+\eta)F(\eta)H(\xi)}{(1+|\eta|)^{s-1}(|\xi|+|\eta|-|\xi+\eta|)^{1-\gamma}}.\tag{2.31}$$

The problem is essentially reduced to the previous case. If

$$\xi \cdot \eta \le 0,$$

then

$$\begin{aligned}|\xi| + |\eta| - |\xi + \eta| &= \frac{|\xi||\eta| - \xi \cdot \eta}{|\xi| + |\eta| + |\xi + \eta|} \\ &\ge \frac{|\xi||\eta|}{|\xi| + |\eta| + |\xi + \eta|}.\end{aligned}$$

Thus

$$I_3 \le Ct^\gamma \iint \frac{F(\xi+\eta)F(\eta)H(\xi)d\xi d\eta}{(1+|\eta|)^{s-\gamma}} \tag{2.32}$$

provided that $|\eta| \le |\xi + \eta|$, and

$$I_3 \le Ct^\gamma \iint \frac{F(\xi+\eta)F(\eta)H(\xi)d\xi d\eta}{(1+|\xi+\eta|)^{s-1}(1+|\eta|)^{1-\gamma}} \tag{2.33}$$

provided that $|\xi + \eta| \le |\eta|$. In both cases, the desired result follows from the following

Lemma 2.1. *Let f, g, h be functions defined on R^n. Then*

$$\iint \frac{f(x)g(y)h(x+y)}{(1+|x|)^a(1+|y|)^b(1+|x+y|)^c} dxdy \le C\|f\|\|g\|\|h\| \tag{2.34}$$

provided that $a, b, c \ge 0$ and $a + b + c > \frac{n}{2}$.

Proof. By

$$\begin{aligned}&\frac{1}{(1+|x|)^a(1+|y|)^b(1+|x+y|)^c} \\ &\le \frac{1}{(1+|x|)^{a+b+c}} + \frac{1}{(1+|y|)^{a+b+c}} + \frac{1}{(1+|x+y|)^{a+b+c}},\end{aligned}$$

(2.34) follows easily from Schwartz's inequality.

Therefore, we finish the proof of Proposition 2.1.

We now prove Proposition 2.2. We write

$$\phi = \frac{1}{2}(\phi_+ + \phi_-), \tag{2.35}$$

where

$$\phi_\pm(t, x) = (2\pi)^{-4} \int_{R^4} e^{i(x\cdot\xi \pm t|\xi|)} \hat{f}(\xi) d\xi. \tag{2.36}$$

To estimate $|\nabla\phi|^2$, it suffices to estimate $\nabla\phi_+ \cdot \nabla\phi_+$ and $\nabla\phi_+ \cdot \nabla\phi_-$. By duality, we only need to prove that

$$\begin{aligned}&\iint (|\tau| + |\xi|)^{s-1} ||\tau| - |\xi||^{-(\frac{5}{2}-s)} \nabla\widetilde{\phi_+ \cdot \nabla}\phi_\pm(\tau, \xi) H(\tau, \xi) d\tau d\xi \\ &\le C|f|^2_{H^s} \|H\|.\end{aligned} \tag{2.37}$$

A simple calculation shows that the left-hand side is equal to

$$\begin{aligned}&\iint (||\xi - \eta| \pm |\eta|| + |\xi|)^{s-1} (||\xi - \eta| \pm |\eta|| - |\xi|)^{-(\frac{5}{2}-s)} (\xi - \eta) \cdot \eta \\ &\cdot \hat{f}(\xi - \eta) \hat{f}(\eta) H(|\xi - \eta| \pm |\eta|, \xi) d\xi d\eta.\end{aligned} \tag{2.38}$$

Let

$$F(\xi) = (1 + |\xi|)^s \hat{f}(\xi).$$

Then we only need to prove

$$I = \iint \frac{(\xi-\eta)\cdot\eta(||\xi-\eta| \pm |\eta|| + |\xi|)^{s-1}}{(||\xi-\eta| \pm |\eta|| - |\xi|)^{\frac{5}{2}-s}|\xi-\eta|^s|\eta|^s} F(\xi-\eta)F(\eta)H(|\xi-\eta| \pm |\eta|, \xi)d\xi d\eta$$
$$\leq C\|F\|^2\|H\|. \tag{2.39}$$

Without loss of generality, we may assume

$$|\xi-\eta| \geq |\eta|. \tag{2.40}$$

Then

$$||\xi-\eta| \pm |\eta|| + |\xi| \leq 4|\xi-\eta|. \tag{2.41}$$

It follows that

$$I \leq C \iint \frac{F(\xi-\eta)F(\eta)H(|\xi-\eta| \pm |\eta|, \xi)d\xi d\eta}{(||\xi-\eta| \pm |\eta|| - |\xi|)^{\frac{5}{2}-s}|\eta|^{s-1}}. \tag{2.42}$$

Thus, the desired conclusions follows from the following

Lemma 2.2. *The following estimate holds for any* $\frac{9}{4} < s < \frac{5}{2}$,

$$J = \iint \frac{F(\xi-\eta)G(\eta)H(|\xi-\eta| \pm |\eta|, \xi)d\xi d\eta}{(||\xi-\eta| \pm |\eta|| - |\xi|)^{\frac{5}{2}-s}|\eta|^{s-1}} \leq C\|F\|\|G\|\|H\|. \tag{2.43}$$

Proof. See [4, Theorem 1.1].

We now prove Theorem 2.1. By duality, it suffices to prove that

$$\iint w_+^{s-1}(\tau,\xi)w_-^{-(\frac{5}{2}-s)}(\tau,\xi)\widetilde{|\nabla\phi|^2}(\tau,\xi)H(\tau,\xi)d\tau d\xi$$
$$\leq CN^2_{s,s-\frac{3}{2}}(\phi)\|H\|. \tag{2.44}$$

Let

$$F = w_+^s(\tau,\xi)w_-^{s-\frac{3}{2}}(\tau,\xi)\tilde{\phi}(\tau,\xi). \tag{2.45}$$

Then an easy calculation shows that (2.44) is equivalent to

$$I = \iint \frac{w_+^{s-1}(\tau+\lambda,\xi+\eta)(\xi\cdot\eta)F(\tau,\xi)F(\lambda,\eta)H(\tau+\lambda,\xi+\eta)d\tau d\lambda d\xi d\eta}{w_-^{\frac{5}{2}-s}(\tau+\lambda,\xi+\eta)w_+^s(\tau,\xi)w_-^{s-\frac{3}{2}}(\tau,\xi)w_+^s(\lambda,\eta)w_-^{s-\frac{3}{2}}(\lambda,\eta)}$$
$$\leq C\|F\|^2\|H\|. \tag{2.46}$$

Without loss of generality, we may assume

$$w_+(\lambda,\eta) \leq w_+(\tau,\xi),$$

then

$$w_+(\tau+\lambda,\xi+\eta) \leq 2w_+(\tau,\xi).$$

Therefore, noting

$$|\xi| \leq w_+(\tau,\xi), \quad |\eta| \leq w_+(\lambda,\eta),$$

we can get

$$I \leq \iint \frac{F(\tau,\xi)F(\lambda,\eta)H(\tau+\lambda,\xi+\eta)d\tau d\lambda d\xi d\eta}{w_-^{\frac{5}{2}-s}(\tau+\lambda,\xi+\eta)|\eta|^{s-1}w_-^{s-\frac{3}{2}}(\tau,\xi)w_-^{s-\frac{3}{2}}(\lambda,\eta)}. \tag{2.47}$$

Without loss of generality, we take

$$|\tau| = \tau, \quad |\lambda| = \pm\lambda. \tag{2.48}$$

We claim that

$$w_-(\tau+\lambda,\xi+\eta)+w_-(\tau,\xi)+w_-(\lambda,\eta)\geq \frac{1}{2}||\xi|\pm|\eta||-|\xi+\eta||. \tag{2.49}$$

Let

$$u=|\tau|-|\xi|,\quad v=|\lambda|-|\eta|, \tag{2.50}$$

then

$$\tau=u+|\xi|,\quad \lambda=\pm(v+|\eta|). \tag{2.51}$$

If

$$|u|+|v|\geq\frac{1}{2}|||\xi|\pm|\eta||-|\xi+\eta||,$$

then (2.49) will be right. Otherwise

$$w_-(\tau+\lambda,\xi+\eta)=1+||u\pm v+|\xi|\pm|\eta||-|\xi+\eta||. \tag{2.52}$$

In the "+" case, we have

$$\begin{aligned}|u+v+|\xi|+|\eta||-|\xi+\eta| &\geq -|u|-|v|+|\xi|+|\eta|-|\xi+\eta|\\ &\geq\frac{1}{2}(|\xi|+|\eta|-|\xi+\eta|)\geq 0.\end{aligned} \tag{2.53}$$

In the "−" case, we have

$$\begin{aligned}|\xi+\eta|-|u-v+||\xi|-|\eta|||&\geq -|u|-|v|+|\xi+\eta|-||\xi|-|\eta||\\ &\geq\frac{1}{2}(|\xi+\eta|-||\xi|-|\eta||)\geq 0.\end{aligned} \tag{2.54}$$

Therefore, (2.49) is always valid. It then follows from (2.49) that

$$\begin{aligned}|||\xi|\pm|\eta||-|\xi+\eta||^{\frac{5}{2}-s}&\leq C(w_-(\tau+\lambda,\xi+\eta)+w_-(\tau,\xi)+w_-(\lambda,\eta))^{\frac{5}{2}-s}\\ &\leq 3Cw_-(\tau+\lambda,\xi+\eta)^{\frac{5}{2}-s}w_-(\tau,\xi)^{\frac{5}{2}-s}w_-(\lambda,\eta)^{\frac{5}{2}-s}.\end{aligned}$$

Thus

$$\begin{aligned}I&\leq C\iint\frac{F(\tau,\xi)F(\lambda,\eta)H(\tau+\lambda,\xi+\eta)d\tau d\lambda d\xi d\eta}{|||\xi|\pm|\eta||-|\xi+\eta||^{\frac{5}{2}-s}|\eta|^{s-1}w_-^{2s-4}(\tau,\xi)w_-^{2s-4}(\lambda,\eta)}\\ &=C\iint\frac{F(u+|\xi|,\xi)F(\pm v\pm|\eta|,\eta)H(u\pm v+|\xi|\pm|\eta|,\xi+\eta)d\xi d\eta}{|||\xi|\pm|\eta||-|\xi+\eta||^{\frac{5}{2}-s}|\eta|^{s-1}}\\ &\quad\times\iint\frac{dudv}{(1+|u|)^{2s-4}(1+|v|)^{2s-4}}.\end{aligned} \tag{2.55}$$

Let

$$f_u(\xi)=F(u+|\xi|,\xi),\quad g_v(\eta)=F(\pm v\pm|\eta|,\eta). \tag{2.56}$$

Then it follows from Lemma 2.2 that

$$\iint\frac{f_u(\xi)g_v(\eta)H(u\pm v+|\xi|\pm|\eta|,\xi+\eta)d\xi d\eta}{|||\xi|\pm|\eta||-|\xi+\eta||^{\frac{5}{2}-s}|\eta|^{s-1}}\leq C\|H\|\|f_u\|\|g_v\|. \tag{2.57}$$

Thus

$$I\leq C\|H\|\iint\frac{\|f_u\|\|f_v\|dudv}{(1+|u|)^{2s-4}(1+|v|)^{2s-4}}. \tag{2.58}$$

Noting $2s-4>\frac{1}{2}$, it follows from Schwartz's inequality that

$$I\leq C\|H\|\Big(\int\|f_u\|^2du\Big)^{\frac{1}{2}}\Big(\int\|g_v\|^2dv\Big)^{\frac{1}{2}}=C\|H\|\|F\|^2. \tag{2.59}$$

This finishes the proof of Theorem 2.1.

§3. Two Space Dimensional Case

In this section, we consider the Cauchy problem (1.1),(1.2) in two space dimensions. Our main result can be summarized in the following

Theorem 3.1. *Suppose that* $(f,g)\in H^s(R^2)\times H^{s-1}(R^2)$ *for* $s>\frac{7}{4}$. *Then there exists a* $T>0$ *depending on* s *and* $|f|_{H^s}+|g|_{H^{s-1}}$ *such that* (1.1),(1.2) *has a unique solution* ϕ *satisfying*

$$\phi\in C([0,T];H^s(R^2))\cap C^1([0,T];H^{s-1}(R^2)), \tag{3.1}$$

$$\int_0^T |(-\triangle)^{\frac{\sigma}{2}}\partial\phi(t,\cdot)|_{L^q}^r dt<+\infty, \tag{3.2}$$

where $\frac{2}{4s-7}<q<+\infty$, $r=\frac{4q}{q-2}$, *and* $\sigma=s-\frac{7}{4}+\frac{3}{2q}$.

To obtain the result of Theorem 3.1, we need the following Strichartz inequality for the linear wave equation (cf. [9]).

Lemma 3.1. *Let* ϕ *be a function defined on* $R\times R^2$ *which solves the following initial value problem for the linear wave equation*

$$\Box\phi(t,x)=F(t,x), \tag{3.3}$$

$$\phi(0,x)=f(x),\quad \partial_t\phi(0,x)=g(x). \tag{3.4}$$

Then for any $2\le q<+\infty$, $r=\frac{4q}{q-2}$, $\alpha=\frac{3}{r}$, *there holds*

$$\Big(\int_0^T |\partial\phi(t,\cdot)|_{L^q}^r dt\Big)^{\frac{1}{r}}\le C\Big(|f|_{H^{\alpha+1}}+|g|_{H^\alpha}+\int_0^T |F(t,\cdot)|_{H^\alpha}dt\Big). \tag{3.5}$$

We are now ready to prove Theorem 3.1. For $T,a>0$, define the space

$$X_T^a=\{\phi\in C([0,T];H^s(R^2))\cap C^1([0,T];H^{s-1}(R^2)):\|\phi\|\le a\}, \tag{3.6}$$

where

$$\|\phi\|=\sup_{0\le t\le T}(|\phi(t,\cdot)|_{H^s}+|\partial_t\phi(t,\cdot)|_{H^{s-1}})+\Big(\int_0^T |(-\triangle)^{\frac{\sigma}{2}}\partial\phi(t,\cdot)|_{L^q}^r dt\Big)^{\frac{1}{r}} \tag{3.7}$$

with σ,q,r as in Theorem 3.1. For any $\phi\in X_T^a$, define $\psi=\Lambda\phi$ by solving the following Cauchy problem

$$\Box\psi=F(\partial\phi), \tag{3.8}$$

$$\psi(0,x)=f(x),\quad \partial_t\psi(0,x)=g(x). \tag{3.9}$$

We shall show that Λ is a contraction of X_T^a into itself.

By energy estimate, we get

$$\sup_{0\le t\le T}|\partial\psi(t,\cdot)|_{H^{s-1}}\le C_1\Big(|f|_{H^s}+|g|_{H^{s-1}}+\int_0^T |F(\partial\phi)(t,\cdot)|_{H^{s-1}}dt\Big). \tag{3.10}$$

By Strichartz's inequality, we get

$$\begin{aligned}&\Big(\int_0^T |(-\triangle)^{\frac{\sigma}{2}}\partial\phi(t,\cdot)|_{L^q}^r dt\Big)^{\frac{1}{r}}\\ &\le C_2\Big(|f|_{H^s}+|g|_{H^{s-1}}+\int_0^T |F(\partial\phi)(t,\cdot)|_{H^{s-1}}dt\Big).\end{aligned} \tag{3.11}$$

Thus, we get

$$\begin{aligned}
&\sup_{0\le t\le T}|\partial\psi(t,\cdot)|_{H^{s-1}}+\Big(\int_0^T|(-\triangle)^{\frac{\sigma}{2}}\partial\phi(t,\cdot)|^r_{L^q}dt\Big)^{\frac{1}{r}}\\
&\le C_0\Big(|f|_{H^s}+|g|_{H^{s-1}}+\int_0^T|F(\partial\phi)(t,\cdot)|_{H^{s-1}}dt\Big)\\
&\le \frac{a}{4}+C_0\int_0^T|F(\partial\phi)(t,\cdot)|_{H^{s-1}}dt, \qquad (3.12)
\end{aligned}$$

where we take

$$a=4C_0(|f|_{H^s}+|g|_{H^{s-1}}). \qquad (3.13)$$

Without loss of generality, we assume $C_0>1$. As in the three dimensional case, we have

$$|F(\partial\phi)|_{H^{s-1}}\le C|\partial\phi|_{L^\infty}|\partial\phi|_{H^{s-1}}\le Ca|\partial\phi|_{L^\infty}.$$

Noting $\sigma>\frac{2}{q}$, it follows from Sobolev inequality that

$$|\partial\phi|_{L^\infty}\le C(|\partial\phi|_{L^2}+|(-\triangle)^{\frac{\sigma}{2}}\partial\phi|_{L^q})\le Ca+C|(-\triangle)^{\frac{\sigma}{2}}\partial\phi|_{L^q}.$$

Thus

$$|F(\partial\phi)|_{H^{s-1}}\le Ca^2+Ca|(-\triangle)^{\frac{\sigma}{2}}\partial\phi|_{L^q}. \qquad (3.14)$$

It then follows from Hölder's inequality that

$$\begin{aligned}
\int_0^T|F(\partial\phi(t,\cdot))|_{H^{s-1}}dt&\le Ca^2T+CaT^{1-\frac{1}{r}}\Big(\int_0^T|(-\triangle)^{\frac{\sigma}{2}}\partial\phi(t,\cdot)|^r_{L^q}dt\Big)^{\frac{1}{r}}\\
&\le Ca^2T+Ca^2T^{1-\frac{1}{r}}\le\frac{a}{4C_0} \qquad (3.15)
\end{aligned}$$

provided that T is taking suitably small. By (3.12), we get

$$\sup_{0\le t\le T}|\partial\psi(t,\cdot)|_{H^{s-1}}+\Big(\int_0^T|(-\triangle)^{\frac{\sigma}{2}}\partial\phi(t,\cdot)|^r_{L^q}dt\Big)^{\frac{1}{r}}\le\frac{a}{2}. \qquad (3.16)$$

We have

$$\begin{aligned}
|\psi(t,\cdot)|_{L^2}&\le|f|_{L^2}+\int_0^t|\partial_t\psi(\tau,\cdot)|_{L^2}d\tau\\
&\le\frac{a}{4}+Ta\le\frac{a}{2} \qquad (3.17)
\end{aligned}$$

provided that T is taken suitably small. Therefore, it follows from (3.16), (3.17) that

$$\|\psi\|\le a$$

and thus Λ is a map from X_T^a to X_T^a. In a similar fashion, it can be shown that the mapping Λ is a contraction on X_T^a. Thus, there exists a unique fixed point of Λ which is a solution of (1.1),(1.2). This finishes the proof of Theorem 3.1.

Acknowledgement. The author would like to thank Prof. Yang Han for helpful discussions.

References

[1] Beals, M., Self-spreading and strength of singularities for solution to semilinear wave equations, *Ann. Math.,* **118**(1983), 187–214.

[2] Beals, M. & Bézard, M., Low regularity local solutions for field equations, *Comm. Partial Differential Equations,* **21**(1996), 79–124.

[3] Bourgain, J., Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations I, *Geom. Funct. Anal.,* **3**(1993), 107–156.

[4] Foschi, D. & Klainerman, S., Homogeneous L^2 bilinear estimates for wave equations, *Ann. Scient. ENS* 4^e *Serie,* **23**(2000), 211–274.

[5] Klainerman, S. & Machedon, M., Space-time estimate for null forms and the local existence theorem, *Comm. Pure Appl. Math.,* **46**(1993), 1221–1268.

[6] Klainerman, S. & Machedon, M., Smoothing estimates for null forms and applications, *Duke Math. J.,* **81**(1995), 99–133.

[7] Klainerman, S. & Selberg, S., Bilinear estimates and applications to nonlinear wave equations, *Bull. A.M.S.*, to appear.

[8] Lindblad, H., Counterexamples to local existence for semilinear wave equations, *Amer. J. Math.,* **118** (1996), 1–16.

[9] Pecher, H., Nonlinear small data scattering for the wave and Klein-Gordan equations, *Math. Z.,* **185** (1984), 261–270.

[10] Ponce, G. & Sideris, T., Local regularity of nonlinear wave equations in three space dimensions, *Comm. Partial Differential Equations,* **18**(1993), 169–177.

[11] Tataru, D., On the equation $\Box u = |\nabla u|^2$ in 5+1 dimension, *Math. Research Letters,* **6**(1999), 469–485.

[12] Zhou, Y., Local existence with minimal regularity for nonlinear wave equations, *Amer. J. Math.,* **119** (1997), 671–703.

[13] Zhou, Y., Uniqueness of generalized solutions to nonlinear wave equations, *Amer. J. Math.,* **122**(2000), 939–965.