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SOME RESULTS ON INFINITE DIMENTIONAL NOVIKOV ALGEBRAS

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Abstract

This paper gives some sufficient conditions for determining the simplicity of infinite dimentional Novikov algebras of characteristic 0, and also constructs a class of simple Novikov algebras by extending the base field. At last, the deformation theory of Novikov algebras is introduced.

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§1. Introduction

Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic type[1]. They were also introduced in connection with Hamiltonian operators in the formal variational calculus[2]. Geometrically, a Novikov algebra corresponds to a left invariant torsion free flat connection of the Lie group whose Lie algebra is isomorphic to the commutator Lie algebra of the Novikov algebra. The abstract study of Novikov algebras began with Zelmanov[3], who showed that simple finite dimentional Novikov algebras of characteristic 0 are one-dimensional. Osborn classified simple Novikov algebras with an idempotent element and some modules over such algebras[4]. A class of simple Novikov algebra without idempotent elements was constructed though Novikov-Poisson algebras by Xiaoping Xu[5]. For further understanding and physical applications, Bai Chengming[6] gave a classification of Novikov algebras over the complex field in dimension 2 and 3.

We call a nonassociative algebra left Novikov if it satisfies the two identities

$$(x,y,z)=(y,x,z), \quad (xy)z=(xz)y,$$

where $(x,y,z)=(xy)z-x(yz)$. The operators L_x and R_x for some $x\in A$,

$$L_x : A\to A,$$
$$y\mapsto xy,$$
$$R_x : A\to A,$$
$$y\mapsto yx$$

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are called respectively the left multiplication operator and the right multiplication operator. The beauty of a Novikov algebra is that the left multiplication operators form a Lie algebra and the right multiplication operators are commutative. A Novikov-Poission algebra is a vector space A with two operations "$\cdot$, $\circ$" such that $(A, \cdot)$ forms a commutative associative algebra and $(A, \circ)$ forms a Novikov algebra for which

$$(x \cdot y) \circ z = x \cdot (y \circ z), (x \circ y) \cdot z - x \circ (y \cdot z) = (y \circ x) \cdot z - y \circ (x \cdot z).$$

The main theorem in [4] is: if A is a simple infinite dimentional Novikov algebra over a field of characteristic 0 containing an idempotent e with the property that $A = \sum\limits_{\alpha} A'_\alpha$, where

$$A'_\alpha = \{x \in A |\ [L_e - (\alpha + b)id]^n x = 0 \text{ for some } n\},$$

then A is described by one of the following:

(1.1) A has a basis $\{x_i\}_{i \geq -1}$, where products are given by $x_i x_j = (j+1) x_{i+j}$.

(1.2) A has a basis $\{x_\alpha\}$, where α ranges over an additive subgroup $\triangle$ of F, and products are given by $x_\alpha x_\beta = (\beta + b) x_{\alpha+\beta}$.

(1.3) A has a basis $\{x_{\alpha,k}\}$, where products are given by

$$x_{\alpha,k} x_{\beta,l} = (\beta + b) \binom{k+l}{k} x_{\alpha+\beta,k+l} + \binom{l+k-1}{k} x_{\alpha+\beta,k+l-1}.$$

A natural question is whether the inverse of the statement is true. In this paper we give the positive answer. To prove this ,we will first show that the Novikov algebras of algebraic structure (1.1), (1.2) or (1.3) are all simple.

Algebraic deformation theory was introduced for associative algebras by Nijenhuis and Richardson[7]. In [8, 9], Bai gave the deformation theory of Novikov algebras and proved that the Novikov algebras in dimension ≤ 3 can be realized as the algebras defined by I. M. Gel'fand[10] and their compatible infinitesimal deformations. Whether this result can be extended to higher dimensions remains an open problem. In this paper, we introduce the general deformation theory of Novikov algebras.

The paper is organized as follows. In Section 2, we prove that the Novikov algebras of algebraic structure (1.1), (1.2) or (1.3) are all simple. In Section 3, we construct a class of simple Novikov algebras by extending the base field . The general deformation theory of Novikov algebra is introduced in Section 4.

§2. Simplicity of Novikov Algebras

In this section, we give some sufficient conditions for determining the simplicity of Novikov algebras. Throughout this section, let $(A\cdot, \circ)$ be an infinite dimentional Novikov algebra over a field of characteristic 0. The set of positive integers will be denoted by N.

Theorem 2.1. *If A has a basis $\{x_i\}_{i \geq -1}$, where products are given by $x_i \circ x_j = (j+1) x_{i+j}$, then $(A, \circ)$ is simple.*

Proof. Suppose I is a nonzero ideal of A. Let i be an arbitrary nonzero element of I. We can write $i = a_{i_1} x_{i_1} + \cdots + a_{i_j} x_{i_j}$, where $a_{i_k} \neq 0$ and $i_m \neq i_n$, if $m \neq n$. $k, m, n = 1, \cdots, j$.

Since $x_0 \circ x_{i_k} = (i_k + 1)x_{i_k}$, we obtain

$$\begin{aligned} i &= a_{i_1}x_{i_1} + \cdots + a_{i_j}x_{i_j}, \\ L_{x_0}i &= a_{i_1}(1+i_1)x_{i_1} + \cdots + a_{i_j}(1+i_j)x_{i_j}, \\ &\cdots\cdots, \\ L_{x_0}^{j-1}i &= a_{i_1}(1+i_1)^{j-1}x_{i_1} + \cdots + a_{i_j}(1+i_j)^{j-1}x_{i_j}. \end{aligned}$$

The matrix of this system has the form

$$M = \begin{pmatrix} a_{i_1} & a_{i_2} & \cdots & a_{i_j} \\ a_{i_1}(1+i_1) & a_{i_2}(1+i_2) & \cdots & a_{i_j}(1+i_j) \\ \vdots & \vdots & \ddots & \vdots \\ a_{i_1}(1+i_1)^{j-1} & a_{i_2}(1+i_2)^{j-1} & \cdots & a_{i_j}(1+i_j)^{j-1} \end{pmatrix}_{j\times j}.$$

Notice that $\det M = \Big(\prod_{k=1}^{j} a_{i_k}\Big)\det B$, where $\det B$ is the determinant of Vandermonde and we have $\det M \neq 0$ if $i_m \neq i_n (m \neq n)$. Hence, $x_{i_1}, \cdots, x_{i_j}$ are linear combinations of the vectors $i, L_{x-0}i, \cdots, L_{x_0}^{j-1}i$. So, $x_{i_1}, \cdots, x_{i_j}$ are all in I. Since $i \neq 0$, we have at least one element $x_k \in I$. If $k = -1$, i.e., $x_{-1} \in I$, for arbitrary $i \in N, x_{-1} \circ x_i = (i+1)x_{i-1} \in I$. So, $x_{i-1} \in I$. Hence, $I = A$. If $k \neq -1$, we have $x_{-1} \circ x_k = (k+1)x_{k-1} \in I$. So $x_{k-1} \in I$. Using induction we can get $x_{-1} \in I$. Hence, $I = A$. From the above proof, we can see that A has no nontrivial ideal, i.e. A is simple.

Theorem 2.2. *If A has a basis $\{x_\alpha\}$, where α ranges over an additive group $\triangle$ of F, and products are given by $x_\alpha x_\beta = (\beta + b)x_{\alpha+\beta}$, then A is simple.*

Proof Suppose I is a nonzero ideal of A. Let i be an arbitrary element of I. We can write $i = a_1x_{\alpha_1} + \cdots + a_jx_{\alpha_j}$, where $a_k \neq 0$ and $\alpha_m \neq \alpha_n$. Using the similar device as in Theorem 2.1, we can get $x_{\alpha_k} \in I, k = 1, \cdots, j$. Since $i \neq 0$, we have at least one element $x_\alpha \in I$.

Case 1. Suppose $b \neq 0$. If $\alpha \neq -b, \forall \beta \in \triangle, \beta - \alpha \in \triangle$, we have $x_{\beta-\alpha}x_\alpha = (\alpha+b)x_\beta \in I$. So $x_\beta \in I$. If $\alpha = -b$, i.e. $x_{-b} \in I$, since $-b \in \triangle$, we get $b \in \triangle$, $x_{-b} \circ x_b = 2bx_0 \in I$. Hence, $x_0 \in I$. For all $\alpha \neq -b, x_0 \circ x_\alpha = (\alpha + b)x_\alpha \in I$. So $x_\alpha \in I$. Summarizing, we have $I = A$.

Case 2. Suppose $b = 0$. If $\alpha = 0$, i.e., $x_0 \in I$, for arbitrary $\alpha \in \triangle - \{0\}$, $x_0x_\alpha = \alpha x_\alpha \in I$. So we have $x_\alpha \in I$. If $\alpha \neq 0, x_\alpha \in I$. Since $x_{-\alpha} \circ x_\alpha = \alpha x_0$, we have $x_0 \in I$. For arbitrary $\beta \in \triangle - \{0\}, x_0x_\beta = \beta x_\beta \in I$. So we have $x_\beta \in I$. Summarizing we get $I = A$. From the above we can see that A has no nontrivial ideal, i.e., A is simple.

Theorem 2.3. *If A has a basis $\{x_{\alpha,k}\}$, where α ranges over an additive group $\triangle$ of F and k ranges over the nonnegative integers Z_+, where products are given by*

$$x_{\alpha,k}x_{\beta,l} = (\beta + b)\binom{k+l}{k}x_{\alpha+\beta,k+l} + \binom{k+l-1}{k}x_{\alpha+\beta,k+l-1},$$

then A is simple.

Proof. For any $\omega = \sum_{\alpha\in\triangle, j\in Z_+} a_{\alpha,j}x_{\alpha,j} \in A$, we define

$$d_\beta(\omega) = \begin{cases} x\,Max\{j \in J | a_{\beta,j} \neq 0\}, & \text{if some } a_{\beta,j} \neq 0; \\ -1, & \text{otherwise.} \end{cases}$$

Put $\widetilde{A}_\gamma = \sum_{j\in Z_+} Fx_{\gamma,j}$ for $\gamma \in \triangle$ and put $\partial = L_{x_{0,0}} - b\,\mathrm{id}$. It is obvious that $\partial(x_{\alpha,j}) =$

$\alpha x_{\alpha,j} + x_{\alpha,j-1}$. Note that, for any $\beta \in \triangle$,

$$(\partial - \beta id)(\omega) = \sum a_{\alpha,j}[(\alpha - \beta)x_{\alpha,j} + x_{\alpha,j-1}],$$
$$d_\beta[(\partial - \beta id)(\omega)] = d_\beta(\omega) - 1,$$
$$d_\gamma[(\partial - \beta id)(\omega)] = d_\gamma(\omega), \text{ for } \beta \neq \gamma \in \triangle.$$

Moreover, if $d_\beta(\omega) \geq 0$, we have

$$d_\beta[(\partial - \beta id)^{d_\beta(\omega)+1}(\omega)] = -1,$$
$$d_\gamma[(\partial - \beta id)^{d_\beta(\omega)+1}(\omega)] = d_\gamma(\omega) \quad \text{for} \quad \beta \neq \gamma \in \triangle.$$

Therefore $(\partial - \beta id)^{d_\beta(\omega)+1}(\omega) \in \sum_{\beta\neq\gamma\in\triangle} \widetilde{A}_\gamma$.

Suppose I is a nonzero ideal of A, and let ω be an arbitrary nonzero element of I. We can write

$$\begin{aligned}\omega = {} & b_{11}x_{\alpha_1,a_{11}} + \cdots + b_{1i_1}x_{\alpha_1,a_{1i_1}} \\ & + b_{21}x_{\alpha_2,a_{21}} + \cdots + b_{2i_2}x_{\alpha_2,a_{2i_2}} \\ & + \cdots\cdots \\ & + b_{s1}x_{\alpha_s,a_{s1}} + \cdots + b_{si_s}x_{\alpha_s,a_{si_s}},\end{aligned}$$

where $b_{mn} \in F - \{0\}, a_{k1} \geq a_{k2} \geq \cdots \geq a_{ki_k}, k = 1, \cdots, s$. From the above proof, we get

$$Q = \prod_{i\neq j}(\partial - \alpha_i id)^{d_{\alpha_i}(\omega)+1}(\omega) \in \widetilde{A}_{\alpha_j},$$
$$d_{\alpha_j}(Q) = d_{\alpha_j}(\omega).$$

Using the similar device as in Theorem 2.1, we get $x_{\alpha_j,a_{j,p}} \in I,\ p = 1, \cdots, i_j,\ j = 1, \cdots, s$. Since $\omega \neq 0$, we have at least one nonzero element $x_{\alpha,k} \in I$.

Case 1. Suppose $b \neq 0$.

If $\alpha = 0$, i.e. $x_{0,k} \in I$, since $x_{0,0}x_{0,k} = x_{0,k-1} \in I$, we get $x_{0,k-1} \in I$. Using induction, we have $x_{0,0}, x_{0,2} \in I$. Since $x_{0,k}x_{0,2} = \binom{k+1}{k}x_{0,k+1} \in I$, we have $x_{0,k+1} \in I$. Using induction we have $x_{0,i} \in I$, for arbitrary $i \in Z_+$. For arbitrary $\beta \in \triangle - \{0\}, l \in Z_+$. Since $x_{0,l}x_{\beta,0} = \beta x_{\beta,l} \in I$, we have $x_{\beta,l} \in I$. Hence, $I = A$.

If $\alpha \neq 0$, since $x_{0,0}x_{\alpha,k} = \alpha x_{\alpha,k} + x_{\alpha,k-1} \in I$, we have $x_{\alpha,k-1} \in I$. Using induction we have $x_{\alpha,0} \in I$. Since $\alpha \in \triangle, -\alpha \in \triangle$ and $x_{-\alpha}, x_{\alpha,0} = \alpha x_{0,0} \in I$, we have $x_{0,0} \in I$. Since $x_{0,0}x_{0,k} = x_{0,k-1} \in I$, we have $x_{0,k-1} \in I$. Using induction, we can get $x_{0,i} \in I$ for arbitrary $i \in Z_+$. For arbitrary $\beta \in \triangle -\{0\}, k \in Z_+$, we have $x_{0,k}x_{\beta,0} = \beta x_{\beta,k} \in I$. So $x_{\beta,k} \in I$. Hence, $I = A$.

Case 2. Suppose $b \neq 0$.

If $\alpha \neq -b$, since $x_{0,0}x_{\alpha,k} = (\alpha + b)\ x_{\alpha,k} + x_{\alpha,k-1} \in I$, we have $x_{\alpha,k-1} \in I$. Using induction we can get $x_{\alpha,0} \in I$. Since $\alpha \in \triangle, -\alpha \in \triangle$, and $x_{-\alpha,o}\ x_{\alpha,0} = (\alpha + b)x_{0,0} \in I$, we have $x_{0,0} \in I$. Since $x_{0,0}\ x_{\alpha,k+1} = (\alpha + b)x_{\alpha,k+1} + x_{\alpha,k} \in I$, we have $x_{\alpha,k+1} \in I$. Using induction we have $x_{\alpha,i} \in I$, for arbitrary $i \in Z_+$. For arbitrary $\beta \in \triangle, i \in Z_+, -\alpha + \beta \in \triangle$, $x_{-\alpha+\beta,i}x_{\alpha,0} = (\alpha + b)x_{\beta,i} \in I$, we have $x_{\beta,i} \in I$. Hence, $I = A$.

If $\alpha = -b$, i.e. $x_{-b,k} \in I$, $\alpha = -b \in \triangle$, then $b \in \triangle$. Since $x_{-b,k}x_{b,0} = 2bx_{0,k} \in I$, we have $x_{0,k} \in I$.

Since $x_{0,0}x_{0,k} = bx_{0,k} + x_{0,k-1}$, we have $x_{0,k-1} \in I$. Using induction we have $x_{0,0} \in I$. Since $x_{0,k}x_{0,1} = bx_{0,k+1} + x_{0,k} \in I$, we have $x_{0,k+1} \in I$. Using induction we have $x_{0,i} \in I$, for arbitrary $i \in Z_+$. For arbitrary $\beta \in \triangle - \{-b\}, i \in Z_+$, $x_{0,i}x_{\beta,0} = (\beta + b)x_{\beta,i} \in I$. $\alpha = -b \in \triangle$, then $2\alpha = -2b \in \triangle$. For arbitrary $i \in Z_+$, $x_{b,i}x_{-2b,0} = -bx_{-b,i} \in I$. So we have $x_{-b,i} \in I$. Hence, $I = A$.

From the above proof we can see A has no nontrivial ideal, i.e. A is simple.

Corrollary 2.1. *If A is a simple infinite dimentional Novikov algebra over a field of characteristic 0 containing an idempotent e with the property that $A = \sum_\alpha A'_\alpha$, where $A'_\alpha = \{x \in A|\ [L_e - (\alpha + b)id]^n x = 0$ for some $n\}$, then the subalgebra $\sum_{\alpha\in\triangle} A_\alpha$ is a simple Novkov algebra over A_0.*

Theorem 2.4. *A is a simple infinite dimentional Novikov algebra of characteristic 0 containing an idempotent e with the property that $A = \sum_\alpha A'_\alpha$, where $A'_\alpha = \{x \in A|\ [L_e - (\alpha + b)id]^n x = 0$ for some $n\}$, if and only if A is described by one of the following*:

(1) *A has a basis $\{x_i\}_{i\geq -1}$, where products are given by $x_i x_j = (j + 1)x_{i+j}$.*

(2) *A has a basis $\{x_\alpha\}$ where α ranges over an additive subgroup $\triangle$ of F, and products are given by $x_\alpha x_\beta = (\beta + b)x_{\alpha+\beta}$*

(3) *A has a basis $\{x_{\alpha,k}\}$ where products are given by*

$$x_{\alpha,k}x_{\beta,l} = (\beta + b)\binom{k+l}{k} x_{\alpha+\beta,k+l} + \binom{l+k-1}{k} x_{\alpha+\beta,k+l-1}.$$

Proof. Based on the above results, to prove the theorem we need only to prove that there exists an element $e \in A$ such that $e^2 = be$ for some $b \in F$ and $A = \sum_\alpha A'_\alpha$, where $A'_\alpha = \{x \in A|\ [L_e - (\alpha + b)id]^n x = 0$ for some $n\}$.

In Case (1), since $x_0x_0 = x_0$, putting $e = x_0$ and $A'_i = Fx_i$, we can easily verify that $A'_i = \{x \in A|\ [L_e - (i + 1)id]^n x = 0$ for some $n\}$ and $A = \sum A'_i$.

In Case (2), since $x_0x_0 = bx_0$, putting $e = x_0$ and $A'_\alpha = Fx_\alpha$,then we can easily verify that $A'_\alpha = \{x \in A|\ [L_e - (\alpha + b)id]^n x = 0$ for some $n\}$ and $A = \sum_{\alpha\in\triangle} A'_\alpha$.

In Case (3), since $x_{0,0}x_{0,0} = bx_{0,0}$, putting $e = x_{0,0}$ and $A'_\alpha = \widetilde{A}_\alpha$, we can easily verify that $A'_\alpha = \{x \in A|\ [L_e - (\alpha + b)id]^n x = 0$ for some $n\}$ and $A = \sum_{\alpha\in\triangle} A'_\alpha$.

§3. New Simple Novikov Algebras

In this section, we consider the simplicity of Novikov algebra $A \otimes_F E := A^E$ over E, where E is an extension field of F.

Let $\{\xi_i\}_{i\in I}$ be a basis of A. Then A^E has $\{\xi_i \otimes 1\}_{i\in I}$ as a basis. The products of Novikov algebra A^E are given by

$$(\xi_i \otimes 1)(\xi_j \otimes 1) = \xi_i\xi_j \otimes 1.$$

By Theorems 2.1–2.3, we can easily obtain

Theorem 3.1. *If A is a Novikov algebra satisfying one of the conditions* (1.1), (1.2) *and* (1.3), *then A^E is simple.*

§4. The Deformation Theory of Novikov Algebras

In this section, we shall give the definition of deformation of a Navikov algebra A over a commutative ring F. Since A is F-module, we can obtain an $F[t]$-module $A[t] = A \otimes_F F[t]$.

As a matter of fact, $A[t]$ is an $F[t]-$module of formal power-series with coefficients in the $F-$ module A, i.e. $A[t] = \Big\{ \sum\limits_{i=0}^{n} a_i t^i \big| n \in Z_+, a_i \in A \Big\}$. The algebra A is submodule of $A[t]$ and we could make $A[t]$ an algebra by bilinearly extending the multiplication of A, but we may also impose other multiplications on $A[t]$ that agree with that of A when we specialize $t = 0$.

Suppose a multiplication $\alpha : \ A[t] \otimes_{F[t]} A[t] \to A[t]$ is given by a formal power-series of the form

$$\alpha(a, b) = \alpha_0(a, b) + \alpha_1(a, b)t + \alpha_2(a, b)t^2 + \cdots .$$

Since we are defining α over $F[t]$, it is enough to consider a and b in A, and we further presume that each α_n is a linear map $A \otimes A \to A$. Since we want the specialization $t = 0$ to give the original multiplication on A, we insist that $\alpha_0(a, b) = ab$ (multiplication in A).

Definition 4.1. *A one-parameter formal deformation of a Novikov algebra A over a commutative ring F is a formal power-series $\alpha = \sum\limits_{n=0}^{\infty} \alpha_n t^n$ with coefficients in $Hom_F(A \otimes A, A)$ such that $\alpha_0 : A \otimes A \to A$ is a multiplication in A. The deformation is called Novikov if $\alpha(\alpha(a, b), c) - \alpha(a, \alpha(b, c)) = \alpha(\alpha(b, a), c) - \alpha(b, \alpha(a, c))$ and $\alpha(\alpha(a, b), c) = \alpha(\alpha(a, c), b)$ for all $a, b, c \in A$.*

Definition 4.2. *We call $A[t] = A \otimes_F F[t]$ with the multiplication defined by α the deformation of Novikov algebra A.*

Example 4.1. Let (A, α_1, α_0) be a Novikov-Poisson algebra. (A, α_1) forms a commutative associative algebra, (A, α_0) forms a Novikov algebra and define $\alpha : A[t] \otimes_{F[t]} A[t] \to A[t]$ by

$$\alpha = \alpha_0 + \alpha_1 t.$$

It can be easily verified that $(A[t], \alpha)$ forms a Novikov algebra, i.e. $(A[t], \alpha)$ is the deformation of (A, α_0).

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