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TENSOR PRODUCTS OF JACOBSON RADICALS IN NEST ALGEBRAS

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Abstract

This paper studies the tensor product $\mathcal{R}_N \otimes_w \mathcal{R}_M$ of Jacobson radicals in nest algebras, and obtains that $\mathcal{R}_N \otimes_w \mathcal{R}_M = \{T \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) : T(N \otimes M) \subseteq N_- \otimes M_-, \forall N \in \mathcal{N}, M \in \mathcal{M}\}$; and based on the characterization of rank-one operators in $\mathcal{R}_N \otimes_w \mathcal{R}_M$, it is proved that if $\mathcal{N}, \mathcal{M}$ are non-trivial then $\mathcal{R}_N \otimes_w \mathcal{R}_M = \mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}}$ if and only if $\mathcal{N}, \mathcal{M}$ are continuous.

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§1. Introduction

One of the central results in the theory of tensor products of von Neumann algebras is Tomita's commutation formula:

$$\mathcal{A}' \otimes_w \mathcal{B}' = (\mathcal{A} \otimes_w \mathcal{B})', \tag{1.1}$$

where $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras. It was observed in [3] that if we let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the projection lattices of $\mathcal{A}$ and $\mathcal{B}$ respectively, then (1.1) can be rewritten as

$$\mathrm{Alg}\mathcal{L}_1 \otimes_w \mathrm{Alg}\mathcal{L}_2 = \mathrm{Alg}(\mathcal{L}_1 \otimes \mathcal{L}_2). \tag{1.2}$$

This version of Tomita's theorem makes sense for any pair of reflexive algebras $\mathrm{Alg}\mathcal{L}_1$ and $\mathrm{Alg}\mathcal{L}_2$. It remains a deep open question whether the tensor product formula (1.2) is valid for general reflexive algebras, or even general CSL algebras. However, (1.2) has been verified in a number of special cases[3,5,6,7]. In particular, it is known that if $\mathcal{N}, \mathcal{M}$ are nests, then $\mathrm{Alg}\mathcal{N} \otimes_w \mathrm{Alg}\mathcal{M} = \mathrm{Alg}(\mathcal{N} \otimes \mathcal{M})$ (see [3]). Since then one has been interested in the relationship between $\mathcal{R}_N \otimes_w \mathcal{R}_M$ and $\mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}}$. The technique employed in this paper is different from the other papers about tensor products. We use rank-one operators to study tensor products and the technique shows its power in this paper.

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Let us introduce some notation and terminology. $\mathcal{H}$ represents a complex Hilbert space, $\mathcal{B}(H)$ the algebra of bounded operators on $\mathcal{H}$ and $\mathcal{F}(H)$ the set of finite -rank operators on $\mathcal{H}$. A sublattice $\mathcal{L}$ of the projection lattice of $\mathcal{B}(H)$ is said to be a subspace lattice if it contains 0 and I and is strongly closed, where we identify projections with their ranges. If the elements of $\mathcal{L}$ pairwise commute, $\mathcal{L}$ is a commutative subspace lattice (CSL). A subspace lattice is completely distributive if distributive laws are valid for families of arbitrary cardinality (see [9]). A nest $\mathcal{N}$ is a totally ordered subspace lattice. For $L \in \mathcal{L}$, we define

$$L_- = \vee\{E \in \mathcal{L} : L \not\leq E\}.$$

In the case of nests, either N_- is the the immediate predecessor of N or $N = N_-$. If $N = N_-$ for any $N \in \mathcal{N}$, $\mathcal{N}$ is called a continuous nest. If $\mathcal{L}$ is a subspace lattice, Alg$\mathcal{L}$ denotes the set of operators in $\mathcal{B}(H)$ that leave the elements of $\mathcal{L}$ invariant. If $\mathcal{L}$ is a CSL, Alg$\mathcal{L}$ is said to be a CSL algebra. If $\mathcal{L}$ is a nest, Alg$\mathcal{L}$ is said to be a nest algebra.

Recall that the Jacobson radical of a Banach algebra coincides with these elements T such that AT is quasinilpotent for every A in the algebra and it is a closed ideal of the Banach algebra. For a subspace lattice $\mathcal{L}$, we denote $\mathcal{R}_{\mathcal{L}}$ the Jacobson radical of Alg$\mathcal{L}$. In [10], Ringrose characterized the Jacobson radical of a nest algebra. In [1], Davidson and Orr pushed the characterization further to the case of all width two CSL algebras. The result is essential to our paper.

Let $\mathcal{H}_i (i = 1, 2)$ be complex Hilbert spaces. If $\mathcal{L}_i \subseteq \mathcal{B}(\mathcal{H}_i)$ $(i = 1, 2)$ are subspace lattices, $\mathcal{L}_1 \otimes \mathcal{L}_2$ is the subspace lattice in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ generated by $\{L_1 \otimes L_2 : L_i \in \mathcal{L}_i,\ i = 1, 2\}$. If $\mathcal{S}_i \subseteq \mathcal{B}(\mathcal{H}_i)(i = 1, 2)$ are subspaces, then $\mathcal{S}_1 \otimes \mathcal{S}_2$ denotes the linear span of $\{S_1 \otimes S_2 : S_i \in \mathcal{S}_i\}$; $\mathcal{S}_1 \otimes_n \mathcal{S}_2$ denotes the norm closure of $\mathcal{S}_1 \otimes \mathcal{S}_2$; $\mathcal{S}_1 \otimes_w \mathcal{S}_2$ denotes the weak closure of $\mathcal{S}_1 \otimes \mathcal{S}_2$ in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. It is easy to show that $\mathcal{S}_1 \otimes_n \mathcal{S}_2 = \overline{\mathcal{S}}_1 \otimes_n \overline{\mathcal{S}}_2$, where $\overline{\mathcal{S}}_i$ denotes the norm closure of $\mathcal{S}_i$; however in general case, $\mathcal{S}_1 \otimes_w \mathcal{S}_2 \neq \mathcal{S}_2^w \otimes_w \mathcal{S}_2^w$. The reason lies in that the map $(A, B) \to A \otimes B$ from $\mathcal{B}(\mathcal{H}_1) \times \mathcal{B}(\mathcal{H}_2)$ to $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is not weakly continuous in general.

§2. Tensor Products of Jacobson Radicals

In the following we suppose that $\mathcal{N}$ and $\mathcal{M}$ are nests on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively; and that $\mathcal{N} \otimes \mathcal{M}$ is the tensor product of $\mathcal{N}$ and $\mathcal{M}$. $\mathcal{R}_N$, $\mathcal{R}_M$ and $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ denote Jacobson radicals of Alg$\mathcal{N}$, Alg$\mathcal{M}$ and Alg$(\mathcal{N} \otimes \mathcal{M})$ respectively.

For $x, y \in \mathcal{H}$, the rank-one operator xy^* is defined by the equation

$$(xy^*)(z) = \langle z, y\rangle x, \quad \forall z \in \mathcal{H}.$$

Proposition 2.1. $\mathcal{R}_{\mathcal{N}}^w = \{A \in \mathcal{B}(\mathcal{H}_1) : AN \subseteq N_-, \forall N \in \mathcal{N}\}$.

Proof. Set $\mathcal{U} = \{A \in \mathcal{B}(\mathcal{H}_1) : AN \subseteq N_-, \forall N \in \mathcal{N}\}$. Then $\mathcal{U}$ is a weakly closed Alg$\mathcal{N}$-module determined by the order homomorphism $N \to N_-$ from $\mathcal{N}$ into itself. By virtue of [2] Lemma 1.1, a rank-one operator $xy^* \in \mathcal{U}$ if and only if there exists an element $N \in \mathcal{N}$ such that $x \in N$ and $y \in N_{\sim}^{\perp}$, where

$$N_{\sim} = \vee\{N' : N'_- < N\} = N.$$

We also know that a rank-one operator $xy^* \in \mathcal{R}_N$ if and only if there exists an element $N \in \mathcal{N}$ such that $x \in N$ and $y \in N^{\perp}$. Thus the Jacobson radical $\mathcal{R}_N$ and $\mathcal{U}$ have the same rank-one operators. Since each finite-rank operator in $\mathcal{R}_N$ or $\mathcal{U}$ can be represented as a finite sum of rank-one operators in itself respectively, $\mathcal{R}_N$ and $\mathcal{U}$ have the same finite-rank operators.

It follows from Erdos Density Theorem that there is a net $\{F_\alpha\}$ of finite-rank contractions in Alg$\mathcal{N}$ such that $F_\alpha \xrightarrow{w} I$. Thus for any $A \in \mathcal{R}_N$,

$$F_\alpha A \xrightarrow{w} A,$$

and $\{F_\alpha A\}$ are finite-rank operators in $\mathcal{R}_N$. So

$$(\mathcal{R}_N \cap \mathcal{F}(H))^w \supseteq \mathcal{R}_N$$

and $(\mathcal{R}_N \cap \mathcal{F}(H))^w \supseteq \mathcal{R}_\mathcal{N}^w$, $(\mathcal{R}_N \cap \mathcal{F}(H))^w = \mathcal{R}_\mathcal{N}^w$. Combining [2] Corollary 1.6 with the result in the preceding paragraph, we obtain that

$$\mathcal{R}_\mathcal{N}^w = (\mathcal{R}_N \cap \mathcal{F}(H))^w = (\mathcal{U} \cap \mathcal{F}(H))^w = \mathcal{U}.$$

This completes the proof.

Lemma 2.1. *Suppose that $A \in \mathcal{R}_\mathcal{N}^w$, then there exists a net of finite-rank operators $F_\alpha \subseteq \mathcal{R}_N$ such that $\| F_\alpha \| \leq \| A \|$ and $F_\alpha \xrightarrow{w} A$.*

Proof. By Erdos Density Theorem, there is a net of finite-rank contractions $\{F'_\alpha\} \subseteq$ Alg$\mathcal{N}$ such that $F'_\alpha \xrightarrow{w} I$. So

$$F_\alpha = F'_\alpha A \xrightarrow{w} A.$$

Since $\mathcal{R}_\mathcal{N}^w$ is a weakly closed Alg$\mathcal{N}$-module, $\{F_\alpha\} \subseteq \mathcal{R}_\mathcal{N}^w \cap \mathcal{F}(H)$ and $\| F_\alpha \| \leq \| A \|$. From the proof of the preceding proposition, we know that $\mathcal{R}_\mathcal{N}^w \cap \mathcal{F}(H) = \mathcal{R}_N \cap \mathcal{F}(H)$. So $\{F_\alpha\} \subseteq \mathcal{R}_N \cap \mathcal{F}(H)$ and satisfies the condition in the lemma.

Lemma 2.2. $\mathcal{R}_N \otimes_w \mathcal{R}_M = \mathcal{R}_\mathcal{N}^w \otimes_w \mathcal{R}_\mathcal{M}^w$.

Proof. Suppose that $A \in \mathcal{R}_\mathcal{N}^w$ and $B \in \mathcal{R}_\mathcal{M}^w$. it follows from Lemma 2.1 that there exist nets of finite-rank operators $\{A_\alpha\} \subseteq \mathcal{R}_N$, $\{B_\beta\} \subseteq \mathcal{R}_M$ such that

$$\| A_\alpha \| \leq \| A \|, \quad \| B_\beta \| \leq \| B \| \quad \text{and} \quad A_\alpha \xrightarrow{w} A, \quad B_\beta \xrightarrow{w} B.$$

For any $x_i, y_i \in \mathcal{H}_i (i = 1, 2)$, we have that

$$\begin{aligned} &\langle (A_\alpha \otimes B_\beta)(x_1 \otimes x_2), y_1 \otimes y_2 \rangle = \langle A_\alpha x_1, y_1 \rangle \langle B_\beta x_2, y_2 \rangle \\ \longrightarrow\ &\langle A x_1, y_1 \rangle \langle B x_2, y_2 \rangle = \langle (A \otimes B)(x_1 \otimes x_2), y_1 \otimes y_2 \rangle. \end{aligned}$$

Since $\mathcal{H}_1 \otimes \mathcal{H}_2$ is the completion of span$\{x_1 \otimes x_2 : x_i \in \mathcal{H}_i\}$ and

$$\| A_\alpha \otimes B_\beta \| = \| A_\alpha \| \cdot \| B_\beta \| \leq \| A \| \cdot \| B \|,$$

it is routine to prove that

$$\langle (A_\alpha \otimes B_\beta) z, w \rangle \longrightarrow \langle (A \otimes B) z, w \rangle \quad \text{for any } z, w \in \mathcal{H}_1 \otimes \mathcal{H}_2.$$

So $A_\alpha \otimes B_\beta \xrightarrow{w} A \otimes B$ and $A \otimes B \in \mathcal{R}_N \otimes_w \mathcal{R}_M$, thus

$$\begin{aligned} \mathcal{R}_\mathcal{N}^w \otimes \mathcal{R}_\mathcal{M}^w &\subseteq \mathcal{R}_N \otimes_w \mathcal{R}_M, \\ \mathcal{R}_\mathcal{N}^w \otimes_w \mathcal{R}_\mathcal{M}^w &\subseteq \mathcal{R}_N \otimes_w \mathcal{R}_M. \end{aligned}$$

The converse inequality is obviouse, so $\mathcal{R}_N \otimes_w \mathcal{R}_M = \mathcal{R}_\mathcal{N}^w \otimes_w \mathcal{R}_\mathcal{M}^w$.

Lemma 2.3. *Suppose that $\mathcal{U}_\tau$ is a weakly closed* $\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$*- module determined by an order homomorphism τ from $\mathcal{N}\otimes\mathcal{M}$ into itself. Then a rank-one operator $xy^*\in\mathcal{U}_\tau$ if and only if there exists an element $L\in\mathcal{N}\otimes\mathcal{M}$ such that $x\in L$ and $y\in L_\sim^\perp$, where $L_\sim=\vee\{G\in\mathcal{N}\otimes\mathcal{M}:L\not\leq\tau(G)\}$.*

Proof. Suppose that there exists an element $L\in\mathcal{N}\otimes\mathcal{M}$ such that $x\in L$ and $y\in L_\sim^\perp$. For any $G\in\mathcal{N}\otimes\mathcal{M}$, if $L\leq\tau(G)$, then

$$(xy^*)G=L(xy^*)L_\sim^\perp G\subseteq L\subseteq\tau(G);$$

if $L\not\leq\tau(G)$, then $G\leq L_\sim$ and

$$(xy^*)G=L(xy^*)L_\sim^\perp G=(0)\subseteq\tau(G).$$

Thus the rank-one operator $xy^*\in\mathcal{U}_\tau$.

Conversely, suppose that $xy^*\in\mathcal{U}_\tau$. Set $L=\wedge\{G\in\mathcal{N}\otimes\mathcal{M}:Gx=x\}$, certainly $x\in L$. For any $G\in\mathcal{N}\otimes\mathcal{M}$ and $L\not\leq\tau(G)$, it follows from the definition of L that $\tau(G)x\neq x$. If $Gy\neq 0$, since $(xy^*)G=\tau(G)(xy^*)G$, we have

$$[(xy^*)G](Gy)=[\tau(G)(xy^*)G](Gy),$$
$$\| Gy\|^2\,x=\| Gy\|^2\,\tau(G)x.$$

This cotradicts $\tau(G)x\neq x$, so $Gy=0$ for any $L\not\leq\tau(G)$, From the definition of $L_\sim$, we have $L_\sim y=0$ and $y\in L_\sim^\perp$.

Certainly, Lemma 2.3 is true for any subspace lattice $\mathcal{L}$.

Theorem 2.1. $\mathcal{R}_N\otimes_w\mathcal{R}_M=\{T\in\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2):T(N\otimes M)\subseteq N_-\otimes M_-,\forall N\in\mathcal{N},M\in\mathcal{M}\}=\{T\in\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2):TL\subseteq\tau(L),\forall L\in\mathcal{N}\otimes\mathcal{M}\}$*, where* $\tau(L)=\vee\{N_-\otimes M_-:N\otimes M\leq L\}$.

Proof. Set $\mathcal{U}_\tau=\{T\in\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2):TL\subseteq\tau(L),\forall L\in\mathcal{N}\otimes\mathcal{M}\}$. $\mathcal{U}_\tau$ is a weakly closed $\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$-module determined by the order homomorphism $L\to\tau(L)$ from $\mathcal{N}\otimes\mathcal{M}$ into itself. By virtue of [3, Proposition 2.4],

$$L=\vee\{N\otimes M:N\otimes M\leq L\}\quad\text{for any } L\in\mathcal{N}\otimes\mathcal{M}.$$

Thus it is easy to show that

$$\mathcal{U}_\tau=\{T\in\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2):T(N\otimes M)\subseteq N_-\otimes M_-,\forall N\in\mathcal{N},M\in\mathcal{M}\}.$$

Since $\mathcal{R}_N\otimes_w\mathcal{R}_M=\mathcal{R}_\mathcal{N}^w\otimes_w\mathcal{R}_\mathcal{M}^w$, it follows from Proposition 2.1 that $\mathcal{R}_N\otimes_w\mathcal{R}_M\subseteq\mathcal{U}_\tau$.

Define $\tau_1:\mathcal{N}\otimes I\to\mathcal{N}\otimes I$ by

$$\tau_1(N\otimes I)=N_-\otimes I,\quad\forall N\in\mathcal{N}.$$

τ_1 is an order homomorphism from $\mathcal{N}\otimes I$ into $\mathcal{N}\otimes I$. Define $\mathcal{U}_{\tau_1}=\{T\in\mathcal{B}(\mathcal{H}_1\otimes\mathcal{H}_2):T(N\otimes I)\subseteq N_-\otimes I,\forall N\in\mathcal{N}\}$. Similarly we define $\tau_2:I\otimes\mathcal{M}\to I\otimes\mathcal{M}$ and $\mathcal{U}_{\tau_2}$. Thus we have the equation $\mathcal{U}_\tau=\mathcal{U}_{\tau_1}\cap\mathcal{U}_{\tau_2}$. In fact, $\mathcal{U}_\tau\subseteq\mathcal{U}_{\tau_1}\cap\mathcal{U}_{\tau_2}$ is obvious. For $T\in\mathcal{U}_{\tau_1}\cap\mathcal{U}_{\tau_2}$ we have that for any $N\in\mathcal{N}$, $M\in\mathcal{M}$,

$$T(N\otimes M)\subseteq T(N\otimes I)\subseteq N_-\otimes I,$$
$$T(N\otimes M)\subseteq T(I\otimes M)\subseteq I\otimes M_-.$$

Thus $T(N\otimes M)\subseteq(N_-\otimes I)\cap(I\otimes M_-)=N_-\otimes M_-$ and $T\in\mathcal{U}_\tau$.

Since $\mathcal{N}\otimes\mathcal{M}$ is a completely distributive CSL ([3, Proposition 2.7]), it follows from [8, Theorem 3] that the rank-one subalgebra of $\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$ is weakly dense in $\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$. So it is routine to show that the linear span of rank-one operators in $\mathcal{U}_\tau$ is also weakly dense in $\mathcal{U}_\tau$. Accordingly, to show $\mathcal{U}_\tau\subseteq\mathcal{R}_N\otimes_w\mathcal{R}_M$, it suffices to consider rank-one operators in $\mathcal{U}_\tau$.

From Lemma 2.3, it follows that a rank-one operator $xy^*\in\mathcal{U}_{\tau_1}$ if and only if there exists $N\otimes I\in\mathcal{N}\otimes I$ such that $x\in N\otimes I$ and $y\in(N\otimes I)^\perp_\sim$, where

$$\begin{aligned}(N\otimes I)^\perp_\sim&=(\vee\{N'\otimes I: N\otimes I\not\leq N'_-\otimes I\})^\perp\\&=(\vee\{N'\otimes I: N'_-\otimes I<N\otimes I\})^\perp\\&=(\vee\{N'\otimes I: N'_-<N\})^\perp\\&=(N\otimes I)^\perp=N^\perp\otimes I.\end{aligned}$$

Similarly, a rank-one operator $xy^*\in\mathcal{U}_{\tau_2}$ if and only if there exists $I\otimes M\in I\otimes\mathcal{M}$ such that $x\in I\otimes M$ and $y\in I\otimes M^\perp$. Hence a rank-one operator $xy^*\in\mathcal{U}_\tau=\mathcal{U}_{\tau_1}\cap\mathcal{U}_{\tau_2}$ if and only if there exist $N\in\mathcal{N}$ and $M\in\mathcal{M}$ such that $x\in N\otimes M$ and $y\in N^\perp\otimes M^\perp$. If $x=x_1\otimes x_2$ and $y=y_1\otimes y_2$, we can obtain that

$$\begin{aligned}xy^*&=(N\otimes M)[(x_1\otimes x_2)(y_1\otimes y_2)^*](N^\perp\otimes M^\perp)\\&=(N\otimes M)[(x_1y_1^*)\otimes(x_2y_2^*)](N^\perp\otimes M^\perp)\\&=N(x_1y_1^*)N^\perp\otimes M(x_2y_2^*)M^\perp\in\mathcal{R}_N\otimes\mathcal{R}_M\end{aligned}$$

(here the second equality follows from $(x_1\otimes x_2)(y_1\otimes y_2)^*=(x_1y_1^*)\otimes(x_2y_2^*)$, it is casy to prove). In general, there exist nets $\{z_\alpha\}$, $\{w_\beta\}$ such that

$$z_\alpha\xrightarrow{\|\cdot\|}x\quad\text{and}\quad w_\beta\xrightarrow{\|\cdot\|}y,$$

where z_α, w_β are finite linear combinations of simple tensors in $\mathcal{H}_1\otimes\mathcal{H}_2$. Thus

$$\begin{aligned}&(N\otimes M)(z_\alpha w_\beta^*)(N^\perp\otimes M^\perp)\xrightarrow{\|\cdot\|}(N\otimes M)(xy^*)(N^\perp\otimes M^\perp)=xy^*,\\&xy^*\in\mathcal{R}_N\otimes_n\mathcal{R}_M\subseteq\mathcal{R}_N\otimes_w\mathcal{R}_M.\end{aligned}$$

Hence each rank-one operator in $\mathcal{U}_\tau$ belongs to $\mathcal{R}_N\otimes_w\mathcal{R}_M$, so $\mathcal{U}_\tau\subseteq\mathcal{R}_N\otimes_w\mathcal{R}_M$ and $\mathcal{R}_N\otimes_w\mathcal{R}_M=\mathcal{U}_\tau$.

Corollary 2.1. *The following statements are equivalent:*

(1) $xy^*\in\mathcal{R}_N\otimes_n\mathcal{R}_M$;

(2) $xy^*\in\mathcal{R}_N\otimes_w\mathcal{R}_M$;

(3) *there exist* $N\in\mathcal{N}$ *and* $M\in\mathcal{M}$ *such that* $x\in N\otimes M$ *and* $y\in N^\perp\otimes M^\perp$.

Proof. (1)⇒(2) It is obvious.

(2)⇒(3) and (3)⇒(1) They follow from the proof of Theorem 2.1.

Lemma 2.4. $xy^*\in\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$ *if and only if there exist* $N\in\mathcal{N}$ *and* $M\in\mathcal{M}$ *such that* $x\in N\otimes M$ *and* $y\in N_-^\perp\otimes M_-^\perp$.

Proof. Since $\mathcal{N}\otimes\mathcal{M}=(\mathcal{N}\otimes I)\vee(I\otimes\mathcal{M})$, we have that

$$\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})=\mathrm{Alg}(\mathcal{N}\otimes I)\cap\mathrm{Alg}(I\otimes\mathcal{M}).$$

Just like the proof in Theorem 2.1, a rank-one operator $xy^* \in \mathrm{Alg}(\mathcal{N}\otimes I)$ if and only if there exists an element $N \in \mathcal{N}$ such that $x \in N\otimes I$ and $y \in N_-^\perp \otimes I$. Similarly, $xy^* \in \mathrm{Alg}(I\otimes\mathcal{M})$ if and only if there exists $M \in \mathcal{M}$ such that $x \in I\otimes M$ and $y \in I\otimes M_-^\perp$. Therefore $xy^* \in \mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$ if and only if there exist $N\in\mathcal{N}$ and $M\in\mathcal{M}$ such that $x \in N\otimes M$ and $y \in N_-^\perp \otimes M_-^\perp$. Conversely, if there exist $N\in\mathcal{N}, M\in\mathcal{M}$ such that $x\in N\otimes M$ and $\in N_-^\perp\otimes M_-^\perp$, we have

$$xy^* \in \mathrm{Alg}(\mathcal{N}\otimes I)\cap \mathrm{Alg}(I\otimes\mathcal{M}) = \mathrm{Alg}(\mathcal{N}\otimes\mathcal{M}).$$

Lemma 2.5. *$0 \neq (N\ominus N_-)\otimes(M\ominus M_-)$ is an atom of $\mathcal{N}\otimes\mathcal{M}$.*

Proof. Recall that an atom P of $\mathcal{N}\otimes\mathcal{M}$ is an interval projection from $\mathcal{N}\otimes\mathcal{M}$ such that for any $E\in\mathcal{N}\otimes\mathcal{M}$, either $P\le E$ or $PE=0$ (see [4]). Set $P=(N\ominus N_-)\otimes(M\ominus M_-)$. $P = N\otimes M-[(N_-\otimes M)\vee(N\otimes M_-)]$ is an interval projection. For any $E=E_1\otimes E_2\in\mathcal{N}\otimes\mathcal{M}$, since $\mathcal{N}$ is totally ordered, either $E_1\le N_-$ or $E_1\ge N$. If $E_1\le N_-$, then $P(E_1\otimes E_2)=0$; if $E_1\ge N$, since $\mathcal{M}$ is also totally ordered, either $E_2\le M_-$ or $E_2\ge M$. If $E_2\le M_-$, then $P(E_1\otimes E_2)=0$; and if $E_2\ge M$, then $P\le E_1\otimes E_2$. Hence for any $E=E_1\otimes E_2$, either $P\le E_1\otimes E_2$ or $P(E_1\otimes E_2)=0$.

Now for any $E\in\mathcal{N}\otimes\mathcal{M}$, by virtue of [3, Proposition 2.4] we have

$$E=\vee\{E_1\otimes E_2 : E_1\otimes E_2\le E\}.$$

If for any $E_1\otimes E_2\le E$, $P(E_1\otimes E_2)=0$, then $PE=0$; if there exists some $E_1\otimes E_2\le E$ such that $P(E_1\otimes E_2)\neq 0$. It follows from the result of the preceding paragraph that $P\le E_1\otimes E_2$ and $P\le E$.

Proposition 2.2. *If a rank-one operator $xy^*\in\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$, then the following statements are equivalent:*

(1) *$xy^*\in\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$;*

(2) *there exists $L\in\mathcal{N}\otimes\mathcal{M}$ such that $x\in L$ and $y\in L^\perp$.*

Proof. (1)⇒(2) Since $xy^*\in\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$, it follows from Lemma 2.4 that there exist $N\in\mathcal{N}$ and $M\in\mathcal{M}$ such that $x\in N\otimes M$ and $y\in N_-^\perp\otimes M_-^\perp$. Set

$$\begin{aligned}
G_1 &= (N\ominus N_-)\otimes(M\ominus M_-),\\
G_2 &= (N\otimes M)\ominus G_1 = (N_-\otimes M)\vee(N\otimes M_-),\\
G_3 &= (N_-^\perp\otimes M_-^\perp)\ominus G_1 = (N^\perp\otimes M_-^\perp)\vee(N_-^\perp\otimes M^\perp).
\end{aligned}$$

If $G_1=0$ then $N\ominus N_-=0$ or $M\ominus M_-=0$. In this case $L=N\otimes M$ satisfies the condition in 2). Now we suppose that $G_1\neq 0$. Since $N\otimes M=G_1+G_2$ and $N_-^\perp\otimes M_-^\perp=G_1+G_3$, we have

$$\begin{aligned}
xy^* &= (G_1+G_2)(xy^*)(G_1+G_3)\\
&= (N\otimes M)(xy^*)G_3+G_2(xy^*)G_1+G_1(xy^*)G_1.
\end{aligned}$$

Since $xy^*\in\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ and G_1 is an atom of $\mathcal{N}\otimes\mathcal{M}$, it follows from [1, Theorem 4.8] that $G_1(xy^*)G_1=0$. Hence $x\in G_1^\perp$ or $y\in G_1^\perp$. If $x\in G_1^\perp$ then $x\in G_2$ and $y\in G_1+G_3 = N_-^\perp\otimes M_-^\perp\subseteq G_2^\perp$; if $y\in G_1^\perp$, then $y\in G_3\subseteq (N\otimes M)^\perp$ and $x\in N\otimes M$.

(2)⇒(1) If there exists $L\in\mathcal{N}\otimes\mathcal{M}$ such that $x\in L$ and $y\in L^\perp$, then for any $T\in$

$\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$ we have $L^\perp TL=0$ and

$$[(xy^*)T]^n=[L(xy^*)L^\perp T]^n=0,\quad \forall n\geq 2.$$

So $(xy^*)T$ is quasinilpotent. It follows from the definition of $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ and $xy^*\in\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$ that $xy^*\in\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$.

Lemma 2.6. $\mathcal{R}_N\otimes_n\mathcal{R}_M\subseteq\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$.

Proof. For any $A\in\mathcal{B}(\mathcal{H}_1), B\in\mathcal{B}(\mathcal{H}_2), N\in\mathcal{N}$ and $M\in\mathcal{M}$, we have

$$\begin{aligned}(N\otimes M)(A\otimes B)(N^\perp\otimes M^\perp)&=NAN^\perp\otimes MBM^\perp\\ &\in\mathcal{R}_N\otimes\mathcal{R}_M\\ &\subseteq\mathrm{Alg}(\mathcal{N})\otimes_w\mathrm{Alg}(\mathcal{M})=\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M}).\end{aligned}$$

For any $T\in\mathrm{Alg}(\mathcal{N}\otimes\mathcal{M})$, $T(N\otimes M)\subseteq N\otimes M$. Thus

$$[(N\otimes M)(A\otimes B)(N^\perp\otimes M^\perp)T]^n=0,\quad \forall n\geq 2.$$

It follows from the definition of Jacobson radical that $NAN^\perp\otimes MBM^\perp\in\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$. Recall that $\mathcal{S}_1\otimes_n\mathcal{S}_2=\overline{\mathcal{S}}_1\otimes_n\overline{\mathcal{S}}_2$, where $\mathcal{S}_i$ are subspaces of $\mathcal{H}_i$. Since

$$\begin{aligned}\mathcal{R}_N&=\overline{\mathrm{span}}\{NAN^\perp: A\in\mathcal{B}(\mathcal{H}_1), N\in\mathcal{N}\},\\ \mathcal{R}_M&=\overline{\mathrm{span}}\{MBM^\perp: B\in\mathcal{B}(\mathcal{H}_2), M\in\mathcal{M}\},\end{aligned}$$

we have

$$\begin{aligned}\mathcal{R}_N\otimes_n\mathcal{R}_M&=\mathrm{span}\{NAN^\perp: A\in\mathcal{B}(\mathcal{H}_1), N\in\mathcal{N}\}\otimes_n\mathrm{span}\{MBM^\perp: B\in\mathcal{B}(\mathcal{H}_2), M\in\mathcal{M}\}\\ &\subseteq\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}.\end{aligned}$$

Theorem 2.2. *If $\mathcal{N},\mathcal{M}$ are non-trivial, then the following statements are equivalent:*

(1) $\mathcal{R}_N\otimes_w\mathcal{R}_M=\mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}}$;

(2) $\mathcal{R}_N\otimes_n\mathcal{R}_M$ *and* $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ *have the same rank-one operators;*

(3) $\mathcal{N}$, $\mathcal{M}$ *are continuous.*

Proof. (1)⇒(2) It follows from Lemma 2.6 that we only need to prove that each rank-one operator in $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ belongs to $\mathcal{R}_N\otimes_n\mathcal{R}_M$. Suppose that

$$xy^*\in\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}\subseteq\mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}}=\mathcal{R}_N\otimes_w\mathcal{R}_M.$$

It follows from Corollary 6 that $xy^*\in\mathcal{R}_N\otimes_n\mathcal{R}_M$.

(2)⇒(3) If at least one of $\mathcal{N},\mathcal{M}$ is not continuous, without loss of genenrality, we suppose that $\mathcal{N}$ is not continuous. Thus, there exists $N\in\mathcal{N}$ such that $N>0$ and $N\neq N_-$. Since $\mathcal{M}$ is non-trivial, choose $M\in\mathcal{M}$ such that $0<M<I$. We choose non-zero vectors $x_1\in N\ominus N_-$, $x_2\in M$ and $y_2\in M^\perp$, then

$$x=x_1\otimes x_2\in N\otimes M,$$

and

$$y=x_1\otimes y_2\in(N\ominus N_-)\otimes M^\perp\subseteq(N\otimes M)^\perp\cap(N_-^\perp\otimes M_-^\perp).$$

Then it follows from Lemma 2.4 and Proposition 2.2 that $xy^*\in\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$. By the hypothesis of (2), $xy^*\in\mathcal{R}_N\otimes_n\mathcal{R}_M$. Thus, it follows from Corollary 2.1 that there exist $E_1\in\mathcal{N}$ and $E_2\in\mathcal{M}$ such that

$$x=x_1\otimes x_2\in E_1\otimes E_2\quad\text{and}\quad y=x_1\otimes y_2\in E_1^\perp\otimes E_2^\perp.$$

Since $\mathcal{N}$ is totally ordered, either $E_1 \leq N_-$ or $E_1 \geq N$. If $E_1 \leq N_-$, then

$$\begin{aligned} x_1 \otimes x_2 &= (E_1 \otimes E_2)(x_1 \otimes x_2) \\ &= (E_1 \otimes E_2)[(N \ominus N_-) \otimes M](x_1 \otimes x_2) = 0; \end{aligned}$$

if $E_1 \geq N$, then

$$x_1 \otimes y_2 = (E_1^{\perp} \otimes E_2^{\perp})[(N \ominus N_-) \otimes M^{\perp}](x_1 \otimes y_2) = 0.$$

This contradicts that x_1, x_2 and y_2 are non-zero vectors. Hence both $\mathcal{N}, \mathcal{M}$ are continuous.

(3)⇒(1) Since $\mathcal{N}, \mathcal{M}$ are continuous, it follows from Proposition 2.1 that we have

$$\begin{aligned} \mathcal{R}^w_{\mathcal{N}} &= \{A \in \mathcal{B}(\mathcal{H}_1) : AN \subseteq N, \forall N \in \mathcal{N}\} = \mathrm{Alg}\mathcal{N}, \\ \mathcal{R}^w_{\mathcal{M}} &= \{B \in \mathcal{B}(\mathcal{H}_2) : BM \subseteq M, \forall M \in \mathcal{M}\} = \mathrm{Alg}\mathcal{M}. \end{aligned}$$

Hence it follows from Lemma 2.2 and Lemma 2.6 that

$$\begin{aligned} \mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}} &\supseteq \mathcal{R}_N \otimes_w \mathcal{R}_M = \mathcal{R}^w_{\mathcal{N}} \otimes_w \mathcal{R}^w_{\mathcal{M}} \\ &= \mathrm{Alg}\mathcal{N} \otimes_w \mathrm{Alg}\mathcal{M} = \mathrm{Alg}(\mathcal{N} \otimes \mathcal{M}) \supseteq \mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}}. \end{aligned}$$

So $\mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}} = \mathcal{R}_N \otimes_w \mathcal{R}_M$.

Corollary 2.2. *If $\mathcal{N}, \mathcal{M}$ are non-trivial and at least one of $\mathcal{N}, \mathcal{M}$ is not continuous, then $\mathcal{R}_N \otimes_n \mathcal{R}_M \subset \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$.*

Proof. If $\mathcal{R}_N \otimes_n \mathcal{R}_M = \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$, then $\mathcal{R}_N \otimes_w \mathcal{R}_M = \mathcal{R}^w_{\mathcal{N}\otimes\mathcal{M}}$ and $\mathcal{N}, \mathcal{M}$ continuous. This is a contradiction.

Remark 2.1. If $\mathcal{N}, \mathcal{M}$ are trivial, then $\mathcal{R}_N = \mathcal{R}_M = \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}} = (0)$. In this case we have $\mathcal{R}_N \otimes_n \mathcal{R}_M = \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$. If $\mathcal{N}$ is trivial and $\mathcal{M}$ is not, then $\mathcal{R}_N \otimes_n \mathcal{R}_M = (0)$ and the Jacobson radical $\mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$ of $\mathrm{Alg}(\mathcal{N} \otimes \mathcal{M})$ is not zero (it follows from Proposition 2.2). Thus in this case, $\mathcal{R}_N \otimes_n \mathcal{R}_M \subset \mathcal{R}_{\mathcal{N}\otimes\mathcal{M}}$.

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