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LIFE SPAN OF A SMOOTH SOLUTION FOR THE SURFACE DIFFUSION FLOW**

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Abstract

Consider the motion of immersed hypersurfaces driven by surface diffusion flow and give an lower bound on the life span of a smooth immersed solution, which depends only on how much the curvature of the initial surface is concentrated in space.

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§1. Introduction

Let $\varphi_0 : M \to \mathbb{R}^{n+1}$ be a closed immersed orientable hypersurface. We consider the surface diffusion flow $\varphi : M \times [0,T] \to \mathbb{R}^{n+1}$, i.e. the geometric evolution equation is

$$\frac{\partial}{\partial t}\varphi = (\triangle H)\nu, \quad \varphi(\cdot,0) = \varphi_0, \tag{1.1}$$

where $\triangle$ and H stand for the Laplacian and the mean curvature of $M_t = \{\varphi(x,t) : x \in M\}$, respectively, ν denotes the unit vector field normal to the hypersurface M_t.

The surface diffusion flow (1.1) was first proposed by Mullins[15] to model surface dynamics for phase interfaces when the evolution is only governed by mass diffusion in the interface. It has also been examined in a more general mathematical and physical context by Davi and Gurtin[5], and by Cahn and Taylor[3]. More recently, Cahn, Elliot, and Novick-Cohen[2] showed by formal asymptotic that the surface diffusion flow is the singular limit of the zero level set of the solution to the Cahn-Hillard equation with a concentration-dependent mobility.

In two dimensions and for strip-like domains, the surface flow was investigated by Baras, Duchon, and Robert[1]. They proved global existence of weak solutions. Also in two dimensions, the surface flow for closed embedded curves was analytically investigated by Elliot and Garcke[6]. They showed local existence and regularity for C^4-initial curves, and global existence for small perturbations of circles. Furthermore, assuming global existence, they showed that any closed curve would become circular under this evolution. They did not

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obtain uniqueness of solutions. Recently, Giga and Ito[8] established the existence of unique (local) solutions for immersed H^4-initial curves. Moreover, they proved that the surface diffusion flow could drive an initially embedded curve to a self intersection. In any dimension, Escher, Mayer and Simonett[7] showed the long time existence and convergence of the flow for initial data which are $C^{2,\alpha}$-close to a sphere. By numerical simulations, Escher, Mayer and Simonett showed that an immersed curve could develop singularities under the surface diffusion flow. To the author's knowledge, up to now there is neither a proof of regularity of the flow, no an example showing the development of a singularity. In this paper, as a first step in this research, we obtain the following theorem.

Theorem 1.1. *Let $\varphi : M \to \mathbb{R}^{n+1}$ be a smooth immersion. There are constants $\varepsilon_0 > 0, C < +\infty$ depending only on n, such that if $\rho > 0$ is chosen with*

$$\int_{B_\rho(x)} |A_0|^2 d\mu_0 \le \varepsilon < \varepsilon_0 \quad \text{for any} \ \ x \in \mathbb{R}^{n+1}, \tag{1.2}$$

the maximal time T of smooth existence for (1.1) *satisfies*

$$T \ge \frac{1}{C}\rho^4, \tag{1.3}$$

and one has the estimate

$$\int_{B_\rho(x)} |A|^2 d\mu \le C\varepsilon \quad \text{for} \ \ 0 \le t \le \frac{1}{C}\rho^4, \tag{1.4}$$

where $A = (h_{ij})$ denotes the second fundamental form.

In the statement of the theorem the integrals should be interpreted as integrals over the preimage of $B_\rho(x)$ under φ_0 and φ, respectively.

This paper is organized as follows. In Section 2, we collect some notations and basic results needed; In Section 3, we derive energy estimates; In Section 4, we derive the local estimates of the curvature by concentration of curvature; In Section 5, we give a proof of Theorem 1.1.

Despite the large literature on the mean curvature flow, fourth or even higher order flows appeared only recently. Besides the cited works above, we quote the work of Simonett[19] on the gradient flow of the Willmore functional $W(\varphi) = \int_M |A|^2 d\mu$ defined on surfaces immersed in $\mathbb{R}^3$. In this paper it is shown the long time existence and convergence of the flow for initial data which are $C^{2,\alpha}$-close to a sphere. In [12], Kuwert and Schatzle studied the global existence and regularity of the gradient flow of the Willmore functional for general initial data. Finally, in a very recent paper [13], Mantegazza studied the global existence of the gradient flow of the functional $F_m(\varphi) = \int_M [1 + |\nabla^m \nu|^2] d\mu$ in the case $m > [\frac{n}{2}]$, where n is the dimension of M. For other geometric evolution equations of higher order, refer to [4, 11, 16–18]. Our work borrows from [10, 12, 13, 16, 17] the basic idea of using interpolation inequalities as a tool to derive a-priori estimates.

§2. Notations and Preliminaries

We devote this section to introduce the basic notations and facts about differentiable and Riemannian manifolds which will be needed later in the paper. The main objects of the paper are n-dimensional closed hypersurfaces immersed in $\mathbb{R}^{n+1}$, that is, pairs (M, φ) where M is an n-dimensional smooth manifold, connected with empty boundary, and a smooth map $\varphi : M \to \mathbb{R}^{n+1}$ such that the rank of $d\varphi$ is everywhere equal to n. The manifold M gets in a natural way a metric tensor g turning it in a Riemannian manifold (M, g), by pulling back the standard scalar product of $\mathbb{R}^{n+1}$ with the immersion map φ. Taking local coordinates around $p \in M$ given by a chart $F : \mathbb{R}^n \supset U \to M$, we identify the map φ with its expression in coordinates $\varphi \circ F : \mathbb{R}^n \supset U \to \mathbb{R}^{n+1}$, then we have local basis of T_pM and T_p^*M, respectively given by vectors $\{\frac{\partial}{\partial x_i}\}$ and covectors $\{dx_j\}$. We will denote vectors on M by $X = X^i$, which means $X = X^i \frac{\partial}{\partial x_i}$, covectors by $Y = Y_j$, that is , $Y = Y_j dx_j$ and a general mixed tensor with $T = T^{i_1 \cdots i_k}_{j_1 \cdots j_l}$, where the indices refer to the local basis. The scalar product in $\mathbb{R}^3$ will be denoted by $\langle \, , \, \rangle$. As the metric g is obtained by pulling it back with φ, we have $g_{ij} = \langle \frac{\partial \varphi(x)}{\partial x_i}, \frac{\partial \varphi(x)}{\partial x_j} \rangle$. The canonical measure induced by the metric g is given by $\mu = \sqrt{G}\mathcal{L}^n$, where $G = \det(g_{ij})$ and $\mathcal{L}^n$ is the standard Lebesgue measure on $\mathbb{R}^n$.

The second fundamental form $A = h_{ij}$ of M is the 2-tensor defined as follows:

$$h_{ij}(x) = -\Big\langle \nu(x), \frac{\partial^2 \varphi(x)}{\partial x_i \partial x_j} \Big\rangle, \tag{2.1}$$

the mean curvature H is the trace of A,

$$H(x) = g^{ij} h_{ij}. \tag{2.2}$$

The induced covariant derivative on (M, g) of a vector field X is given by

$$\nabla_j X^i = \frac{\partial}{\partial x_j} X^i + \Gamma^i_{jk} X^k, \tag{2.3}$$

where the Christoffel symbols $\Gamma = \Gamma^i_{ij}$. In all the paper the covariant derivative ∇T of a tensor $T = T^{i_1 \cdots i_k}_{j_1 \cdots j_l}$ will be denoted by $\nabla_s T^{i_1 \cdots i_k}_{j_1 \cdots j_l} = (\nabla T)^{i_1 \cdots i_k}_{s j_1 \cdots j_l}$. With $\nabla^m T$ we will mean the k-th iterated covariant derivative of a tensor T.

We recall that the gradient ∇f of a function and the divergence $\operatorname{div} X$ of a vector field at a point $p \in (M, g)$ are defined respectively by

$$g(\nabla f(p), v) = df_p(v), \qquad \forall v \in T_pM$$

and

$$\operatorname{div} X = \operatorname{trace} \nabla X = \nabla_i X^i = \frac{\partial}{\partial x_i} X^i + \Gamma^i_{ik} X^k. \tag{2.4}$$

The Laplacian $\triangle T$ of a tensor T is

$$\triangle T = g^{ij} \nabla_i \nabla_j T. \tag{2.5}$$

The Riemann tensor, the Ricci tensor and the scalar curvature are expressible via the second fundamental form as follows:

$$R_{ijkl} = h_{ik} h_{jl} - h_{il} h_{jk}, \quad Ric_{ij} = H h_{ij} - h_{il} g^{lk} h_{kj}, \quad R = H^2 - |A|^2. \tag{2.6}$$

Hence, the formulas for the interchange of covariant derivatives, which involve the Riemann tensor, become

$$\begin{aligned} &\nabla_i \nabla_j X^s - \nabla_j \nabla_i X^s = R_{ijkl} g^{ks} X^l = R^s_{ijl} X^l = (h_{ik} h_{jl} - h_{il} h_{jk}) g^{ks} X^l, \\ &\nabla_i \nabla_j Y_k - \nabla_j \nabla_i Y_k = R_{ijkl} g^{ls} Y_s = R^s_{ijk} Y_s = (h_{ik} h_{jl} - h_{il} h_{jk}) g^{ls} Y_s. \end{aligned} \tag{2.7}$$

The Codazzi equations

$$\nabla_i h_{jk} = \nabla_j h_{ik} = \nabla_k h_{ij}$$

imply the following identity which will be crucial in the sequel,

$$\triangle h_{ij} = \nabla_i \nabla_j H + H h_{il} g^{ls} h_{sj} - |A|^2 h_{ij}, \tag{2.8}$$

and the following inequalities

$$|\nabla A| \le C |\nabla A^0|, \quad |\nabla^2 A| \le C |\nabla^2 A^0|. \tag{2.9}$$

The following Gauss-Weingarten relations are fundamental

$$\frac{\partial^2\varphi}{\partial x_i\partial x_j}=\Gamma_{ij}^k\frac{\partial\varphi}{\partial x_k}-h_{ij}\nu,\quad \frac{\partial}{\partial x_j}\nu=h_{jl}g^{ls}\frac{\partial\varphi}{\partial x_s}, \tag{2.10}$$

which imply that $|\nabla\nu|=|A|$.

In all this paper we will write $T*S$, following Hamilton[9], to denote a tensor formed by contraction on some indices of the tensor T and S using the coefficients g^{ij}. We use the notation $P_r^m(A)$ for any term of the type

$$P_r^m(A)=\sum_{i_1+\cdots+i_r=m}\nabla^{i_1}A*\nabla^{i_2}A*\cdots*\nabla^{i_r}A. \tag{2.11}$$

Now we derive some evolution equations from the flow (1.1) which will be useful for the estimates of the following sections.

Lemma 2.1. *We have*

$$\frac{\partial}{\partial t}g_{ij}=-2\triangle Hh_{ij}, \tag{2.12}$$

$$\frac{\partial}{\partial t}g^{ij}=2\triangle Hh^{ij}, \tag{2.13}$$

$$\frac{\partial}{\partial t}\nu=-\nabla(\triangle H), \tag{2.14}$$

$$\frac{\partial}{\partial t}\Gamma_{jk}^i=\nabla(\triangle H)*A+(\triangle H)*A, \tag{2.15}$$

$$\frac{\partial}{\partial t}(d\mu)=-H\triangle Hd\mu, \tag{2.16}$$

$$\frac{\partial}{\partial t}h_{ij}=-\triangle^2h_{ij}+P_3^2(A), \tag{2.17}$$

$$\frac{\partial}{\partial t}\nabla^kh_{ij}=-\triangle^2\nabla^kh_{ij}+P_3^{k+2}(A). \tag{2.18}$$

In the end of this section, we discuss the small time existence of the surface diffusion flow (1.1). Suppose that $\varphi_0:M\to\mathbb{R}^{n+1}$ is a smooth immersion of an n-dimensional hypersurface M which is compact, connected and has empty boundary. We look for a smooth function $\varphi:M\times[0,T]$ such that

(1) the map $\varphi_t=\varphi(\cdot,t):M\to\mathbb{R}^{n+1}$ is an immersion;

(2) the following partial differential equation is satisfied $\frac{\partial}{\partial t}\varphi(p,t)=\triangle H(p,t)\nu(p,t)$.

The small time existence of such problem is standard[11,17].

Lemma 2.2. *For any smooth immersion $\varphi_0:M\to\mathbb{R}^{n+1}$, there exists a unique smooth φ to the surface diffusion flow* (1.1) *defined on some interval $0\le t<T$ and taking φ_0 as its initial value.*

§3. Energy Inequalities

In this section, we will derive energy inequalities from the evolution equations obtained in previous section.

Lemma 3.1. *Let $\eta:M\times[0,T]\to\mathbb{R}$ be a C^2-smooth function. The following formula* *holds,*

$$\frac{d}{dt}\int_M \frac{1}{2}\eta|\nabla^k A|^2 d\mu = -\int_M \frac{1}{2}\eta H\triangle H|\nabla^k A|^2 d\mu + \int_M \frac{1}{2}\eta_t|\nabla^k A|^2 d\mu$$
$$- \int_M g^{is}g^{jz}\nabla_{i_{k+2}}\nabla_{i_{k+1}}\nabla_{i_1\cdots i_k}h_{ij}\nabla^{i_{k+2}}\nabla^{i_{k+1}}(\eta\nabla^{i_1\cdots i_k}h_{sz})d\mu$$
$$+ \int_M \eta P_3^{k+2}(A) * \nabla^k A d\mu. \tag{3.1}$$

Proof. By using Lemma 2.1, we have

$$\frac{d}{dt}\int_M \frac{1}{2}\eta|\nabla^k A|^2 d\mu = -\int_M \frac{1}{2}\eta H\triangle H|\nabla^k A|^2 d\mu + \int_M \frac{1}{2}\eta_t|\nabla^k A|^2 d\mu$$
$$+ \int_M \frac{1}{2}\eta\frac{\partial}{\partial t}|\nabla^k A|^2 d\mu. \tag{3.2}$$

Note that

$$\frac{\partial}{\partial t}|\nabla^k A|^2 = -2g^{is}g^{jz}\nabla^{i_{k+1}}\nabla^{i_{k+2}}\nabla_{i_{k+2}}\nabla_{i_{k+1}}\nabla_{i_1\cdots i_k}h_{ij}\nabla^{i_1\cdots i_k}h_{sz}$$
$$+ P_3^{k+2}(A) * \nabla^k A.$$

Hence, using divergence theorem, we have

$$\int_M \frac{1}{2}\eta\frac{\partial}{\partial t}|\nabla^k A|^2 d\mu = \int_M [-g^{is}g^{jz}\nabla_{i_{k+2}}\nabla_{i_{k+1}}\nabla_{i_1\cdots i_k}h_{ij}\nabla^{i_{k+2}}\nabla^{i_{k+1}}(\eta\nabla^{i_1\cdots i_k}h_{sz})$$
$$+ \eta P_3^{k+2}(A) * \nabla^k A]d\mu.$$

Combining this relation with (3.2), we complete the proof of Lemma 3.1.

Lemma 3.2. *Let* $\eta = \gamma^5$ *with* $\gamma \in C^2(M \times [0, T])$ *and* $s \geq 4$. *Then for* $C = C(n, s)$ *we have*

$$\frac{d}{dt}\int_M |\nabla^k A|^2\gamma^s d\mu + \int_M |\nabla^{k+2}A|^2\gamma^s d\mu$$
$$\leq \int_M s|\nabla^k A|^2\gamma^{s-1}\gamma_t d\mu + C\int_M |\nabla^k A|^2\gamma^{s-4}(|\nabla\gamma|^4 + \gamma^2|\nabla^2\gamma|^2)d\mu$$
$$+ \int_M P_3^{k+2}(A) * \nabla^k A\gamma^s d\mu. \tag{3.3}$$

Proof. First we deal with the third term on the right of (3.1).

$$I = -\int_M g^{is}g^{jz}\nabla_{i_{k+2}}\nabla_{i_{k+1}}\nabla_{i_1\cdots i_k}h_{ij}\nabla^{i_{k+2}}\nabla^{i_{k+1}}(\gamma^s\nabla^{i_1\cdots i_k}h_{sz})d\mu$$
$$\leq -\int_M |\nabla^{k+2}A|^2\gamma^s d\mu + \frac{1}{4}\int_M |\nabla^{k+2}A|^2\gamma^s d\mu$$
$$+ C\int_M |\nabla^{k+1}A|^2\gamma^{s-2}|\nabla\gamma|^2 d\mu + C\int_M |\nabla^k A|^2\gamma^{s-4}(|\nabla\gamma|^4 + \gamma^2|\nabla^2\gamma|^2)d\mu. \tag{3.4}$$

For the third term on the right of (3.4), integrating by parts, we have

$$\int_M |\nabla^{k+1}A|^2\gamma^{s-2}|\nabla\gamma|^2 d\mu$$
$$\leq \varepsilon\int_M |\nabla^{k+2}A|^2\gamma^s d\mu + C(\varepsilon)\int_M |\nabla^k A|^2\gamma^{s-4}|\nabla\gamma|^4 d\mu$$
$$+ \frac{1}{2}\int_M |\nabla^{k+1}A|^2\gamma^{s-2}|\nabla\gamma|^2 d\mu + C\int_M |\nabla^k A|^2\gamma^{s-2}|\nabla^2\gamma|^2 d\mu.$$

Combining this relation with (3.4), we have

$$I \le -\frac{1}{2}\int_M |\nabla^{k+2}A|^2\gamma^s d\mu + C\int_M |\nabla^k A|^2\gamma^{s-4}(|\nabla\gamma|^4 + \gamma^2|\nabla^2\gamma|^2)d\mu.$$

Substituting this relation to (3.1) we have (3.3). The proof of Lemma 3.2 is completed.

In the following we assume that $\gamma = \widetilde{\gamma}\circ\varphi$, where $\|\widetilde{\gamma}\|_{C^2(\mathbb{R}^{n+1})} \le C$. This implies $\nabla\gamma = (\nabla\widetilde{\gamma}\circ\varphi)\nabla\varphi$ and $\nabla^2\gamma = (\nabla^2\widetilde{\gamma}\circ\varphi)(\nabla\varphi,\ \nabla\varphi) + (\nabla\widetilde{\gamma}\circ\varphi)A$, and therefore we have

$$|\nabla\gamma| \le C, \quad |\nabla^2\gamma| \le C(1+|A|). \tag{3.5}$$

Lemma 3.3. *Let $\gamma = \widetilde{\gamma}\circ\varphi$ satisfy* (3.5). *Then we have*

$$\begin{aligned}&\frac{d}{dt}\int_M |\nabla^k A|^2\gamma^s d\mu + \frac{3}{4}\int_M |\nabla^{k+2}A|^2\gamma^s d\mu\\ \le& \int_M (P_3^{k+2}(A) * \nabla^k A\gamma^s + P_3^k(A) * \nabla^k A\gamma^{s-2} + P_3^k(A) * \nabla^k A\gamma^s)d\mu\\ &+ C\int_M |A|^2\gamma^{s-4-k}d\mu.\end{aligned}$$

Proof. We will estimate the terms in (3.3). First, we have, by integrating by parts, that

$$\begin{aligned}\int_M |\nabla^k A|^2\gamma^{s-1}\gamma_t d\mu &= \int_M |\nabla^k A|^2\gamma^{s-1}[(\nabla\widetilde{\gamma}\circ\varphi)\nu]\triangle H d\mu\\ &\le C\int_M |\nabla^{k+1}A|^2\gamma^{s-2}d\mu + C\int_M |\nabla^k A|^2\gamma^{s-4}d\mu\\ &\quad + \int_M (P_3^{k+2}(A) * \nabla^k A\gamma^s + P_3^k(A) * \nabla^k A\gamma^{s-2})d\mu.\end{aligned}\tag{3.6}$$

Now using the following interpolation inequality[12]

$$\Big(\int_M |\nabla^k\phi|^p\gamma^s d\mu\Big)^{\frac{1}{p}} \le \varepsilon\Big(\int_M |\nabla^{k+1}\phi|^p\gamma^{s+p}d\mu\Big)^{\frac{1}{p}} + C_\varepsilon\Big(\int_M |\phi|^p\gamma^{s+kp}d\mu\Big)^{\frac{1}{p}}, \tag{3.7}$$

where $2 \le p \le \infty, k \in \mathbb{N}, s \ge kp$ and $C_\varepsilon = C_\varepsilon(d,p,m,k)$, we have

$$\begin{aligned}&\int_M |\nabla^k A|^2\gamma^{s-4}d\mu + \int_M |\nabla^{k+1}A|^2\gamma^{s-2}d\mu\\ &\le \varepsilon\int_M |\nabla^{k+2}A|^2\gamma^s d\mu + C(\varepsilon)\int_M |A|^2\gamma^{s-4-2k}d\mu.\end{aligned}\tag{3.8}$$

Combining this relation with (3.6) we obtain

$$\begin{aligned}\int_M |\nabla^k A|^2 s\gamma^{s-1}\gamma_t d\mu &\le \varepsilon\int_M |\nabla^{k+2}A|^2\gamma^s d\mu + C(\varepsilon)\int_M |A|^2\gamma^{s-4-2k}d\mu\\ &\quad + \int_M (P_3^{k+2}(A) * \nabla^k A\gamma^s + P_3^k(A) * \nabla^k A\gamma^{s-2})d\mu.\end{aligned}\tag{3.9}$$

Now we deal with the following term

$$\begin{aligned}&\int_M |\nabla^k A|^2\gamma^{s-4}(|\nabla\gamma|^4 + \gamma^2|\nabla^2\gamma|^2)d\mu\\ \le& \varepsilon\int_M |\nabla^{k+2}A|^2\gamma^s d\mu + C(\varepsilon)\int_M |A|^2\gamma^{s-4-2k}d\mu + C\int_M P_5^k(A) * \nabla^k A\gamma^s d\mu.\end{aligned}\tag{3.10}$$

Substituting (3.9) and (3.10) to (3.3), we follow the conclusion of Lemma 3.3.

Corollary 3.1. *Suppose* $\chi_{B_\rho(x_0)} \le \widetilde{\gamma} \le \chi_{B_{2\rho}(x_0)}$ *and* $\|\nabla^j\widetilde{\gamma}\|_{L^\infty} \le C\rho^{-j}$ *for* $j = 1, 2$. *Under the assumptions of Lemma* 3.3, *we have*

$$\begin{aligned}
&\frac{d}{dt}\int_M |\nabla^k A|^2\gamma^s d\mu + \frac{3}{4}\int_M |\nabla^{k+2}A|^2 d\mu \\
&\le \int_M (P_3^{k+2}(A) * \nabla^k A\gamma^s + P_3^k(A) * \nabla^k A\gamma^{s-2} + P_5^k(A) * \nabla^k A\gamma^s)d\mu \\
&\quad + \frac{C}{\rho^{4+2k}}\int_M |A|^2\gamma^{s-4-2k}d\mu.
\end{aligned}$$

§4. Control by Concentration of Curvature

For simplicity, we denote $\|\phi\|_{p,U} = (\int_U |\phi|^p d\mu)^{\frac{1}{p}}$.

Lemma 4.1. *Let* $\varphi : M \times [0, T] \to \mathbb{R}^{n+1}$ *be a surface diffusion flow,* γ *as in* (3.5) *and let*

$$\varepsilon = \sup_{0\le t\le t} \|A\|^2_{2,[\gamma>0]} \le \varepsilon_0 \tag{4.1}$$

for some ε_0 *small enough depending on the constants in* (3.5). *Then for any* $t \in [0, T]$, *we have*

$$\begin{aligned}
&\int_{[\gamma=1]} |A|^2 d\mu + \frac{1}{2}\int_0^t \int_{[\gamma=1]} (|\nabla^2 A|^2 + |A|^2|\nabla A|^2 + |A|^6)d\mu d\tau \\
&\le \int_{[\gamma_0>0]} |A_0|^2 d\mu_0 + C\varepsilon t.
\end{aligned} \tag{4.2}$$

Proof. From Lemma 3.3 for $s = 4$ and $k = 0$, we have

$$\begin{aligned}
&\frac{d}{dt}\int_M |A|^2\gamma^4 d\mu + \frac{3}{4}\int_M |\nabla^2 A|^2\gamma^4 d\mu \\
&\le \frac{1}{4}\int_M |\nabla^2 A|^2\gamma^4 d\mu + C\int_M [|A|^6\gamma^4 + |\nabla A|^2|A|^2\gamma^4]d\mu + C\int_{[\gamma>0]} |A|^2 d\mu.
\end{aligned} \tag{4.3}$$

Here we have used the facts that

$$\int_M |\nabla^2 A||A|^3\gamma^4 d\mu \le \frac{1}{4}\int_M |\nabla^2 A|^2\gamma^4 d\mu + C\int_M |A|^6\gamma^4 d\mu,$$

$$\int_M |A|^4\gamma^2 d\mu \le \frac{1}{2}\int_M |A|^6\gamma^4 d\mu + \frac{1}{2}\int_{[\gamma>0]} |A|^2 d\mu.$$

Now we deal with the terms $\int_M |A|^6\gamma^4 d\mu$ and $\int_M |A|^2|\nabla A|^2\gamma^4 d\mu$. Recall the Michael-Simon Sobolev inequality[14]

$$\Big(\int_M u^2 d\mu\Big)^{\frac{1}{2}} \le C\Big(\int_M |\nabla u|d\mu + \int_M |H||u|d\mu\Big) \tag{4.4}$$

with $C = C(n)$. Letting $u = |A|^3\gamma^2$ we have

$$\begin{aligned}
&\int_M |A|^6\gamma^4 d\mu \\
&\le C\Big(\int_M |\nabla A|^2\gamma^2 d\mu\Big)^2 + C\int_{[\gamma>0]} |A|^2 d\mu \int_M |A|^6\gamma^4 d\mu + C\Big(\int_{[\gamma>0]} |A|^2 d\mu\Big)^2.
\end{aligned} \tag{4.5}$$

By integrating by parts, we have

$$\begin{aligned}&\int_M |\nabla A|^2\gamma^2 d\mu\\ &\le \Big(\int_{[\gamma>0]}|A|^2 d\mu\int_M|\nabla^2 A|^2\gamma^4 d\mu\Big)^{\frac{1}{2}}+C\int_{[\gamma>0]}|A|^2 d\mu+\int_M|\nabla A|^2\gamma^2 d\mu.\end{aligned} \tag{4.6}$$

From (4.5) and (4.6) we have

$$\begin{aligned}&\int_M |A|^6\gamma^4 d\mu\\ &\le C\int_{[\gamma>0]}|A|^2 d\mu\Big(\int_M[|\nabla^2 A|^2\gamma^4+|A|^6\gamma^4]d\mu\Big)+C\Big(\int_{[\gamma>0]}|A|^2 d\mu\Big)^2.\end{aligned} \tag{4.7}$$

Letting $u=|A||\nabla A|\gamma^2$ we obtain

$$\begin{aligned}\int_M |A|^2|\nabla A|^2\gamma^4 d\mu &\le C\int_{[\gamma>0]}|A|^2 d\mu\Big(\int_M[|\nabla^2 A|^2\gamma^4+|A|^6\gamma^4]d\mu\Big)\\ &\quad +C\Big(\int_{[\gamma>0]}|A|^2 d\mu\Big)^2+\Big(\int_M|\nabla A|^2\gamma^2 d\mu\Big)^2.\end{aligned} \tag{4.8}$$

Substituting (4.6) to (4.8) we have

$$\begin{aligned}&\int_M |A|^2|\nabla A|^2\gamma^4 d\mu\\ &\le C\int_{[\gamma>0]}|A|^2 d\mu\Big(\int_M[|\nabla^2 A|^2\gamma^4+|A|^6\gamma^4]d\mu\Big)+C\Big(\int_{[\gamma>0]}|A|^2 d\mu\Big)^2.\end{aligned} \tag{4.9}$$

Combining (4.3), (4.7) with (4.9) we have

$$\begin{aligned}&\frac{d}{dt}\int_M |A|^2\gamma^4 d\mu+\frac{1}{2}\int_M(|\nabla^2 A|^2+|A|^2|\nabla A|^2+|A|^6)\gamma^4 d\mu\\ &\le C\int_{[\gamma>0]}|A|^2 d\mu\Big(\int_M[|\nabla^2 A|^2\gamma^4+|A|^6\gamma^4]d\mu\Big)+C\Big(\int_{[\gamma>0]}|A|^2 d\mu\Big)^2+C\int_{[\gamma>0]}|A|^2 d\mu.\end{aligned}$$

Since $\int_{[\gamma>0]}|A|^2 d\mu\le\varepsilon_0$, we have

$$\frac{d}{dt}\int_M |A|^2\gamma^4 d\mu+\frac{1}{2}\int_M(|\nabla^2 A|^2+|A|^2|\nabla A|^2+|A|^6)\gamma^4 d\mu\le C\varepsilon.$$

Now (4.2) follows by integration over $[0,t]$.

Lemma 4.2. *Let $m\in\mathbb{N}$ and γ as in* (3.5)*. We have for any $s\ge 2m+4$,*

$$\begin{aligned}&\frac{d}{dt}\int_M |\nabla^m A|^2\gamma^s d\mu+\frac{1}{2}\int_M|\nabla^{m+2}A|^2\gamma^s d\mu\\ &\le C(\|A\|^4_{\infty,[\gamma>0]}+\|A\|^3_{\infty,[\gamma>0]})\int_M|\nabla^m A|^2\gamma^s d\mu\\ &\quad +C(1+\|A\|^4_{\infty,[\gamma>0]})\|A\|^2_{2,[\gamma>0]}.\end{aligned} \tag{4.10}$$

Proof. To prove this lemma, we need the following interpolation inequalities[12]

(i) Let $0\le i_1,\cdots,i_r\le k, i_1+\cdots+i_r=2k$ and $s\ge 2k$. Then

$$\Big|\int_M|\nabla^{i_1}\phi*\cdots*\nabla^{i_r}\phi\gamma^s|d\mu\Big|\le C\|\phi\|^{r-2}_\infty\Big(\int_M|\nabla^k\phi|^2\gamma^s d\mu+\|\phi\|^2_{L^2}\Big), \tag{4.11}$$

where $C=C(k,n,s,\|\nabla\gamma\|_\infty)$.

(ii) Let $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$, $1\le p,q,r\le\infty$ and $\alpha+\beta=1$, $\alpha,\beta\ge 0$. For $s\ge\max\{\alpha q,\beta p\}$ and $-\frac{1}{p}\le t\le\frac{1}{q}$, we have

$$\Big(\int_M|\nabla\phi|^{2r}\gamma^s d\mu\Big)^{\frac{1}{r}}\le C\Big(\int_M|\phi|^q\gamma^{s(1-tq)}d\mu\Big)^{\frac{1}{q}}\Big(\int_M|\nabla^2\phi|^p\gamma^{s(1+tp)}d\mu\Big)^{\frac{1}{p}}+Cs\Big(\int_M|\phi|^q\gamma^{s-\alpha q}d\mu\Big)^{\frac{1}{q}}\Big(\int_M|\nabla\phi|^p\gamma^{s-\beta p}d\mu\Big)^{\frac{1}{p}},\tag{4.12}$$

where $C=C(n,r)$. By Lemma 3.3, we have

$$\frac{d}{dt}\int_M|\nabla^m A|^2\gamma^s d\mu+\frac{3}{4}\int_M|\nabla^{m+2}A|^2\gamma^s d\mu\le\int_M(P_3^{m+2}(A)*\nabla^m A\gamma^s+P_5^m(A)*\nabla^m A\gamma^s)d\mu+C\int_M|A|^2\gamma^{s-4-2m}d\mu.\tag{4.13}$$

Using (4.11) with $r=6$ and $k=m$, we get

$$\Big|\int_M P_5^m(A)*\nabla^m A\gamma^s d\mu\Big|\le C\|A\|^4_{\infty,[\gamma>0]}\Big(\int_M|\nabla^m A|^2\gamma^s d\mu+\|A\|^2_{2,[\gamma>0]}\Big).\tag{4.14}$$

Using (4.11) with $r=4$ and $k=m$, we get

$$\Big|\int_M P_3^m(A)*\nabla^m A\gamma^{s-2} d\mu\Big|\le C\|A\|^2_{\infty,[\gamma>0]}\Big(\int_M|\nabla^m A|^2\gamma^{s-2} d\mu+\|A\|^2_{2,[\gamma>0]}\Big).\tag{4.15}$$

Using (4.11) with $r=4, k=m+1$, we get

$$\Big|\int_M P_3^{m+2}(A)*\nabla^m A\gamma^s d\mu\Big|\le\varepsilon\int_M|\nabla^{m+2}A|^2\gamma^s d\mu+C(\varepsilon)\|A\|^4_{\infty,[\gamma>0]}\int_M|\nabla^m A|^2\gamma^s d\mu+C\|A\|^2_{\infty,[\gamma>0]}\|A\|^2_{2,[\gamma>0]}+C\|A\|^2_{\infty,[\gamma>0]}\int_M|\nabla^{m+1}A|^2\gamma^s d\mu.\tag{4.16}$$

By (4.12) with $p=q=2r=2,\alpha=0,\beta=1,t=0$, we have

$$\|A\|^2_{\infty,[\gamma>0]}\int_M|\nabla^{m+1}A|^2\gamma^s d\mu\le\varepsilon\int_M|\nabla^{m+2}A|^2\gamma^s d\mu+C(\varepsilon)\|A\|^4_{\infty,[\gamma>0]}\int_M|\nabla^m A|^2\gamma^s d\mu+C\int_M|\nabla^{m+1}A|^2\gamma^{s-2}d\mu.\tag{4.17}$$

Combining (4.13)–(4.17), we prove Lemma 4.2.

Lemma 4.3. *Under the assumptions of Lemma* 4.1, *let*

$$\sup_{0\le t\le T}\int_{[\gamma>0]}|A|^2 d\mu\le\varepsilon_0,\tag{4.18}$$

where ε_0 *is small enough depending on the constants in* (3.5). *Then*

$$\|\nabla^m A\|_{\infty,[\gamma>1]}\le C(m,T,\alpha_0(m+2)),\tag{4.19}$$

where $\alpha_0(m)=\sum\limits_{j=0}^m\|\nabla^j A\|_{2,[\gamma_0>0]}$.

Proof. Recalling Lemma 4.3 in [12], we have for any tensor ϕ on M and γ as in (3.5) that

$$\|\phi\|^4_{\infty,[\gamma=1]}\le C\|\phi\|^2_{2,[\gamma>0]}(\|\nabla^2\phi\|^2_{2,[\gamma>0]}+\||\phi|^2|A|^4\|_{1,[\gamma>0]}+\|\phi\|^2_{2,[\gamma>0]}).\tag{4.20}$$

Moreover, if $\phi = A$ and if

$$\|A\|^2_{2,[\gamma>0]} \le \varepsilon_0$$

for some ε_0 small enough depending on the constants in (3.5), then

$$\|A\|^4_{\infty,[\gamma=1]} \le C\|A\|^2_{2,[\gamma>0]}(\|\nabla^2 A\|^2_{2,[\gamma>0]} + \|A\|^2_{2,[\gamma>0]}). \tag{4.21}$$

For $0 \le \sigma < \tau \le 1$, let $\gamma_{\sigma,\tau} = \psi_{\sigma,\tau} \circ \gamma$ satisfy $\gamma_{\sigma,\tau} = 0$ for $\gamma \le \sigma$ and $\gamma_{\sigma,\tau} = 1$ for $\gamma \ge \tau$. With $\sigma = 0, \tau = \frac{1}{2}$ we deduce from Lemma 4.1 that

$$\int_0^T \int_{[\gamma\ge 1/2]} (|\nabla^2 A|^2 + |A|^6) d\mu \le C\varepsilon_0(1+T). \tag{4.22}$$

Substituting (4.22) to (4.21), we let $\sigma = \frac{1}{2}, \tau = \frac{3}{4}$ and obtain

$$\int_0^T \|A\|^4_{\infty,[\gamma\ge 3/4]} dt \le C\varepsilon_0(C\varepsilon_0(1+T) + \varepsilon_0 T) \le C\varepsilon_0^2(1+T). \tag{4.23}$$

From Lemma 4.2 with $\sigma = \frac{3}{4}, \tau = \frac{7}{8}$ and $s = 2m+4$, we have

$$\begin{aligned}
&\int_M |\nabla^m A|^2 \gamma^s_{\sigma,\tau} d\mu + \int_0^t \int_{[\gamma\ge 7/8]} |\nabla^{m+2} A|^2 d\mu dt' \\
&\le \int_M |\nabla^m A_0|^2 (\gamma_0)^s_{\sigma,\tau} d\mu_0 + C(m)\varepsilon_0 \Big(T + \int_0^T \|A\|^4_{\infty,[\gamma\ge 3/4]} dt + \int_0^T \|A\|^3_{\infty,[\gamma\ge 3/4]} dt\Big) \\
&\quad + C(m) \int_0^t \big(\|A\|^4_{\infty,[\gamma\ge 3/4]} + \|A\|^3_{\infty,[\gamma\ge 3/4]}\big) \cdot \Big(\int_M |\nabla^m A|^2 \gamma^s_{\sigma,\tau}\Big) dt'
\end{aligned}$$

for any $t \in [0,T]$. Using Gronwall's inequality and (4.22) we have

$$\sup_{0\le t\le T} \int_{[\gamma\ge 7/8]} |\nabla^m A|^2 d\mu + \int_0^T \int_{[\gamma\ge 7/8]} |\nabla^{m+2} A|^2 d\mu dt \le C(m, T, \alpha_0(m)).$$

From this inequality and (4.21) we have

$$\|A\|^4_{\infty,[\gamma\ge 15/16]} \le C\varepsilon_0(C(2,T,\alpha_0(2)) + \varepsilon_0) \le C(T, \alpha_0(2)).$$

Using (4.20) for $\phi = \nabla^m A$, we get

$$\|\nabla^m A\|^4_{\infty,[\gamma=1]} \le C(m, T, \alpha_0(m+2)).$$

The proof of this lemma is completed.

§5. Proof of Theorem 1.1

For simplicity, we denote $\|\phi\|_{p,U} = (\int_{\varphi^{-1}(V)} |\phi|^p)^{\frac{1}{p}}$ for any $V \subseteq \mathbb{R}^{n+1}$. Now we prove Theorem 1.1.

Without loss of generality, we may assume that $\rho = 1$. Let

$$\varepsilon(t) = \sup_{x\in\mathbb{R}^{n+1}} \int_{B_1(x)} |A|^2 d\mu. \tag{5.1}$$

By compactness of $\varphi(M \times [0,t])$ for $t < T$, we have that the function $\varepsilon(t) : [0,T) \to \mathbb{R}$ is continuous. It is easy to check by covering argument that

$$\varepsilon(t) \le \Gamma \sup_{x\in\mathbb{R}^{n+1}} \int_{B_{1/2}(x)} |A|^2 d\mu \tag{5.2}$$

for some Γ depending only on n. We define

$$t_0 = \sup\{0 \le t \le \min(T,\lambda) : \varepsilon(t) \le 3\Gamma\varepsilon \ \text{ for } \ 0 \le \tau \le t\}, \tag{5.3}$$

where $\lambda > 0$ be a parameter. By the continuity of $\varepsilon(t)$, we have $t_0 > 0$ and

$$\varepsilon(t_0) = 3\Gamma\varepsilon \quad \text{if } \ t_0 < \min(T, \lambda). \tag{5.4}$$

Choose a cutoff function $\widetilde{\gamma} \in C^2(\mathbb{R}^{n+1})$ with $\|\widetilde{\gamma}\|_{C^2(\mathbb{R}^{n+1})} \le C(n)$ and $\chi_{B_{1/2}(x)} \le \widetilde{\gamma} \le \chi_{B_1(x)}$, then $\gamma = \widetilde{\gamma} \circ \varphi$ satisfies (3.5). Using Lemma 4.1 on $[0, t_0]$, we have

$$\int_{B_{1/2}(x)} |A|^2 d\mu \le \int_{B_1(x)} |A_0|^2 d\mu_0 + C\Gamma\varepsilon t \le 2\varepsilon \quad \text{for } \ 0 \le t \le t_0,$$

if we take $\lambda = (C\Gamma)^{-1}$. From (5.2) we have

$$\varepsilon(t) \le 2\Gamma\varepsilon \quad \text{for } \ 0 \le t \le t_0, \tag{5.5}$$

and thus (5.4) implies $t_0 = \min(T, (C\Gamma)^{-1})$. Now if $t_0 = (C\Gamma)^{-1}$, then (1.3) holds and (5.5) implies (1.4). Hence, we only need to lead a contradiction from the assumption

$$t_0 = T. \tag{5.6}$$

First, by (5.5) and $T = t_0 \le (C\Gamma)^{-1}$ we can apply Lemma 4.3 to obtaining

$$\|\nabla^m A\|_\infty \le C(n, m, \varphi_0). \tag{5.7}$$

Since $\nabla^m A$ are uniformly bounded in time, we have

$$|\varphi(p,t) - \varphi(p,s)| \le \int_s^t |\triangle H(p,t')| dt' \le C(t-s)$$

for any $0 \le s < t < T$. Then $\varphi(\cdot, t)$ uniformly converges to a continuous limit $\varphi(\cdot, T)$ as $t \to T$. We recall Lemma 8.2 in [10]:

Lemma 5.1.[10] *Let g_{ij} be a time dependent metric on a compact manifold M for $0 \le t < T \le +\infty$. Suppose that*

$$\int_0^T \max_{M_t} \Big|\frac{\partial}{\partial t} g_{ij}\Big| dt \le C.$$

Then the metric $g_{ij}(t)$ are all equivalent, and they converge as $t \to T$ uniformly to a positive definite metric tensor $g_{ij}(T)$ which is continuous and also equivalent.

In our situation, the hypotheses of this lemma are clearly satisfied, hence $\varphi(\cdot, T)$ represents a hypersurface. Moreover, it also follows that there exists a positive constant C depending only on n and φ_0, such that for every $0 \le t < T$, we have $\frac{1}{C} \le g_{ij} \le C$. Since $\frac{\partial}{\partial t} g_{ij} = -\triangle H h_{ij}$, for every $k \in \mathbb{N}$ we get

$$\Big\|\nabla^k \frac{\partial}{\partial t} g_{ij}\Big\|_\infty \le C(n, k, \varphi_0).$$

Analogously, as the time derivative of the Christoffel symbols is given by

$$\frac{\partial}{\partial t}\Gamma^i_{jk} = \nabla\triangle H * A + \triangle H * A,$$

it follows that

$$\Big\|\nabla^k \frac{\partial}{\partial t}\Gamma^i_{jk}\Big\|_\infty \le C(n, k, \varphi_0).$$

With an induction argument, we can prove the following formula relating the iterated covariant and coordinate derivatives of a tensor T,

$$\nabla^m T = \partial^m T + \sum_{l=1}^{m} \sum_{k_1 + \cdots + k_l + k \le m-1} \partial^{k_1}\Gamma \cdots \partial^{k_l}\Gamma \cdot \partial^k T. \tag{5.8}$$

By this relation and induction, it follows that

$$\|\partial^k \Gamma^i_{jl}\|_\infty, \quad \|\partial^k \partial_t \Gamma^i_{jl}\|_\infty \le C(n, k, \varphi_0)$$

for $t \in [0,T]$. Applying formula (5.8) to $T = \nabla^s A$, we see that

$$\partial^k \nabla^s A - \nabla^{k+s} A = \sum_{i=1}^{k} \sum_{j_1+\cdots+j_i+l \le k-1} \partial^{j_1}\Gamma \cdots \partial^{j_i}\Gamma \cdot \partial^l \nabla^s A,$$

and by induction and estimate (5.7) we have $\|\partial^k \nabla^s A\|_\infty \le C(k,s,\varphi_0)$. Since we already know that $|\varphi|$ is bounded and $|\partial\varphi| = 1$, by the Gauss-Weingarden relation $\partial^2\varphi = \Gamma\partial\varphi + A\nu$, $\partial\nu = A * \partial\varphi$ and previous estimates, we can conclude that

$$\|\partial^k \varphi\|_\infty \le C(k,\varphi_0) \tag{5.9}$$

for any $k \in \mathbb{N}$ and $t \in [0,T]$. Since $\partial_t\varphi = \triangle H\nu$, we get from (5.9) that $\|\partial^k \partial_t \varphi\|_\infty \le C(k,n,\varphi_0)$. Hence, the convergence $\varphi(\cdot,t) \to \varphi(\cdot,T)$, as $t \to T$, is in the C^∞ topology and M_t is smooth. $\varphi(\cdot,T)$ is an immersion as the metric $g(t) \to g(T)$ are uniformly equivalent. By short time existence, we can extend the flow φ to an interval $[0, T+\delta]$, contradicting the maximality of T, hence contradicting (5.6), and the theorem is proven.

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