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THE CODIMENSION FORMULA ON QUASI-INVARIANT SUBSPACES OF THE FOCK SPACE

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Abstract

Let M be an approximately finite codimensional quasi-invariant subspace of the Fock space. This paper gives a formula to calculate the codimension of such spaces and uses this formula to study the structure of quasi-invariant subspaces of the Fock space. Especially, as one of applications, it is showed that the analogue of Beurling's theorem is not true for the Fock space $L_a^2(\mathbb{C}^n)$ in the case of $n \geq 2$.

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§0. Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane $\mathbb{C}$. The present paper was mainly motivated by the recent works of K. Guo[7,9] in the Hardy space $H^2(\mathbb{D}^n)$ and analytic Hilbert space over a bounded domain. In [7], the author get the codimension formula for invariant subspaces of Hardy space over $\mathbb{D}^n$. The analogue formula for analytic Hilbert spaces over a bounded domain $\Omega \subset \mathbb{C}^n$ was established in [9]. In this paper we will be concerned with the Fock space $L_a^2(\mathbb{C}^n)$. The Fock space or the so called Siegel-Bargmann space, defined to be the space of all μ-square-integrable entire functions over $\mathbb{C}^n$, where

$$d\mu(z) = e^{\frac{-|z|^2}{2}} d\nu(z)(2\pi)^{-n}$$

is the Gaussian measure on $\mathbb{C}^n$ ($d\nu$ is the ordinary Lebesgue measure). It is easy to see that $L_a^2(\mathbb{C}^n)$ is a closed subspace of $L^2(\mathbb{C}^n)$ with the reproducing kernel functions $K_\lambda(z) = e^{\bar{\lambda}z/2}$ and the normalized reproducing kernel functions $k_\lambda(z) = e^{\bar{\lambda}z/2-|\lambda|^2/4}$. For general background on Fock space one may consult[5] and the references therein. As proved in [10], there exists no nontrivial invariant subspaces for multiplication operator M_{z_i}. Thus, they introduced a substitute for invariant subspace, the so called quasi-invariant space. Let M be a closed subspace of the Fock space $L_a^2(\mathbb{C}^n)$. We say that M is quasi-invariant if

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$pM \cap L_a^2(\mathbb{C}^n) \subset M$ and $M_0 = \{f \in M | pf \in L_a^2(\mathbb{C}^n)\}$ is dense in M for all polynomial p. Let M be a quasi-invariant subspaces of $L_a^2(\mathbb{C}^n)$. We call M an approximately finite codimensional quasi-invariant subspace (abbr., AFCQS), if M is equal to the intersection of all finite codimensional quasi-invariant subspaces containing M. For a quasi-invariant subspaces M, the AF-envelope of M is defined by the intersection of all finite codimensional quasi-invariant subspaces containing M, and denoted by M^e. Clearly, the definition implies that M^e is an AFCQS. It is the main purpose of the present paper to establish the formula to calculate the codimesion of an AFCQS.

Besides the introduction, the paper has three sections. In Section 1, we mainly review some basic terminologies and some results concerning quasi-invariant subspaces of the Fock space and the so called characteristic space theory. The codimension formula for an AFCQS will be established in Section 2. Section 3 contains two applications of the codimension formula. A direct application of the codimension formula is that zero-based quasi-invariant subspaces have codimension 1 (or index 1) property which will play an important role in the construction of quasi-invariant subspace having codimension great than one. Another application shows that the analogue of Beurling's theorem does not hold in the Fock space $L_a^2(\mathbb{C}^n)$ in the case of $n > 1$.

§1. Preliminaries

For a polynomial $q = \sum a_{m_1\cdots m_n} z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$, let $q(D)$ denote the linear partial differential operator $\sum a_{m_1\cdots m_n} \frac{\partial^{m_1+m_2+\cdots m_n}}{\partial z_1^{m_1} \partial z_2^{m_2} \cdots \partial z_n^{m_n}}$. Let M be a quasi-invariant subspaces of $L_a^2(\mathbb{C}^n)$ and $\mathcal{C}$ the polynomial ring over $\mathbb{C}^n$. For $\lambda \in \mathbb{C}^n$, set

$$M_\lambda = \{q \in \mathcal{C} : q(D)f|_\lambda = 0, \forall f \in M\},$$

where $q(D)f|_\lambda$ denotes $(q(D)f)(\lambda)$. It is easy to check that M_λ is invariant under the action by basic partial differential operators $\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \cdots, \frac{\partial}{\partial z_n}\}$. M_λ is called the characteristic space of M at λ. The envelope of M at λ, M_λ^e, is defined by

$$M_\lambda^e = \{f \in L_a^2(\mathbb{C}^n) : q(D)f|_\lambda = 0, \forall\, q \in M_\lambda\}.$$

It is easy to check that $M_\lambda^e \supseteq M$ and M_λ^e is quasi-invariant.

In what follows we will denote the zero set of M by $Z(M)$, that is, $Z(M) = \{\lambda \in \mathbb{C}^n : f(\lambda) = 0, \forall f \in M\}$. The following algebraic reduction theorem for finite codimension quasi-invariant subspace first appeared in [10] and will be used several times in the present paper.

Lemma 1.1.[10, Theorem 5.5] *Let M be a quasi-invariant subspace of finite codimension. Then $\mathcal{C} \cap M$ is an ideal in the polynomial ring $\mathcal{C}$ and $\mathcal{C} \cap M$ is dense in M. Conversely, if I is an ideal in $\mathcal{C}$ of finite codimension then $[I]$ is quasi-invariant subspace of the same codimension and $[I] \cap \mathcal{C} = I$.*

The next proposition illustrates some basic properties of the characteristic space of quasi-invariant subspaces and their envelope and will be used in establishing the codimensional formula next. The argument in the proof is close to that in [6] where a similar theorem for invariant subspaces of Hardy space is proved.

Proposition 1.1. *Let M be a quasi-invariant subspace of the Fock space $L_a^2(\mathbb{C}^n)$. Then we have*

(1) *if $Z(M) = \emptyset$, then $M^e = L_a^2(\mathbb{C}^n)$;*

(2) *if $Z(M) \neq \emptyset$, then $M \subseteq M^e \neq L_a^2(\mathbb{C}^n)$, $(M^e)^e = M^e$, and $Z(M) = Z(M^e)$;*

(3) *if $Z(M) \neq \emptyset$, then $M^e = \cap_{\lambda \in Z(M)} M_\lambda^e$.*

In particular, suppose M_1, M_2 are two quasi-invariant subspaces of $L_a^2(\mathbb{C}^n)$. Then $M_1^e = M_2^e$ if and only if $Z(M_1) = Z(M_2)$, and for every $\lambda \in Z(M_1)$, $M_{1\lambda} = M_{2\lambda}$.

Proof. (1) Let M_α be a finite co-dimensional quasi-invariant subspaces such that $M_\alpha \supseteq M$. Then $Z(M_\alpha) \subseteq Z(M)$. Hence $Z(M_\alpha) = \emptyset$. By Lemma 1.1 we have $M_\alpha \cap \mathcal{C}$ is an ideal of the polynomial ring $\mathcal{C}$ and $M_\alpha \cap \mathcal{C}$ is dense in M. This implies that $Z(M_\alpha \cap \mathcal{C}) = \emptyset$. Thus $M_\alpha \cap \mathcal{C} = \mathcal{C}$ and $M_\alpha = L_a^2(\mathbb{C}^n)$. Hence $M^e = \cap M_\alpha = L_a^2(\mathbb{C}^n)$.

(2) The proof is obvious.

(3) We first recall that the degree of a monomial $z_1^{m_1} z_2^{m_2} \cdots z_n^{m_n}$ is defined by $m_1 + m_2 + \cdots + m_n$ and the degree of a polynomial p is defined by maximum of the degrees of the monomials which occur in p and denoted by degree(p). For every $\lambda \in Z(M)$ and each positive integer k, set

$$M_\lambda^{(k)} = \{f \in L_a^2(\mathbb{C}^n) : p(D)f|_\lambda = 0, \ p \in M_\lambda \text{ and degree}(p) \le k\}.$$

Then $M_\lambda^{(k)}$ is finite codimensional quasi-invariant subspace and $M_\lambda^{(k)} \supseteq M$. Since

$$M_\lambda^e = \{f \in L_a^2(\mathbb{C}^n) : p(D)f|_\lambda = 0, \quad \forall p \in M_\lambda\} = \bigcap_{k \ge 1} M_\lambda^{(k)}$$

for each $\lambda \in Z(M)$. By the definition of M^e we have

$$M^e \subseteq \bigcap_{\lambda \in Z(M)} M_\lambda^e. \tag{1.1}$$

Let N be a finite codimensional quasi-invariant subspace of $L_a^2(\mathbb{C}^n)$ and $N \supseteq M$. Then $N \cap \mathcal{C}$ is dense in N. Since N is finite codimensional, then $Z(N)$ is finite. Thus we have:

$$\Big(\bigcap_{\lambda \in Z(N)} N_\lambda^e\Big) \bigcap \mathcal{C} = \bigcap_{\lambda \in Z(N)} (N_\lambda^e \cap \mathcal{C}) = \bigcap_{\lambda \in Z(N)} (N \cap \mathcal{C})_\lambda^e = N \cap \mathcal{C}.$$

The second equality is based on the fact that $(N \cap \mathcal{C})_\lambda = N_\lambda$ when N has finite codimensional and the definition of the N_λ^e, while the last equality comes from Theorem 2.1 in [6]. Hence

$$N = \bigcap_{\lambda \in Z(N)} N_\lambda^e.$$

Since for every $\lambda \in Z(N)$, λ also is in $Z(M)$, we have $N_\lambda \subseteq M_\lambda$. Consequently, $M_\lambda^e \subseteq N_\lambda^e$. So, $\bigcap\limits_{\lambda \in Z(N)} M_\lambda^e \subseteq N$. This means that

$$\bigcap_{\lambda \in Z(M)} M_\lambda^e \subseteq N. \tag{1.2}$$

We thus conclude by (1.1) and (1.2) that $\bigcap\limits_{\lambda \in Z(M)} M_\lambda^e = M^e$.

For each $\lambda \in Z(M)$, it is easy to check that $M_\lambda = (M^e)_\lambda$. From (2) and (3), one easily induces that for any two quasi-invariant subspaces M_1, M_2, $M_1^e = M_2^e$ if and only if $Z(M_1) = Z(M_2)$ and $M_{1\lambda} = M_{2\lambda}$ for each $\lambda \in Z(M_1)$. This completes the proof of Proposition 1.1.

Remark 1.1. Proposition 1.1 gives an interesting application of the characteristic space theory to quasi-invariant subspaces. Besides its application in the present paper, we believe it will play an important role in the study of the structure of quasi-invariant subspaces. One may consult [8] where characteristic space theory have been proved to be a powerful tool in studying the structure of invariant subspaces of Hardy space $H^2(\mathbb{D}^n)$.

§2. The Codimensional Formula for AFCQS

In [7], the author got the codimension formula for an approximately finite codimensional invariant subspaces of Hardy space over $\mathbb{D}^n$. The analogue formula in analytic Hilbert spaces over a bounded domain $\Omega \subset \mathbb{C}^n$ was established in [9]. In this section, we will establish the codimension formula for an AFCQS in the Fock space. The main ideal of the proof comes from [7] and [9].

Following the notation in [9], let M_1, M_2 be quasi-invariant subspaces of $L_a^2(\mathbb{C}^n)$ and $\lambda \in \mathbb{C}^n$. We call that M_1,M_2 have the same multiplicity at λ if $M_{1\lambda} = M_{2\lambda}$. We use $Z(M_2)\backslash Z(M_1)$ to denote the set of zeros of M_2 related to M_1, that is, $Z(M_2)\backslash Z(M_1)$ is defined by $\{\lambda \in Z(M_2) : M_{2\lambda} \neq M_{1\lambda}\}$. If $M_1 \supseteq M_2$, the cardinality of zeros of M_2 related to M_1, $\mathrm{card}(Z(M_2)\backslash Z(M_1))$, is defined by $\sum\limits_{\lambda\in Z(M_2)\backslash Z(M_1)} \dim(M_{2\lambda}/M_{1\lambda})$. We call the following proposition the codimension formula throughout this paper.

Theorem 2.1. *Let M_1, and M_2 be quasi-invariant subspaces of $L_a^2(\mathbb{C}^n)$ such that $M_1 \supseteq M_2$ and $dim M_1/M_2 = k < \infty$. If M_2 is an approximately finite codimensional quasi-invariant subspace* (AFCQS)*, then*

$$\dim(M_1/M_2) = \sum_{\lambda\in Z(M_2)\backslash Z(M_1)} \dim M_{2\lambda}/M_{1\lambda} = \mathrm{card}(Z(M_2)\backslash Z(M_1)).$$

Proof. For each $\gamma \in Z(M_2) \setminus Z(M_1)$, let

$$M_2^\gamma = \{g \in M_1 : q(D)g|_\gamma = 0,\ q \in (M_2)_\lambda\} = M_1 \cap (M_2)_\gamma^e.$$

Obviously M_2^γ contains M_2 and M_2^γ is quasi-invariant. Set

$$M_2' = \bigcap_{\gamma\in Z(M_2)\backslash Z(M_1)} M_2^\gamma.$$

We claim that $M_2' = M_2$. In fact, by the fact that $M_2' \supseteq M_2$, we have $(M_2')_\lambda \subseteq (M_2)_\lambda$. Since

$$M_2' = \bigcap_{\gamma\in Z(M_2)\backslash Z(M_1)} M_2^\gamma \supseteq \cap_{\lambda\in\mathbb{C}^n}\{g \in M_1 : q(D)g|_\lambda = 0,\ q \in (M_2)_\lambda\},$$

$(M_2')_\lambda \supseteq (M_2)_\lambda$. Therefore, by Proposition 1.1, we have

$$M_2 \subseteq M_2' \subseteq (M_2')^e = M_2^e = M_2.$$

This completes the proof of the claim.

Since $\dim(M_1/M_2) = k$, the number of elements in $Z(M_2) \setminus Z(M_1)$ is at most k. Assume first that $Z(M_2) \setminus Z(M_1)$ contains only one point λ, then

$$M_2 = \{h \in M_1 : p(D)h|_\lambda = 0, p \in M_{2\lambda}\}.$$

We define a map

$$\phi(\cdot,\cdot) : M_{2\lambda}/M_{1\lambda} \times M_1/M_2 \to \mathbb{C}$$

by $\phi(\tilde{p}, \tilde{h}) = p(D)h|_\lambda$. Clearly, this is well defined and from this and the representation of M_2, one has that $\dim\ M_1/M_2 = \dim\ M_{2\lambda}/M_{1\lambda}$.

Now let $l > 1$ and assume that the proposition has been proved for $Z(M_2) \setminus Z(M_1)$ containing points less than l. Let $Z(M_2) \setminus Z(M_1) = \{\lambda_1, \cdots, \lambda_l\}$, where $\lambda_i \neq \lambda_j$. Writing

$$M_2^\star = \{h \in M_1 : p(D)h|_{\lambda_1} = 0, p \in (M_2)_{\lambda_1}\},$$

then $(M_2^\star)_{\lambda_1} = M_{2\lambda_1}$. Similarly to the preceding proof, we have

$$\dim(M_1/M_2^\star) = \dim(M_{2\lambda_1}/M_{1\lambda_1}).$$

Write $M_{2\lambda_1} = M_{1\lambda_1} \dot{+} R$ with $\dim R = \dim (M_{2\lambda_1}/M_{1\lambda_1})$, and let LR denote the linear space of polynomials generated by R such that it is invariant under the action by $\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_n}\}$. Put

$$Q_R = \{p \in \mathcal{C} : q(D)p|_{\lambda_1} = 0, \, q \in LR\}.$$

Then it is easily verified that Q_R is a finite codimensional ideal of $\mathcal{C}$ with only zero point λ_1 because LR is finite dimensional. From the definition of $M_2^\star$, the following inclusion are easily verified:

$$Q_R M_1 \subseteq M_2^\star \subseteq M_1.$$

Consequently, for $\lambda \neq \lambda_1$, $(M_1)\lambda = (M_2^\star)_\lambda$. So $Z(M_2) \setminus Z(M_2)^\star = \{\lambda_2, \cdots, \lambda_l\}$. By the induction hypothesis, we have

$$dim M_2^\star/M_2 = \sum_{j=2}^{l} \dim (M_2)_{\lambda_j}/(M_2^\star)_{\lambda_j} = \sum_{j=2}^{l} \dim (M_2)_{\lambda_j}/(M_1)_{\lambda_j}.$$

It follows that

$$\begin{aligned} \dim M_1/M_2 &= \dim M_1/M_2^\star + \dim M_2^\star/M_2 \\ &= \sum_{j=2}^{l} \dim (M_2)_{\lambda_j}/(M_1)_{\lambda_j} \\ &= \operatorname{card}(Z(M_2) \setminus Z(M_1)). \end{aligned}$$

This completes the proof.

§3. Applications

We end this paper with two applications of the codimension formula. The simplest nontrivial example of the quasi-invariant subspace that comes to mind is that of zero-based one: Given a sequence Λ in $\mathbb{C}$, if there is an element $f \in L_a^2(\mathbb{C})$ such that $f(\Lambda) = 0$ we say that Λ is a zero sequence. The quasi-invariant subspaces of the form

$$I = \{f \in L_a^2(\mathbb{C}) | \, f = 0 \;\text{ on }\; \Lambda\}$$

counting multiplicities whenever necessary are called zero-based quasi-invariant subspace. It is showed in [13] that such subspaces in Hardy space are of codimension 1 (That is $\dim M/zM = 1$). Let M be a quasi-invariant subspace of Fock space, we use zM to denote the set $\{zf| \; f \in M, \; zf \in L_a^2(\mathbb{C})\}$. Obviously, I is an AFCQS. Thus by Theorem 2.1 we have the following proposition which will play an important role in our next paper [12].

Proposition 3.1. *Let I be a zero-based quasi-invariant subspace of the Fock space. Then* $\dim I/zI = 1$.

In [2], A. Beurling proved that: If $N \neq 0$ be an invariant subspace of the Hardy space $H^2(\mathbb{D})$, then $N \ominus zN$ is a one dimension subspace spanned by an inner function ϕ and

$$N = [\phi] = [N \ominus zN],$$

where $N \ominus zN = N \cap (zN)^\perp$ and $[\phi]$ denotes the smallest invariant subspace containing ϕ. Beurling's Theorem has played an important role in operator theory, function theory and their intersection, function-theoretic operator theory. In [14], Richter proved that the analogue of Beurling's Theorem is true in the Dirichlet space. It is well known that the invariant subspace lattice of the Bergman space $L_a^2(\mathbb{D})$ is very complicated. In fact the dimension of $N \ominus zN$ can be an arbitrary positive integer or ∞ (see [11]). However, a big breakthrough in the study of the analogue of Beurling's theorem on Bergman space was made by A. Aleman S. Richter and C. Sundberg[1]. They proved that any invariant subspace N of the Bergman space $L_a^2(\mathbb{D})$ also has the form $N = [N \ominus zN]$. In a previous paper [4], we proved that the analogue of Beurling's Theorem deos not hold in general in the Fock space $L_a^2(\mathbb{C})$. The following example shows that the analogue of Beurling's Theorem does not hold in Fock space $L_a^2(\mathbb{C}^n)$ when $n > 1$.

Example. Let I be an ideal of polynomial ring $\mathcal{C}$ over $\mathbb{C}^n$ $(n > 1)$ with finite codimension such that $(0, 0, \cdots, 0) \notin Z(I)$. Let

$$\mathcal{C}_0 = \{p|\, p(0, 0, \cdots, 0) = 0,\ p \in \mathcal{C}\}.$$

Then $[I] \ominus [\mathcal{C}_0 I]$ does not generate $[I]$.

Proof. By Proposition 2.1, $\dim([I] \ominus [\mathcal{C}_0 I]) = 1$. Since $[I]$ has finite codimension, $\operatorname{rank}[I] > 1$. Thus $[I] \ominus [\mathcal{C}_0 I]$ does not generate $[I]$.

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