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CHIRAL RING OF Sp(N) AND SO(N) SUPERSYMMETRIC GAUGE THEORY IN FOUR DIMENSIONS**

E. WITTEN*

Abstract

The chiral ring of classical supersymmetric Yang-Mills theory with gauge group Sp(N) or SO(N) is computed, extending previous work (of Cachazo, Douglas, Seiberg, and the author) for SU(N). The result is that, as has been conjectured, the ring is generated by the usual glueball superfield $S \sim \mathrm{Tr}\, W_\alpha W^\alpha$, with the relation $S^h = 0$, h being the dual Coxeter number. Though this proposition has important implications for the behavior of the quantum theory, the statement and (for the most part) the proofs amount to assertions about Lie groups with no direct reference to gauge theory.

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§1. Introduction

In four-dimensional supersymmetric Yang-Mills theory, the basic gauge invariant operator is the superspace field strength W_α, $\alpha = 1, 2$ (and its hermitian conjugate $\overline{W}_{\dot\alpha}$). W_α transforms in the adjoint representation of the gauge group, which we will take to be a simple Lie group G; we denote its Lie algebra as $\mathbf{g}$ and let Tr denote an invariant quadratic form on $\mathbf{g}$. (For classical Lie groups, we will take Tr to be the trace in the fundamental representation.) W_α is a fermionic operator of dimension 3/2, and is chiral, that is, it is annihilated by the supersymmetries of one chirality: $\{\overline{Q}_{\dot\alpha}, W_\beta\} = 0$.

Gauge-invariant polynomials in the W_α, such as $\mathrm{Tr}\, W_{\alpha_1} W_{\alpha_2} \cdots W_{\alpha_s}$, are likewise chiral. In this paper, we consider the "pure" supersymmetric gauge theory without matter multiplets. In this theory, the gauge-invariant polynomials in W_α are the only chiral superfields of importance. However, such a polynomial is considered trivial if it is proportional to a linear combination of expressions $\{W_\alpha, W_\beta\}$ for any α, β. The reason for this is the identity[1]

$$\{W_\alpha, W_\beta\} = \{\overline{Q}^{\dot\alpha}, D_{\alpha\dot\alpha} W_\beta\}, \tag{1.1}$$

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*Institute for Advanced Study, Princeton, NJ 08540, USA. **E-mail:** witten@ias.edu

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[1]This identity follows directly from the superspace definition $W_\alpha = \{\overline{Q}^{\dot\alpha}, D_{\alpha\dot\alpha}\}$.

which implies that any gauge-invariant expression $\sum_{\alpha\beta} X^{\alpha\beta}\{W_\alpha, W_\beta\}$ is a descendant, that is, it can be written as $\sum_{\dot\alpha}\{\overline{Q}_{\dot\alpha}, Y^{\dot\alpha}\}$ for some $Y^{\dot\alpha}$, and hence decouples from the expectation value of a product of chiral operators.

Mathematical Description of the Problem

The problem of classifying modulo descendants the chiral operators in supersymmetric gauge theory is of mathematical as well as physical interest. Before proceeding, let us reformulate the problem mathematically. We introduce a $\mathbf{Z}_2$-graded ring R that is generated by the components of the W_α. Explicitly, picking a basis T_a, $a = 1, \cdots, \dim G$ of $\mathbf{g}$, we write

$$W_\alpha = \sum_a w_\alpha^a T_a,$$

and then R is generated by the (odd) variables w_α^a. In the ring R we define an ideal I that is generated by the components of $\{W_\alpha, W_\beta\}$. In more detail, we write

$$\{W_\alpha, W_\beta\} = \frac{1}{2}\sum_{a,b} w_\alpha^a w_\beta^b [T_a, T_b] = \frac{1}{2}\sum_{a,b,c} w_\alpha^a w_\beta^b f_{ab}^c T_c,$$

where $[T_a, T_b] = \sum_c f_{ab}^c T_c$. Thus, I is generated by the even, nilpotent elements $\sum_{a,b} w_\alpha^a w_\beta^b f_{ab}^c$, for all α, β, and c. Elements of I are descendants; the quotient ring R/I is the ring of chiral operators mod descendants. The ideal I is clearly G-invariant, so G acts on R/I. The G-invariant chiral operators mod descendants form the classical approximation to the physical "chiral ring" of the theory. So in the classical supersymmetric gauge theory, the chiral ring is $\mathcal{R}_{\rm cl} = (R/I)^G$, where $(R/I)^G$ denotes the G-invariant part of R/I, and the subscript "cl" means "classical" (we recall shortly how quantum corrections deform the picture). An element of $\mathcal{R}_{\rm cl}$ can be represented by a G-invariant element of R that is not in I.

If we consider the W_α to be of degree one, then the ideal I is graded — its generators being homogeneous of degree two — and hence the classical chiral ring $\mathcal{R}_{\rm cl}$ is a graded ring. There is no non-trivial element of $\mathcal{R}_{\rm cl}$ in degree one (since there is no gauge-invariant linear function of the W_α). In degree two, since $W_1^2 = \frac{1}{2}\{W_1, W_1\}$ and $W_2^2 = \frac{1}{2}\{W_2, W_2\}$ are contained in I, any element of $\mathcal{R}_{\rm cl}$ is a multiple of

$$\mathrm{Tr}\, W_1 W_2 = -\mathrm{Tr}\, W_2 W_1.$$

The degree two part of $\mathcal{R}_{\rm cl}$ is thus a one-dimensional vector space, generated by

$$S = \mathrm{Tr}\, W_1 W_2 = \frac{1}{2}\sum_\alpha \epsilon^{\alpha\beta}\mathrm{Tr}\, W_\alpha W_\beta. \tag{1.2}$$

(Here $\epsilon^{\alpha\beta}$, $\alpha, \beta = 1, 2$ is the antisymmetric tensor with $\epsilon^{12} = 1$. It is conventional to include a factor of $-1/16\pi^2$ in the definition of S; this factor, which is motivated by instanton considerations, will play no role in the present paper and we will omit it.)

The conjecture [1] that we will be exploring in the present paper is that for any simple Lie group G with dual Coxeter number h, the ring $\mathcal{R}_{\rm cl}$ is generated by S with the relation $S^h = 0$. In [1], this conjecture was proved for SU(N), and certain partial results were obtained for other groups. For the classical Lie groups (SO(N) and Sp(N) as well as SU(N)) it was proved that $\mathcal{R}_{\rm cl}$ is generated by S. (This was also proved in [2].) For any Lie group G of rank r, it was proved that $S^r \neq 0$ in $\mathcal{R}_{\rm cl}$. The purpose of the present paper is to prove the

conjecture for Sp(N) and SO(N). Most of the arguments are similar to those in [1], but for SO(N) one important step in the proof uses arguments of a quite different nature, based on instanton calculations [3] that were reviewed and extended in [4–6]. For exceptional groups, a proof of the conjecture, or even of the fact that $\mathcal{R}_{\rm cl}$ is generated by S, has not yet emerged.

The rings and ideals R, I, and $\mathcal{R}_{\rm cl}$ all admit an action of $SL(2,\mathbf{C})$, under which the W_α, $\alpha = 1,2$ transform in the two-dimensional representation. Physically, this $SL(2,\mathbf{C})$ originates from an SU(2) rotation symmetry of the four-dimensional gauge theory. It will not play an important role in the present paper. The conjecture about the structure of $\mathcal{R}_{\rm cl}$ implies in any case that $SL(2,\mathbf{C})$ acts trivially on $\mathcal{R}_{\rm cl}$.

Significance for Physics

To conclude this introduction, let us recall [1] the main reason for the physical interest of the conjecture. First we must discuss the quantum deformation of the ring $\mathcal{R}_{\rm cl}$. Our definition of $\mathcal{R}_{\rm cl}$ was a classical approximation to the analogous chiral ring $\mathcal{R}$ of the quantum theory. For any G, the operator S^h, which has dimension $3h$ and carries charge $2h$ with respect to the $U(1)_R$ symmetry of the classical theory, has just the quantum numbers of a one-instanton contribution to correlation functions. Hence it is possible that the classical relation $S^h = 0$ in the ring $\mathcal{R}_{\rm cl}$ could be deformed in the quantum ring $\mathcal{R}$ to a relation of the form

$$S^h = c\Lambda^{3h}, \tag{1.3}$$

with Λ the scale factor of the theory and some constant c. (Λ^{3h} is essentially the exponential of minus the one-instanton action and is the standard factor that appears in all one-instanton amplitudes.) This is the only such deformation that is possible if the conjectured structure of $\mathcal{R}_{\rm cl}$ is correct. (A k-instanton amplitude has the quantum numbers of S^{kh} and so for $k > 1$ cannot modify the classical relation $S^h = 0$.) In fact, explicit instanton calculations [3–6] can be interpreted, as we will recall in Section 5, as showing that this deformation does arise.

The quantum deformation of the classical ring $\mathcal{R}_{\rm cl}$ to a quantum ring $\mathcal{R}$ has a perhaps more familiar analog in two dimensions. In the context of two-dimensional supersymmetric sigma models, the classical cohomology ring of (for example) $\mathbf{CP}^{N-1}$, which is generated by a degree two element x obeying $x^N = 0$, is naturally deformed to a quantum cohomology ring in which the relation is $x^N = e^{-I}$. In this case, I is the area of a holomorphic curve of degree one (computed using a Kahler metric on $\mathbf{CP}^{N-1}$ that is introduced to define the sigma model). One difference between the four-dimensional gauge theories and the two-dimensional sigma models is that in four dimensions the starting point, which is the ring $\mathcal{R}_{\rm cl}$ associated with a Lie group, is less familiar mathematically than the classical cohomology of $\mathbf{CP}^{N-1}$.

The quantum relation (1.3) has a striking consequence. It implies that in any supersymmetric vacuum, S must have a nonzero expectation value, equal to $c^{1/h}\Lambda^3$ (for one of the h possible values of $c^{1/h}$). In fact, since

$$\sum_{\dot\alpha} \langle \{\overline{Q}_{\dot\alpha}, Y^{\dot\alpha}\}\rangle = 0$$

for any $Y^{\dot\alpha}$ in any supersymmetric vacuum, (1.3) implies that in a supersymmetric vacuum

$$\langle S^h\rangle = \langle c\Lambda^{3h}\rangle = c\Lambda^{3h}.$$

But expectation values of products of chiral operators factorize [3], so in particular $\langle S^h\rangle = \langle S\rangle^h$. So we get

$$\langle S\rangle^h = c\Lambda^{3h},$$

whence $\langle S\rangle = c^{1/h}\Lambda^3$, as claimed.

From this, we can deduce more. The supersymmetric gauge theory in four dimensions has an anomaly-free discrete chiral symmetry under which S is rotated by an $h^{\rm th}$ root of 1. A nonzero expectation value of S breaks this symmetry, so it follows that in any supersymmetric vacuum, the discrete chiral symmetry is spontaneously broken. If a supersymmetric vacuum exists in which $\langle S\rangle = c^{1/h}\Lambda^3$ for one choice of $c^{1/h}$, then applying the broken symmetry gives additional vacua with the other possible choices of $c^{1/h}$. Hence supersymmetric vacua must come in groups of h, permuted by the spontaneously broken discrete chiral symmetry.

Of course, it is believed that for any G, the theory has precisely h supersymmetric vacua, all with a mass gap, permuted by the broken symmetry. Unfortunately, no satisfactory approximation is known for describing these vacua.

Results in the Present Paper

In Section 2 of this paper, we review the arguments given in [1]. In Section 3, we extend the argument for $\mathrm{Sp}(N)$, and in Section 4, we do so for $\mathrm{SO}(N)$. In Section 5, we briefly review some pertinent aspects of the one-instanton calculations [3–6].

§2. Review

In this section, we briefly review the known arguments.

For the classical groups $\mathrm{SU}(N)$, $\mathrm{Sp}(N)$,[2] and $\mathrm{SO}(N)$, one can prove directly [2, 1] that the ring $\mathcal{R}_{\rm cl}$ is generated by $S = \mathrm{Tr}\, W_1W_2$. In fact, for the classical groups, any invariant polynomial in the W_α's is a polynomial in the traces of words in W_1 and W_2.[3] A typical trace of such a word is

$$\mathrm{Tr}\, W_1^{n_1}W_2^{n_2}W_1^{n_3}\cdots W_2^{n_s}. \tag{2.1}$$

The ideal I contains $\{W_1, W_2\}$, so modulo I, we can take W_1 and W_2 to anticommute. Hence the only traces to consider are $\mathrm{Tr}\, W_1^{n_1}W_2^{n_2}$. But I also contains $W_1^2 = \frac{1}{2}\{W_1, W_1\}$, and likewise $W_2^2 = \frac{1}{2}\{W_2, W_2\}$. So we are reduced to generators $\mathrm{Tr}\, W_1^{n_1}W_2^{n_2}$, $n_1, n_2 \le 1$. As

$$\mathrm{Tr}\, W_1 = \mathrm{Tr}\, W_2 = 0$$

for simple G, it follows that $\mathcal{R}_{\rm cl}$ is generated for G a classical Lie group by

$$S = \mathrm{Tr}\, W_1W_2 = -\mathrm{Tr}\, W_2W_1.$$

The conjecture under discussion asserts more precisely that $\mathcal{R}_{\rm cl}$ is generated by S with the relation $S^h = 0$. So among other things one would like to prove that $S^{h-1} \neq 0$ in $\mathcal{R}_{\rm cl}$,

[2] Our notation for symplectic groups is such that $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. For classical groups, we take the symbol Tr to refer to a trace in the fundamental representation of $\mathrm{SU}(N)$, $\mathrm{Sp}(N)$, or $\mathrm{SO}(N)$ (of dimension N, $2N$, and N, respectively).

[3] For $\mathrm{SO}(2k)$, there is an antisymmetric tensor of order $2k$, but because of anticommutativity of W_1 and W_2, it cannot be used to make an invariant polynomial in these variables if $k > 2$. The groups $\mathrm{SO}(2k)$ with $k \le 2$ are of course not simple, so we need not consider them.

or equivalently, that as an element of R, $S^{h-1} \notin I$. In [1], denoting the rank of G as r, the weaker statement that $S^r \notin I$ was proved. In fact, let $\mathbf{g} = \mathbf{t} \oplus \mathbf{k}$, where $\mathbf{t}$ is the Lie algebra of a maximal torus in G, and $\mathbf{k}$ is its orthocomplement. Let I' be the ideal generated by the matrix elements of the projection of W_α to $\mathbf{k}$. Then $I \subset I'$, since if W_α take values in $\mathbf{t}$, we have $\{W_\alpha, W_\beta\} = 0$. To prove that $S^r \notin I$, it suffices to show that $S^r \notin I'$. The projection of W_α to $\mathbf{t}$ can be written $W_\alpha = \sum_{a=1}^{r} w_\alpha^a T'_a$, where the sum runs over an orthonormal basis T'_a, $a = 1, \cdots, r$, of $\mathbf{t}$. The $\mathbf{Z}_2$ graded ring R/I' is freely generated by the odd elements w_α^a, $a = 1, \cdots, r$ (with no relations, that is, except anticommutativity). Moreover, $S = \sum_{a=1}^{r} w_1^a w_2^a$ as an element of R/I'. This formula and the absence of relations among the w_i^a make clear that $S^r \neq 0$ in R/I'; indeed,

$$S^r = r! \prod_{a=1}^{r} w_1^a w_2^a.$$

For the Lie groups SU(N) and Sp(N), one has $h - 1 = r$. So for these groups, the result $S^r \neq 0$ in $\mathcal{R}_{\rm cl}$ is equivalent to the desired $S^{h-1} \neq 0$. For other groups, $h - 1 > r$, and a direct algebraic proof that $S^{h-1} \neq 0$ is not yet known. It is possible to use four-dimensional instantons to prove this result; an indication of how to do so is given in Section 5.

The remaining step in [1] was to complete the proof of the conjecture for SU(N) by showing that $S^N = 0$ for this group. (For SU(N), $h = N$.) In sketching the proof, and making similar arguments for other groups, we will make the formulas less clumsy by writing A and B for W_1 and W_2.[4] We regard A and B as $N \times N$ traceless matrices, and construct the following $N^{\rm th}$ order polynomial in A:

$$F^{i_1 i_2 \cdots i_N}(A) = \epsilon^{j_1 j_2 \cdots j_N} A^{i_1}{}_{j_1} A^{i_2}{}_{j_2} \cdots A^{i_N}{}_{j_N}. \tag{2.2}$$

Here $\epsilon^{j_1 j_2 \cdots j_N}$ is the completely antisymmetric tensor, so the right hand side of (2.2) is antisymmetric in the "lower" indices of the A's and hence (as the matrix elements of A anticommute) F is completely symmetric in its indices $i_1, i_2, \cdots, i_N$. As we explain in a moment, $F^{i_1 i_2 \cdots i_N}$ is contained in the ideal generated by matrix elements of A^2. Suppose that this is known. Define the dual function of B,

$$G_{i_1 i_2 \cdots i_N}(B) = \epsilon_{k_1 k_2 \cdots k_N} B^{k_1}{}_{i_1} B^{k_2}{}_{i_2} \cdots B^{k_N}{}_{i_N}. \tag{2.3}$$

It is likewise contained in the ideal generated by B^2. Since F and G are both contained in the ideal I, so is

$$\begin{aligned} F(A) \cdot G(B) &= F^{i_1 i_2 \cdots i_N}(A) G_{i_1 i_2 \cdots i_N}(B) \\ &= \epsilon^{j_1 j_2 \cdots j_N} A^{i_1}{}_{j_1} A^{i_2}{}_{j_2} \cdots A^{i_N}{}_{j_N} \epsilon_{k_1 k_2 \cdots k_N} B^{k_1}{}_{i_1} B^{k_2}{}_{i_2} \cdots B^{k_N}{}_{i_N}. \end{aligned} \tag{2.4}$$

But a direct evaluation of the right hand side of (2.4) can be made using the identity

$$\epsilon^{j_1 j_2 \cdots j_N} \epsilon_{k_1 k_2 \cdots k_N} = \delta^{j_1}_{k_1} \delta^{j_2}_{k_2} \cdots \delta^{j_N}_{k_N} \pm \text{permutations of } k_1, k_2, \cdots, k_N. \tag{2.5}$$

When this is done, all indices of A's become contracted with indices of B's, implying that $F(A) \cdot G(B)$ is a sum of terms

$$\mathrm{Tr}(AB)^{r_1} \mathrm{Tr}(AB)^{r_2} \cdots \mathrm{Tr}(AB)^{r_m}$$

[4] In effect, we are here picking a basis for the space of W_α, $\alpha = 1, 2$. This will obscure the $SL(2, \mathbf{C})$ symmetry that was mentioned in the introduction.

with various r_i. The coefficient of $S^N = (\mathrm{Tr}\, AB)^N$ is nonzero — it is 1, coming from the trivial permutation in (2.5). The other terms with some $r_i > 1$ are contained in I, as we have seen in proving that $\mathcal{R}_{\mathrm{cl}}$ is generated by S. Hence $S^N \in I$.

So it remains only to show that $F(A)^{i_1 i_2 \cdots i_N}$ is in the ideal generated by A^2. Without loss of generality, since this tensor is symmetric in the indices i_k, we can set these to a common value, say N. We will show that

$$\epsilon^{j_1 j_2 \cdots j_N} A^N{}_{j_1} A^N{}_{j_2} \cdots A^N{}_{j_N} \tag{2.6}$$

is a nonzero multiple of

$$\epsilon^{N j_1 j_2 \cdots j_{N-1}} (A^2)^N{}_{j_1} A^N{}_{j_2} A^N{}_{j_3} \cdots A^N{}_{j_{N-1}}, \tag{2.7}$$

which is certainly proportional to A^2. We can write (2.7) more explicitly as

$$\sum_{x=1}^{N} \epsilon^{N j_1 j_2 \cdots j_{N-1}} A^N{}_x A^x{}_{j_1} A^N{}_{j_2} A^N{}_{j_3} \cdots A^N{}_{j_{N-1}}. \tag{2.8}$$

The expression

$$A^N{}_x A^N{}_{j_2} A^N{}_{j_3} \cdots A^N{}_{j_{N_1}}, \tag{2.9}$$

being antisymmetric in $x, j_2, \cdots, j_{N-1}$, is a nonzero multiple of

$$\epsilon_{x j_2 j_3 \cdots j_{N-1} r} \epsilon^{r s_1 s_2 \cdots s_{N-1}} A^N{}_{s_1} A^N{}_{s_2} \cdots A^N{}_{s_{N-1}}. \tag{2.10}$$

Now substitute this expression in (2.8), and then use (2.5) to write the product

$$\epsilon^{N j_1 j_2 \cdots j_{N-1}} \epsilon_{x j_2 j_3 \cdots j_{N-1} r}$$

as a multiple of

$$\delta^N_x \delta^{j_1}_r - \delta^N_r \delta^{j_1}_x.$$

We learn that (2.8) is a nonzero multiple of

$$(\delta^N_x \delta^{j_1}_r - \delta^N_r \delta^{j_1}_x) A^x{}_{j_1} \epsilon^{r s_1 s_2 \cdots s_{N-1}} A^N{}_{s_1} A^N{}_{s_2} \cdots A^N{}_{s_{N-1}}. \tag{2.11}$$

The $\delta^{j_1}_x$ terms give a multiple of $\mathrm{Tr}\, A$, which vanishes for A in the Lie algebra of $\mathrm{SU}(N)$, and the $\delta^N_x \delta^r_{j_1}$ term gives (2.6), as promised.

§3. Proof for Sp(N)

In this section, we prove the conjecture for $G = \mathrm{Sp}(N)$. For this group, $h = N + 1$ and $r = h - 1 = N$. Since we have in Section 2 explained why $\mathcal{R}_{\mathrm{cl}}$ is generated by S for $\mathrm{Sp}(N)$, and why $S^N = S^r \neq 0$, we need only prove that

$$S^{N+1} = 0.$$

We recall that a generator A of $\mathrm{Sp}(N)$ can be represented as a $2N \times 2N$ symmetric tensor A_{ij}. Indices are raised and lowered using the invariant antisymmetric tensor γ_{jk} of $\mathrm{Sp}(N)$, and its inverse

$$\gamma^{kr}: \ A^i{}_j = \gamma^{ik} A_{kj}, \quad A_{ij} = \gamma_{ik} A^k{}_j,$$

with $\gamma_{ik}\gamma^{kj} = \delta^j_i$. The definition of S is

$$S = \mathrm{Tr}\, AB = A_{ij} B_{kl} \gamma^{jk} \gamma^{li}.$$

To think of A as a matrix that can be multiplied, one should raise an index and use $A^i{}_j = \gamma^{ik}A_{kj}$. The ideal I is generated by the matrix elements of A^2, or explicitly by the quantities

$$\sum_{kl} A_{ik}A_{jl}\gamma^{kl}, \tag{3.1}$$

as well as similar expressions with one or both A's replaced by B.

The antisymmetric tensor γ^{ij} is nondegenerate and has a nonzero Pfaffian. This implies that the antisymmetric tensor $\epsilon^{i_1 i_2 \cdots i_{2N}}$ can be written in terms of γ:

$$\epsilon^{i_1 i_2 \cdots i_{2N}} = \gamma^{i_1 i_2}\gamma^{i_3 i_4}\cdots\gamma^{i_{2N-1}i_{2N}} \pm \text{permutations}. \tag{3.2}$$

The strategy of the proof will be the same as for SU(N). We will construct a polynomial $F(A)$ which is contained in the ideal generated by A^2, and which when contracted with the analogous polynomial in B is equal to S^{N+1} modulo I. We simply set

$$F^{k_1 k_2 \cdots k_{N-1}}_{i_1 i_2 \cdots i_{N+1}}(A) = \epsilon^{k_1 k_2 \cdots k_{N-1} j_1 j_2 \cdots j_{N+1}} A_{i_1 j_1} A_{i_2 j_2} \cdots A_{i_{N+1} j_{N+1}}. \tag{3.3}$$

F is antisymmetric in the k's and symmetric in the i's. To show that $F(A)$ is contained in the ideal I, we use the identity (3.2) to express the tensor ϵ as a sum of products of N γ's. The γ's have a total of $2N$ indices, $N+1$ of which are $j_1, j_2, \cdots, j_{N+1}$ and are contracted with A's. $N+1$ exceeds the number of γ's, so in each term of the sum, at least one γ has two indices j_m, j_n that are contracted with A's. Since a γ cannot be contracted twice with the same A ($\gamma^{ij}A_{ij} = 0$ as A is symmetric), the γ in question is contracted once each with two different A's, giving $A_{i_m j_m} A_{i_n j_n}\gamma^{j_m j_n}$, for some values of the indices; this is a generator of I. So $F(A) \in I$.

After defining $F^{k_1 k_2 \cdots k_{N-1}}_{i_1 i_2 \cdots i_{N+1}}(B)$ by the same formula, we now want to evaluate

$$F(A)\cdot F(B) = F^{i_1 i_2 \cdots i_{N+1}}_{k_1 k_2 \cdots k_{N-1}}(A) F^{k_1 k_2 \cdots k_{N-1}}_{i_1 i_2 \cdots i_{N+1}}(B). \tag{3.4}$$

(The indices of $F(A)$ have been raised and lowered with γ's to make this contraction.) Clearly, $F(A)\cdot F(B)$ is contained in I, since $F(A)$ and $F(B)$ are. However, we can also evaluate $F(A)\cdot F(B)$ by working directly from the definition:

$$\begin{aligned} F(A)\cdot F(B) = {} & \epsilon_{k_1 k_2 \cdots k_{N-1} j_1 j_2 \cdots j_{N+1}} A^{i_1 j_1} A^{i_2 j_2} \cdots A^{i_{N+1} j_{N+1}} \\ & \cdot \epsilon^{k_1 k_2 \cdots k_{N-1} m_1 m_2 \cdots m_{N+1}} B_{i_1 m_1} B_{i_2 m_2} \cdots B_{i_{N+1} m_{N+1}}. \end{aligned} \tag{3.5}$$

Now upon using the identity (2.5) to evaluate $\epsilon^{k_1 k_2 \cdots k_{N-1} m_1 m_2 \cdots m_{N+1}}\epsilon_{k_1 k_2 \cdots k_{N-1} j_1 j_2 \cdots j_{N+1}}$, all indices of A's are contracted with indices of B's, and we get as in Section 2 a sum of terms each of which is of the form $\operatorname{Tr}(AB)^{r_1}\operatorname{Tr}(AB)^{r_2}\cdots\operatorname{Tr}(AB)^{r_m}$ for some r_i. The coefficient of $S^{N+1} = (\operatorname{Tr} AB)^{N+1}$ is nonzero, and the other terms with some $r_i > 1$ are again all contained in the ideal I. So we have shown that S^{N+1} is contained in I, completing the proof of the conjecture for the symplectic group.

§4. Proof for SO(N)

For SO(N), the dual Coxeter number is $h = N-2$. The proof that $S^{N-2} \in I$ will be similar to what we have already seen, though slightly more elaborate. A novelty for SO(N) is that $h > r+1$ in this case, so the argument using reduction to a maximal torus (which only shows that $S^r \notin I$) does not suffice to show that $S^{h-1} \notin I$. The only proof of this that I know of uses facts about four-dimensional instantons and is deferred to Section 5.

An element of the Lie algebra of SO(N) is an antisymmetric $N \times N$ matrix A_{ij}; indices are raised and lowered and contracted using the invariant metric δ_{ij} and its inverse δ^{ij}. Since indices can be raised and lowered in a unique way without introducing any minus signs, we make no distinction between upper and lower indices. The ideal I is generated by

$$(A^2)_{ij} = \sum_k A_{ik}A_{kj} = -\sum_k A_{ik}A_{jk}, \tag{4.1}$$

and analogous expressions with one or both A's replaced by B. Apart from δ_{ij}, the only independent invariant tensor is the antisymmetric tensor $\epsilon_{i_1 i_2 \cdots i_N}$.

Since the proof that $S^{N-2} \in I$ will be slightly elaborate, we first consider the case of SO(5) (which is isomorphic to Sp(2) so that we could borrow the result of the last section, though the argument will not be expressed in such terms). To prove that $S^3 = 0$ for SO(5), we will construct a cubic polynomial $F(A)$, which is contained in I and when contracted with the analogous cubic polynomial in B is equal to S^3 mod I. This will show that $S^3 \in I$ for SO(5). Then we will generalize the construction to SO(N) with $N > 5$. (Since SO(3) is equivalent to SU(2) and SO(4) to SU(2) $\times$ SU(2), we need not consider those cases.)

To construct $F(A)$, we will begin with a product of three A's, say $A_{rr'}A_{ss'}A_{tt'}$, and a product of two antisymmetric tensors, $\epsilon_{i_1 i_2 \cdots i_5}\epsilon_{j_1 j_2 \cdots j_5}$. Then we will contract all six indices of the A's with some of the ten indices carried by the antisymmetric tensors. There is essentially only one way to do this. We cannot take two A's and contract all four of their indices with the same antisymmetric tensor, since

$$A_{i_1 i_2}A_{i_3 i_4}\epsilon_{i_1 i_2 i_3 i_4 i_5} = 0 \tag{4.2}$$

by anticommutativity. Likewise, we cannot have two A's each with one index contracted with each of the antisymmetric tensors, since again

$$A_{i_1 j_1}A_{i_2 j_2}\epsilon_{i_1 i_2 i_3 i_4 i_5}\epsilon_{j_1 j_2 j_3 j_4 j_5} = 0 \tag{4.3}$$

by anticommutativity. So the only nonzero expression that we can make by contracting all six indices of the three A's with six of the ten indices of the antisymmetric tensors is

$$F_{i_1 i_2 j_1 j_2}(A) = A_{i_3 i_4}A_{j_3 j_4}A_{i_5 j_5}\epsilon_{i_1 i_2 i_3 i_4 i_5}\epsilon_{j_1 j_2 j_3 j_4 j_5}, \tag{4.4}$$

in which one A is contracted twice with the first antisymmetric tensor, one is contracted twice with the second, and one is contracted once with each.

If we insert in the definition of F the identity

$$\epsilon_{i_1 i_2 i_3 i_4 i_5}\epsilon_{j_1 j_2 j_3 j_4 j_5} = \delta_{i_1 j_1}\delta_{i_2 j_2}\cdots\delta_{i_5 j_5} \pm \text{permutations},$$

we get a sum of many terms each proportional to a product of five metric tensors $\delta_{i_m j_n}$. The five metrics have a total of ten indices, six of which are contracted with indices of the three A's. The crucial fact is that six exceeds five, so one metric has both indices contracted with A's. One cannot contract a metric tensor twice with the same A ($\delta_{ij}A_{ij} = 0$, as A is antisymmetric). So inevitably, in each term, one of the δ's is contracted with two different A's, giving an expression

$$A_{mn}\delta_{np}A_{pr} = (A^2)_{mr}$$

that is a generator of the ideal I. So $F(A)$ is contained in I.

On the other hand, consider

$$F(A)\cdot F(B)=A_{i_3i_4}A_{j_3j_4}A_{i_5j_5}\epsilon_{i_1i_2i_3i_4i_5}\epsilon_{j_1j_2j_3j_4j_5}\\ \cdot B_{k_3k_4}B_{t_3t_4}B_{k_5t_5}\epsilon_{i_1i_2k_3k_4k_5}\epsilon_{j_1j_2t_3t_4t_5}. \tag{4.5}$$

We can evaluate this by using (2.5) to express the products

$$\epsilon_{i_1i_2i_3i_4i_5}\epsilon_{i_1i_2k_3k_4k_5}\quad\text{and}\quad\epsilon_{j_1j_2j_3j_4j_5}\epsilon_{j_1j_2t_3t_4t_5}$$

in terms of products of δ's. When we do this, the indices of A's and B's are contracted, and we get a sum of terms, each of which is a product of traces of words in A and B. The sum includes a positive multiple of S^3, and additional terms that are contained in I because the trace of any word with more than two letters is in I. This proves that $S^3\in I$ for SO(5).

To prove in a similar fashion that $S^{N-2}\in I$ for SO(N), we should start with $N-2$ factors of A and contract some of their indices with a product of two ϵ's to define a polynomial $F(A)$. To prove along the above lines that $F(A)\in I$, we need to have at least $N+1$ indices of the product of ϵ's contracted with A's. Let us verify that this is just possible. As we have seen above, three A's can be contracted twice each with the product of ϵ's, giving a total of 6 contractions. The remaining $N-5$ A's can each have only one index contracted with the product of ϵ's, since two contractions will give a vanishing result by virtue of (4.2) or (4.3) (which have obvious analogs for $N>5$). The total number of contractions will hence be

$$6+(N-5)=N+1,$$

exactly what we need. It does not matter with which antisymmetric tensor the last $N-5$ A's are contracted. So we define

$$F(A)_{i_1i_2j_1j_2\cdots j_{N-3}s_1s_2\cdots s_{N-5}}\\ =\epsilon_{i_1i_2t_1t_2\cdots t_{N-2}}\epsilon_{j_1j_2\cdots j_{N-3}k_1k_2k_3}A_{t_1t_2}A_{k_1k_2}A_{t_3k_3}A_{t_4s_1}A_{t_5s_2}\cdots A_{t_{N-2}s_{N-5}}. \tag{4.6}$$

$F(A)$ is contained in I for the familiar reason: upon using (2.5) to replace the product of antisymmetric tensors with a sum of products of δ's, we get a sum of terms in each of which some δ is contracted with two A's, giving a generator of I.

Hence the quantity

$$F(A)\cdot F(B)=F(A)_{i_1i_2j_1j_2\cdots j_{N-3}s_1s_2\cdots s_{N-5}}F(B)_{i_1i_2j_1j_2\cdots j_{N-3}s_1s_2\cdots s_{N-5}} \tag{4.7}$$

is contained in I. On the other hand, explicitly

$$F(A)\cdot F(B)\\ =\epsilon_{i_1i_2t_1t_2\cdots t_{N-2}}\epsilon_{j_1j_2\cdots j_{N-3}k_1k_2k_3}A_{t_1t_2}A_{k_1k_2}A_{t_3k_3}A_{t_4s_1}A_{t_5s_2}\cdots A_{t_{N-2}s_{N-5}}\\ \epsilon_{i_1i_2u_1u_2\cdots u_{N-2}}\epsilon_{j_1j_2\cdots j_{N-3}n_1n_2n_3}B_{u_1u_2}B_{n_1n_2}B_{u_3n_3}B_{u_4s_1}B_{u_5s_2}\cdots B_{u_{N-2}s_{N-5}}. \tag{4.8}$$

Using (2.5) to replace

$$\epsilon_{i_1i_2t_1t_2\cdots t_{N-2}}\epsilon_{i_1i_2u_1u_2\cdots u_{N-2}}$$

and likewise

$$\epsilon_{j_1j_2\cdots j_{N-3}k_1k_2k_3}\epsilon_{j_1j_2\cdots j_{N-3}n_1n_2n_3}$$

with sums of products of δ's, we learn in the familiar fashion that $F(A)\cdot F(B)$ is equal to S^{N-2} plus a sum of terms (proportional to traces of longer words in A and B) that are contained in I. Combining these results, we deduce that $S^{N-2}\in I$.

§5. Implications of Instanton Calculations

For any simple Lie group G, a one-instanton solution on $\mathbf{R}^4$ is obtained by picking a minimal SU(2) subgroup of G, and embedding in G the one-instanton solution of SU(2). Under such a minimal SU(2), the Lie algebra $\mathbf{g}$ of G decomposes as the adjoint representation of SU(2) plus a certain number of pairs of copies of the spin one-half representation, as well as SU(2) singlets. Because the same SU(2) representations arise for any G, the relevant properties of the one-instanton computation are largely independent of G. A computation for all simple Lie groups was performed in [7].

For instanton number one, the instanton moduli are the position and size and "SU(2) orientation" of the instanton and the choice of minimal embedding of SU(2) in G.

In the field of the instanton, the gluino field (which is a fermi field with values in the adjoint representation of G) has $2h$ zero modes, all of one chirality. This is the right number to give an expectation value to an operator with the quantum numbers of a product of h copies of S. In [3], general properties of chiral operators were used to show that in a supersymmetric vacuum the expectation value of a product of chiral operators such as $\langle S(x_1)S(x_2)\cdots S(x_h)\rangle$ is independent of the choice of points $x_i \in \mathbf{R}^4$ as long as the x_i are distinct, ensuring there are no ambiguities in defining the operator products. Moreover, a one-instanton computation was performed on $\mathbf{R}^4$, with the result

$$\langle S(x_1)S(x_2)\cdots S(x_h)\rangle_{1\text{ inst}} = c_0\Lambda^{3h} \tag{5.1}$$

for some constant c_0. The computation is made by evaluating $S(x_i)$ as bilinear expressions in the fermion zero modes (corrections to this vanish by holomorphy) and then integrating over instanton moduli space. The subscript "1 inst" in (5.1) refers to the fact that we are recording here the result of a one-instanton computation, which may or may not give the exact quantum answer.

Our main goal in the present section is to argue from properties of the one-instanton moduli space that $S^{h-1} \notin I$. For this, we do not need to know whether the one-instanton computation gives the exact quantum answer or not; in fact, we do not even need to know if the quantum theory really exists. The argument we will give could be formulated as a conventional mathematical proof that $S^{h-1} \notin I$, using properties of instanton moduli space.

We will also sketch how instantons are used to deduce the quantum anomaly that makes S^h a non-zero multiple of the identity in the quantum chiral ring (rather than vanishing, as it does classically). For this, one does need to know something about the quantum theory, so after arguing that $S^{h-1} \notin I$, we will recall some issues concerning the relation of the instanton computation to the quantum theory.

It is possible to take $h-1$ of the x_i to coincide without running into any difficulty or ambiguity and in particular without running into a singular contribution from small instantons. (See [6, Equation (7.17)], where this choice is made.) So

$$\langle S(x)S^{h-1}(y)\rangle_{1\text{ inst}} = c_0\Lambda^{3h}. \tag{5.2}$$

From this we can deduce the desired result that $S^{h-1} \notin I$. Indeed, because of the $\overline{Q}$-invariance of the one-instanton computation, a formula $S^{h-1} = \{\overline{Q}_{\dot\alpha}, Y^{\dot\alpha}\}$ would imply the vanishing of (5.2) (it would lead to a representation of (5.2) as the integral of a total derivative over instanton moduli space). Since it does not vanish, $S^{h-1} \notin I$.

This completes what we have to say about nonvanishing of S^{h-1} in the classical theory. Now let us discuss how the quantum anomaly in S^h comes about. If we simply set $x = y$ in (5.2), we find that the function that must be integrated over instanton moduli space is identically zero. The reason for this is that at each point $y \in \mathbf{R}^4$ and for any given one-instanton solution, two of the fermion zero modes vanish. (They are a suitable linear combination, depending on y and on the position of the instanton, of the zero modes generated by global supersymmetries and superconformal transformations.) Hence, when $S^h(y)$ is evaluated using the fermion zero modes, one gets identically zero before doing any integral over instanton moduli space. This is compatible with (but stronger than) the kind of behavior of S^h that one would expect from the classical result $S^h \in I$ (this result would make us expect a perhaps not identically zero total derivative on moduli space).

In the quantum theory, however, we should be careful in defining an operator product such as S^h. This is conveniently done by point-splitting, taking a product such

$$S(x_1)S(x_2)\cdots S(x_h)$$

and, after performing the computation, taking the limit as the x_i coincide. In the present case, there is no problem in setting $h-1$ of the x_i equal (since no singular small-instanton contributions appear in (5.2) as long as $x \neq y$). But we should be careful to define $S^h(y)$ as $\lim_{x\to y} S(x)S^{h-1}(y)$. When we do this, clearly we get

$$\langle S^h(y)\rangle_{1\ \mathrm{inst}} = c_0\Lambda^{3h}. \tag{5.3}$$

If we assume that the one-instanton amplitude coincides with the exact quantum answer, we would deduce from this that the classical ring relation $S^h = 0$ is deformed quantum mechanically to $S^h = c_0\Lambda^{3h}$. However, it is believed that the exact quantum answer is actually

$$\langle S^h(y)\rangle = c\Lambda^{3h} \tag{5.4}$$

with a different constant c, so that the quantum ring relation is really $S^h = c\Lambda^{3h}$. The discrepancy between the one-instanton computation of the anomaly coefficient and the exact result is still somewhat surprising; for a detailed analysis and references, see Section 7 of [6]. One simple statement[8] is that if the one-instanton computation is done on $\mathbf{R}^3 \times \mathbf{S}^1$ instead of $\mathbf{R}^4$, with an arbitrary radius for the $\mathbf{S}^1$, one gets the result (5.4), with what is believed to be the correct coefficient c, independent of the radius (as long as the radius is finite). Since the statement $S^h = c\Lambda^{3h} + \{\overline{Q}_{\dot\alpha}, Y^{\dot\alpha}\}$ is an operator statement, independent of any particular choice of state, the coefficient c can be computed, in principle, on any chosen four-manifold and with any chosen boundary conditions. Compactification on $\mathbf{S}^1$ with small radius and a non-trivial Wilson loop expectation value at infinity gives a suitable framework for a reliable computation of the anomaly coefficient in a weakly coupled context. (The proof that $S^{h-1} \notin I$ could also have been carried out in just the same way after compactification on $\mathbf{S}^1$.) The direct computation on $\mathbf{R}^4$ is presumably affected by some infrared divergences in the relation between the perturbative vacuum in which the computation is done and the true quantum vacuum.[5]

[5] Since completion of this paper, the conjecture studied here has been proved for the case of the exceptional Lie group G_2 by P. Etinghof and V. Kac, "On the Cachazo-Douglas-Seiberg-Witten conjecture for simple Lie algebras," math. QA/0305175.

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