Chin. Ann. Math.
24B:4(2003),415–422.

EXACT CONTROLLABILITY FOR FIRST ORDER QUASILINEAR HYPERBOLIC SYSTEMS WITH ZERO EIGENVALUES***

LI Tatsien*   YU Lixin**

Abstract

For a class of mixed initial-boundary value problem for general quasilinear hyperbolic systems with zero eigenvalues, the authors establish the local exact controllability with boundary controls acting on one end or on two ends and internal controls acting on a part of equations in the system.

Keywords Semi-global C^1 solution, Exact controllability, Quasilinear hyperbolic system
2000 MR Subject Classification 35L50, 49J20, 93B05, 93C20
Chinese Library Classification O175.27, O232   **Document Code** A
Article ID 0252-9599(2003)04-0415-08

§1. Introduction

Consider the following first order quasilinear hyperbolic system

$$\frac{\partial u}{\partial t}+A(u)\frac{\partial u}{\partial x}=F(u), \tag{1.1}$$

where $u=(u_1,\cdots,u_n)^T$is the unknown vector function of (t,x), $A(u)$ is an $n\times n$ matrix with suitably smooth elements $a_{ij}(u)$ $(i,j=1,\cdots,n)$ and $F:R^n\to R^n$ is a vector function with suitably smooth components $f_i(u)$ $(i=1,\cdots,n)$ such that

$$F(0)=0. \tag{1.2}$$

By the definition of hyperbolicity, on the domain under consideration, the matrix $A(u)$ has n real eigenvalues $\lambda_i(u)$ $(i=1,\cdots,n)$ and a complete set of left eigenvectors $l_i(u)=(l_{i1}(u),\cdots,l_{in}(u))$ $(i=1,\cdots,n)$:

$$l_i(u)A(u)=\lambda_i(u)l_i(u). \tag{1.3}$$

We have

$$\det|l_{ij}(u)|\neq 0. \tag{1.4}$$

Under the assumption that the eigenvalues satisfy the following conditions:

$$\lambda_r(u)<0<\lambda_s(u)\qquad(r=1,\cdots,m;\ s=m+1,\cdots,n), \tag{1.5}$$

Manuscript received May 9, 2003.
*Department of Mathematics, Fudan University, Shanghai 200433, China. **E-mail:** dqli@fudan.ac.cn
**Institute of Mathematics, Fudan University, Shanghai 200433, China.
***Project supported by the Special Funds for Major State Basic Research Projects of China.

the local exact boundary controllability for the quasilinear hyperbolic system (1.1) with general nonlinear boundary conditions was considered by Li Tatsien and Rao Bopeng in [1]–[2] (see also [3]). In an earlier work [4], M. Cirina also discussed this kind of problem under much stronger assumptions.

In this paper we will discuss the quasilinear hyperbolic system (1.1) with zero eigenvalues. This case is of great importance in applications, but the method used in [1]–[4] can not be applied directly. For fixing the idea, we assume that on the domain under consideration, the eigenvalues of $A(u)$ satisfy the following conditions:

$$\lambda_p(u) < \lambda_q(u) \equiv 0 < \lambda_r(u) \qquad (p = 1, \cdots, l;\ q = l+1, \cdots, m;\ r = m+1, \cdots, n). \tag{1.6}$$

Noting that, in order to solve the simplest equation with zero eigenvalue

$$\frac{\partial u}{\partial t} = 0, \tag{1.7}$$

it is not necessary to have a boundary condition, then any boundary control gives no effect on the solution. Therefore, differently from the situation that the eigenvalues satisfy (1.5), in order to realize the exact controllability for quasilinear hyperbolic systems with zero eigenvalues, we should use not only suitable boundary controls but also suitable internal controls. To this end, it is necessary to rewrite system (1.1) into the corresponding characteristic form

$$l_i(u)\Big(\frac{\partial u}{\partial t} + \lambda_i(u)\frac{\partial u}{\partial x}\Big) = \widetilde{F}_i(u) \triangleq l_i(u)F(u) \qquad (i = 1, \cdots, n), \tag{1.8}$$

in which the i-th equation includes only the directional derivative of unknown function u with respect to t along the i-th characteristic direction $\frac{dx}{dt} = \lambda_i(u)$, and

$$\widetilde{F}_i(0) = 0 \qquad (i = 1, \cdots, n). \tag{1.9}$$

Since in the sequel, we will add some internal controls to those equations which correspond to zero eigenvalues in the characteristic form (1.8), we discuss the following system

$$\begin{cases} l_p(u)\Big(\dfrac{\partial u}{\partial t} + \lambda_p(u)\dfrac{\partial u}{\partial x}\Big) = \widetilde{F}_p(u) & (p = 1, \cdots, l), \\ l_q(u)\dfrac{\partial u}{\partial t} = \widetilde{F}_q(u) + c_q(t,x) & (q = l+1, \cdots, m), \\ l_r(u)\Big(\dfrac{\partial u}{\partial t} + \lambda_r(u)\dfrac{\partial u}{\partial x}\Big) = \widetilde{F}_r(u) & (r = m+1, \cdots, n), \end{cases} \tag{1.10}$$

where $l_i(u), \lambda_i(u)$ and $\widetilde{F}_i(u)$ $(i = 1, \cdots, n)$ are all assumed to be C^1 functions and

$$c_q(t,x) = a_q(t,x)\frac{\partial v_q(t,x)}{\partial t} + b_q(t,x) \qquad (q = l+1, \cdots, m), \tag{1.11}$$

in which a_q and v_q $(q = l+1, \cdots, m)$ are C^1 vector functions of (t,x) and b_q $(q = l+1, \cdots, m)$ are C^1 functions of (t,x).

We give the initial condition

$$t = 0: \quad u = \varphi(x), \quad 0 \le x \le 1 \tag{1.12}$$

and the nonlinear boundary conditions

$$x = 0: \ v_r = G_r(t, v_1, \cdots, v_l, v_{l+1}, \cdots, v_m) + H_r(t) \ \ (r = m+1, \cdots, n), \tag{1.13}$$

$$x = 1: \ v_p = G_p(t, v_{l+1}, \cdots, v_m, v_{m+1}, \cdots, v_n) + H_p(t) \ \ (p = 1, \cdots, l), \tag{1.14}$$

where

$$v_i = l_i(u)u \qquad (i = 1, \cdots, n), \tag{1.15}$$

G_p, G_r, H_p, H_r $(p = 1, \cdots, l;\ r = m+1, \cdots, n)$ and φ are all C^1 functions with respect to their arguments, and, without loss of generality, we assume that

$$G_p(t, 0, \cdots, 0) \equiv 0, \quad G_r(t, 0, \cdots, 0) \equiv 0 \qquad (p = 1, \cdots, l;\ r = m+1, \cdots, n). \tag{1.16}$$

Although $c_q(t, x)$ $(q = l+1, \cdots, m)$ given by (1.11) have a lower regularity, if we assume that the conditions of C^1 compatibility are satisfied at the points (0,0) and (0,1) respectively, the mixed initial-boundary value problem (1.10)–(1.14) admits a unique local C^1 solution $u = u(t, x)$ on the domain

$$R(\delta) = \{(t, x) | 0 \le t \le \delta,\ 0 \le x \le 1\}, \tag{1.17}$$

where $\delta > 0$ is a suitably small constant (see [5]).

In order to get the exact controllability, for a given and probably quite large $T > 0$, in §2 we will give the existence and uniqueness of C^1 solution (called the semi-global C^1 solution) to the corresponding mixed initial-boundary value problem on the domain

$$R(T) = \{(t, x) | 0 \le t \le T,\ 0 \le x \le 1\}. \tag{1.18}$$

Then, by mean of boundary controls $H_p(t)$ $(p = 1, \cdots, l)$ and (or) $H_r(t)$ $(r = m+1, \cdots, n)$ and internal controls $c_q(t, x)$ $(q = l+1, \cdots, m)$, the local exact controllability for system (1.1) will be realized in §3 and §4.

§2. Existence and Uniqueness of Semi-Global C^1 Solution

Lemma 2.1. *Assume that* $l_i, \lambda_i, \widetilde{F}_i, G_p, G_r, H_p,\ H_r$ $(i = 1, \cdots, n;\ p = 1, \cdots, l;\ r = m+1, \cdots, n)$ *and* φ *are all* C^1 *functions with respect to their arguments, and* (1.6), (1.9) *and* (1.16) *hold. Assume furthermore that* c_q $(q = l+1, \cdots, m)$ *are given by* (1.11), *in which* a_q *and* v_q $(q = l+1, \cdots, m)$ *are* C^1 *vector functions of* (t, x) *and* b_q $(q = l+1, \cdots, m)$ *are* C^1 *functions of* (t, x). *Assume finally that the conditions of* C^1 *compatibility are satisfied at the points* $(0, 0)$ *and* $(0, 1)$ *respectively. For any given* $T > 0$, *the mixed initial-boundary value problem* (1.10) *and* (1.12)–(1.14) *admits a unique semi-global* C^1 *solution* $u = u(t, x)$ *with sufficiently small* C^1 *norm on the domain* (1.18), *provided that the* C^1 *norms* $\|\varphi\|_{C^1[0,1]}, \|(H_p, H_r)\|_{C^1[0,T]}$ *and* $\|(v_q, b_q)\|_{C^1[R(T)]}$ $(p = 1, \cdots, l;\ q = l+1, \cdots, m;\ r = m+1, \cdots, n)$ *are small enough* (*depending on* T).

The proof of this lemma can be found in [7] (see also [6]).

§3. Exact Controllability with Boundary Controls Acting on Two Ends

The main result in this section is

Theorem 3.1. *Assume that* $l_i, \lambda_i, \widetilde{F}_i, G_p$ *and* G_r $(i = 1, \cdots, n;\ p = 1, \cdots, l;\ r = m+1, \cdots, n)$ *are all* C^1 *functions with respect to their arguments. Assume furthermore that* (1.6), (1.9) *and* (1.16) *hold. Let*

$$T > \max_{\substack{p=1,\cdots,l \\ r=m+1,\cdots,n}} \Big(\frac{1}{|\lambda_p(0)|}, \frac{1}{\lambda_r(0)} \Big). \tag{3.1}$$

For any given initial data $\varphi \in C^1[0, 1]$ *and final data* $\psi \in C^1[0, 1]$ *with small* C^1 *norm, there exist boundary controls* $H_p(t), H_r(t) \in C^1[0, T]$ $(p = 1, \cdots, l;\ r = m+1, \cdots, n)$ *with small* C^1 *norm, and internal controls* $c_q(t, x)$ $(q = l+1, \cdots, m)$ *given by* (1.11), *in which* $a_q(t, x)$ *and* $v_q(t, x)$ $(q = l+1, \cdots, m)$ *are* C^1 *vector functions and* b_q $(q = l+1, \cdots, m)$ *are* C^1 *functions on* $R(T)$ *and the* C^1 *norms of* b_q *and* v_q $(q = l+1, \cdots, m)$ *are suitably small* , *such that the mixed initial-boundary value problem* (1.10) *and* (1.12)–(1.14) *admits a unique*

C^1 solution $u = u(t,x)$ with small C^1 norm on the domain $R(T)$, which verifies the final condition

$$t = T: \quad u = \psi(x), \quad 0 \le x \le 1. \tag{3.2}$$

In order to prove Theorem 3.1, we construct the following system of characteristic form

$$\begin{cases} l_p(u)\Big(\dfrac{\partial u}{\partial t} + \lambda_p(u)\dfrac{\partial u}{\partial x}\Big) = \widetilde{F}_p(u) & (p = 1, \cdots, l), \\ l_q(u)\Big(\dfrac{\partial u}{\partial t} + \bar{\lambda}_q(u)\dfrac{\partial u}{\partial x}\Big) = \widetilde{F}_q(u) & (q = l+1, \cdots, m), \\ l_r(u)\Big(\dfrac{\partial u}{\partial t} + \lambda_r(u)\dfrac{\partial u}{\partial x}\Big) = \widetilde{F}_r(u) & (r = m+1, \cdots, n), \end{cases} \tag{3.3}$$

in which we assume that

$$\bar{\lambda}_q(u) = \lambda_1(u) \text{ or } \lambda_n(u) \qquad (q = l+1, \cdots, m). \tag{3.4}$$

Since (3.3) is a system without zero eigenvalues, according to the Proposition in [1], we have

Lemma 3.1. *Let $T > 0$ be defined by* (3.1). *Assume that $l_i(u), \lambda_i(u)$ and $\widetilde{F}_i(u)$ $(i = 1, \cdots, n)$ are all C^1 functions with respect to their arguments. For any given initial data $\varphi \in C^1[0,1]$ and final data $\phi \in C^1[0,1]$ with small C^1 norm, the hyperbolic system* (3.3) *admits a unique C^1 solution $u = u(t,x)$ with small C^1 norm on the domain $R(T) = \{(t,x) | 0 \le t \le T,\ 0 \le x \le 1\}$, which satisfies the initial data*

$$t = 0: \quad u = \varphi(x), \quad 0 \le x \le 1 \tag{3.5}$$

and the final data (3.2).

By Lemma 3.1, we can prove Theorem 3.1.

In fact, substituting a C^1 solution $u = u(t,x)$ given by Lemma 3.1 into the boundary conditions (1.13) and (1.14), we get the desired boundary controls

$$H_p(t) = (v_p - G_p(t, v_{l+1}, \cdots, v_m, v_{m+1}, \cdots, v_n))|_{x=1} \qquad (p = 1, \cdots, l), \tag{3.6}$$

$$H_r(t) = (v_r - G_r(t, v_1, \cdots, v_l, v_{l+1}, \cdots, v_m)|_{x=0} \qquad (r = m+1, \cdots, n), \tag{3.7}$$

where v_i $(i = 1, \cdots, n)$ are defined by (1.15). Noting (1.16), the C^1 norms of H_p $(p = 1, \cdots, l)$ and H_r $(r = m+1, \cdots, n)$ are small. On the other hand, substituting $u = u(t,x)$ into the second part of system (1.10), we get the desired internal controls

$$c_q(t,x) = l_q(u(t,x))\frac{\partial u(t,x)}{\partial t} - \widetilde{F}_q(u(t,x)) \qquad (q = l+1, \cdots, m), \tag{3.8}$$

which corresponds to (1.11) in which

$$a_q(t,x) = l_q(u(t,x)), \quad b_q(t,x) = -\widetilde{F}_q(u(t,x)), \quad v_q(t,x) = u(t,x) \ (q = l+1, \cdots, m). \tag{3.9}$$

Noting (1.9), the $C^1[R(T)]$ norms of $b_q(t,x)$ and $v_q(t,x)$ $(q = l+1, \cdots, m)$ are also small.

Obviously, $u = u(t,x)$ verifies the corresponding mixed initial-boundary valued problem (1.10) and (1.12)–(1.14) on the domain $R(T)$. By Lemma 2.1, $u = u(t,x)$ is the semi-global C^1 solution to this mixed problem on the domain $R(T)$, which satisfies also the final condition (3.2). Thus, we obtain the desired exact controllability, in which the boundary controls H_p $(p = 1, \cdots, l)$ and H_r $(r = m+1, \cdots, n)$ are given by (3.6)–(3.7) and the internal controls $c_q(t,x)$ $(q = l+1, \cdots, m)$ are given by (3.8).

In §4 we will prove that for a class of mixed initial-boundary value problem, the number of boundary controls can be diminished, provided that the exact controllability time is doubled.

§4. Exact Controllability with Boundary Controls Acting on One End

Suppose that the number of positive eigenvalues equals that of negative eigenvalues:

$$l = n - m. \tag{4.1}$$

Suppose furthermore that the boundary condition (1.13) [resp. (1.14)] can be equivalently rewritten as

$$x = 0: \ v_p = \overline{G}_p(t, v_{l+1}, \cdots, v_m, v_{m+1}, \cdots, v_n) + \overline{H}_p(t) \ \ (p = 1, \cdots, l) \tag{4.2}$$

$$[\text{resp. } x = 1: \ v_r = \overline{G}_r(t, v_1, \cdots, v_l, v_{l+1}, \cdots, v_m) + \overline{H}_r(t) \ \ (r = m+1, \cdots, n)] \tag{4.2$'$}$$

with

$$\overline{G}_p(t, 0, \cdots, 0) \equiv 0 \qquad (p = 1, \cdots, l) \tag{4.3}$$

$$[\text{resp. } \overline{G}_r(t, 0, \cdots, 0) \equiv 0 \qquad (r = m+1, \cdots, n)]. \tag{4.3$'$}$$

Then

$$\text{small } C^1 \text{ norm of } \|H_r\| \Leftrightarrow \text{small } C^1 \text{ norm of } \|\overline{H}_p\| \tag{4.4}$$

$$[\text{resp. small } C^1 \text{ norm of } \|H_p\| \Leftrightarrow \text{small } C^1 \text{ norm of } \|\overline{H}_r\|], \tag{4.4$'$}$$

in which $p = 1, \cdots, l; r = m+1, \cdots, n$.

Theorem 4.1. *Under the assumptions of Theorem* 3.1, *suppose furthermore that* (4.1)–(4.3) *hold and* $\overline{G}_p\ (t, \cdot) (p = 1, \cdots, l)$ [*resp.* $\overline{G}_r(t, \cdot)\ (r = m+1, \cdots, n)$] *are* C^1 *functions with respect to their arguments. Let*

$$T > 2 \max_{\substack{p=1,\cdots,l \\ r=m+1,\cdots,n}} \Big(\frac{1}{|\lambda_p(0)|}, \frac{1}{\lambda_r(0)} \Big). \tag{4.5}$$

Suppose finally that $H_r(t)\ (r = m+1, \cdots, n)$ [*resp.* $H_p(t)\ (p = 1, \cdots, l)$] *are given* $C^1[0, T]$ *functions with small* C^1*norm. For any given initial data* $\varphi \in C^1[0,1]$ *and final data* $\psi \in C^1[0,1]$ *with small* C^1 *norm, such that the conditions of* C^1 *compatibility are satisfied at the points* $(0,0)$ *and* $(T,0)$ [*resp.* $(0,1)$ *and* $(T,1)$] *respectively, there exist boundary controls* $H_p(t) \in C^1[0,T]\ (p = 1, \cdots, l)$ [*resp.* $H_r(t) \in C^1[0,T]\ (r = m+1, \cdots, n)$] *with small* C^1 *norm and internal controls* $c_q(t,x)\ (q = l+1, \cdots, m)$ *given by* (1.11), *in which* $a_q(t,x)$ *and* $v_q(t,x)\ (q = l+1, \cdots, m)$ *are* C^1 *vector functions and* $b_q(t,x)\ (q = l+1, \cdots, m)$ *are* C^1 *functions on* $R(T)$ *and the* C^1 *norms of* b_q *and* $v_q\ (q = l+1, \cdots, m)$ *are suitably small, such that the mixed initial-boundary value problem* (1.10) *and* (1.12)–(1.14) *admits a unique* C^1 *solution* $u = u(t,x)$ *with small* C^1 *norm on the domain* $R(T)$, *which verifies the final condition* (3.2).

In order to prove Theorem 4.1, we consider system (3.3) in which

$$\bar{\lambda}_q(u) = \lambda_1(u) < 0 \quad [\text{resp. } \ \bar{\lambda}_q(u) = \lambda_n(u) > 0] \qquad (q = l+1, \cdots, m). \tag{4.6}$$

It suffices to establish the following

Lemma 4.1. *Under the assumptions of Theorem* 4.1, *for any given initial data* $\varphi \in C^1[0,1]$ *and final data* $\phi \in C^1[0,1]$ *with small* C^1 *norm, such that the conditions of* C^1 *compatibility are satisfied at the points* $(0,0)$ *and* $(T,0)$ [*resp.* $(0,1)$ *and* $(T,1)$] *respectively, the quasilinear hyperbolic system* (3.3) (*with* (4.6)) *with the boundary condition* (1.13) [*resp.* (1.14)] *admits a unique* C^1 *solution* $u = u(t,x)$ *with small* C^1 *norm on the domain* $R(T)$, *which satisfies the initial condition* (1.12) *and the final condition* (3.2).

In fact, assume that $u = u(t,x)$ is a C^1 solution given by Lemma 4.1, the boundary control can be given by

$$H_p(t) = (v_p - G_p(t, v_{l+1}, \cdots, v_m, v_{m+1}, \cdots, v_n))|_{x=1} \ \ (p = 1, \cdots, l) \tag{4.7}$$

$$[\text{resp. } H_r(t) = (v_r - G_r(t, v_1, \cdots, v_l, v_{l+1}, \cdots, m))|_{x=0} \quad (r = m+1, \cdots, n)]. \tag{4.7$'$}$$

Noting (1.16), the $C^1[0,T]$ norms of $H_p(t)$ $(p = 1, \cdots, l)$ [resp. $H_r(t)$ $(r = m+1, \cdots, n)$] are small. On the other hand, the internal controls can be still given by (3.8)–(3.9). By (1.9), the C^1 norms of $b_q(t,x)$ and $v_q(t,x)$ $(q = 1, \cdots, l)$ are also small.

Since in system (3.3) the number of positive eigenvalues is not equal to that of negative eigenvalues, we should prove Lemma 4.1 in a way slightly different from that in [2]. For fixing the idea, in what follows we consider only the case that the boundary controls are given at the end $x = 1$.

Noting (4.5), there exist an $\varepsilon_0 > 0$ such that

$$T > 2 \max_{\substack{p=1,\cdots,l \\ r=m+1,\cdots,n \\ |u|\le\varepsilon_0}} \Big(\frac{1}{|\lambda_p(u)|}, \frac{1}{\lambda_r(u)}\Big). \tag{4.8}$$

Let

$$T_1 = \max_{\substack{p=1,\cdots,l \\ r=m+1,\cdots,n \\ |u|\le\varepsilon_0}} \Big(\frac{1}{|\lambda_p(u)|}, \frac{1}{\lambda_r(u)}\Big). \tag{4.9}$$

We first consider the forward mixed initial-boundary value problem for system (3.3) (with (4.6)) with the initial data (3.5) and the boundary conditions (1.13) and

$$x = 1: \quad v_{\bar p} = g_{\bar p}(t) \qquad (\bar p = 1, \cdots, m), \tag{4.10}$$

where $g_{\bar p}(t) \in C^1[0, T_1]$ $(\bar p = 1, \cdots, m)$ are any given functions of t with small C^1 norm, such that the conditions of C^1 compatibility are satisfied at the point (0,1). By Lemma 2.1, there exists a unique semi-global C^1 solution $u = u^{(1)}(t,x)$ with small C^1 norm on the domain

$$\{(t,x) | 0 \le t \le T_1, 0 \le x \le 1\}. \tag{4.11}$$

Thus, we can determine the value of $u = u^{(1)}(t,x)$ on $x = 0$ as

$$x = 0: \quad u = a(t), \quad 0 \le t \le T_1 \tag{4.12}$$

and the $C^1[0, T_1]$ norm of $a(t)$ is suitably small.

Similarly, we consider the backward mixed initial-boundary value problem for system (3.3) (with (4.6)) with the initial condition (3.2) and the boundary conditions (4.2) and

$$x = 0: \quad v_q = \tilde g_q(t) \qquad (q = l+1, \cdots, m), \tag{4.13}$$

$$x = 1: \quad v_r = g_r(t) \qquad (r = m+1, \cdots, n), \tag{4.14}$$

where $\tilde g_q(t)$, $g_r(t) \in C^1[T - T_1, T]$ $(q = l+1, \cdots, m;\ r = m+1, \cdots, n)$ are any given functions of t with small C^1 norm, such that the conditions of C^1 compatibility are satisfied at the points $(T,0)$ and $(T,1)$ respectively. Once again by Lemma 2.1, there exists a unique semi-global C^1 solution $u = u^{(2)}(t,x)$ with small C^1 norm on the domain

$$\{(t,x) | T - T_1 \le t \le T, 0 \le x \le 1\}. \tag{4.15}$$

Thus, we can determine the value of $u = u^{(2)}(t,x)$ on $x = 0$ as

$$x = 0: \quad u = b(t), \quad T - T_1 \le t \le T \tag{4.16}$$

and the $C^1[T - T_1, T]$ norm of $b(t)$ is suitably small.

Noting that both $a(t)$ and $b(t)$ satisfy the boundary condition (1.13), we can find a function $c(t) \in C^1[0,T]$ with small C^1 norm, such that

$$c(t) = \begin{cases} a(t), & 0 \le t \le T_1, \\ b(t), & T - T_1 \le t \le T \end{cases} \tag{4.17}$$

and $c(t)$ satisfies the boundary condition (1.13) on the whole interval [0,T].

Now, we change the order of t and x, then system (3.3) (with (4.6)) is rewritten in the following form

$$\begin{cases} l_p(u)\Big(\dfrac{\partial u}{\partial x}+\dfrac{1}{\lambda_p(u)}\dfrac{\partial u}{\partial t}\Big)=\widetilde{\widetilde{F}}_p(u)\triangleq\dfrac{\widetilde{F}_p(u)}{\lambda_p(u)} & (p=1,\cdots,l),\\ l_q(u)\Big(\dfrac{\partial u}{\partial x}+\dfrac{1}{\bar{\lambda}_q(u)}\dfrac{\partial u}{\partial t}\Big)=\widetilde{\widetilde{F}}_q(u)\triangleq\dfrac{\widetilde{F}_q(u)}{\bar{\lambda}_q(u)} & (q=l+1,\cdots,m),\\ l_r(u)\Big(\dfrac{\partial u}{\partial x}+\dfrac{1}{\lambda_r(u)}\dfrac{\partial u}{\partial t}\Big)=\widetilde{\widetilde{F}}_r(u)\triangleq\dfrac{\widetilde{F}_r(u)}{\lambda_r(u)} & (r=m+1,\cdots,n).\end{cases} \tag{4.18}$$

We still have

$$\widetilde{\widetilde{F}}(0)=0 \tag{4.19}$$

and the corresponding eigenvalues satisfy

$$\frac{1}{\lambda_p(u)}<0<\frac{1}{\lambda_r(u)} \qquad (p=1,\cdots,l;\ r=m+1,\cdots,n) \tag{4.20}$$

and

$$\frac{1}{\bar{\lambda}_{l+1}(u)}=\cdots=\frac{1}{\bar{\lambda}_m(u)}=\frac{1}{\lambda_1(u)}<0. \tag{4.21}$$

Moreover, we can still define v_i $(i=1,\cdots,n)$ by the same formula (1.15).

We now consider the mixed initial-boundary value problem for system (4.18) with the initial condition

$$x=0:\quad u=c(t),\quad 0\le t\le T \tag{4.22}$$

and the boundary conditions

$$t=0:\quad v_r=\Phi_r(x) \qquad (r=m+1,\cdots,n),\quad 0\le x\le 1, \tag{4.23}$$

$$t=T:\quad v_{\bar{p}}=\Psi_{\bar{p}}(x) \qquad (\bar{p}=1,\cdots,m),\quad 0\le x\le 1, \tag{4.24}$$

where

$$\Phi_i(x)=l_i(\varphi(x))\varphi(x) \qquad (i=1,\cdots,n), \tag{4.25}$$

$$\Psi_i(x)=l_i(\psi(x))\psi(x) \qquad (i=1,\cdots,n), \tag{4.26}$$

the C^1 norms of which are suitably small. It is easy to see that the conditions of C^1 compatibility are satisfied at the points $(0,0)$ and $(T,0)$ respectively. Therefore, by Lemma 2.1, there exist a unique semi-global C^1 solution $u=u(t,x)$ with small C^1 norm on the domain

$$R(T)=\{(t,x)|0\le t\le T,0\le x\le 1\}. \tag{4.27}$$

In order to finish the proof of Lemma 4.1, it is only necessary to check that

$$t=0:\quad u=\varphi(x),\quad 0\le x\le 1, \tag{4.28}$$

$$t=T:\quad u=\psi(x),\quad 0\le x\le 1. \tag{4.29}$$

In fact, the C^1 solutions $u=u(t,x)$ and $u=u^{(1)}(t,x)$ satisfy simultaneously the system (4.18) (namely (3.3)) with the initial condition

$$x=0:\quad u=c(t),\quad 0\le t\le T_1 \tag{4.30}$$

and the boundary condition (4.23). Because of the uniqueness of C^1 solution (see [8]) and the choice of T_1 given by (4.9), on the domain

$$\{(t,x)|0\le t\le T_1(1-x),\ 0\le x\le 1\} \tag{4.31}$$

we have

$$u(t,x) \equiv u^{(1)}(t,x). \tag{4.32}$$

In particular, we get (4.28). (4.29) can be obtained in a similar way. Thus $u = u(t,x)$ is a C^1 solution desired by Lemma 4.1. The proof of Lemma 4.1 is complete.

§5. Remarks

Remark 5.1. When A and F in system (1.1) depend also on x, Lemma 2.1 (see [7]) and Theorems 3.1 and 4.1 are also valid.

Remark 5.2. The exact controllability time given in Theorem 3.1 or Theorem 4.1 is optimal.

Remark 5.3. In Theorem 3.1, the number of internal controls is equal to the number of zero eigenvalues, while the number of boundary controls is equal to the sum of the numbers of positive and negative eigenvalues.

Remark 5.4. In Theorem 4.1, the number of internal controls is still equal to the number of zero eigenvalues, while the number of boundary controls is equal to half of the sum of the numbers of positive and negative eigenvalues.

Remark 5.5. The boundary controls and internal controls given in Theorem 3.1 or Theorem 4.1 are not unique.

References

[1] Li Tatsien & Rao Bopeng, Exact boundary controllability for quasilinear hyperbolic systems, *SIAM J. Control Optim.,* **41**(2003), 1748–1755.

[2] Li Tatsien & Rao Bopeng, Local exact boundary controllability for a class of quasilinear hyperbolic systems, *Chin. Ann. Math.,* **23B:**2(2002), 209–219.

[3] Li Tatsien, Rao Bopeng & Jin Yi, Solution C^1 semi-globale et contrôlabilité exacte frontière de systèmes hyperboliques quasi linéaires, *C. R. Acad. Sci. Paris,* **333:**Série I(2001), 219–224.

[4] Cirinà, M., Boundary controllability of nonlinear hyperbolic systems, *SIAM J. Control Optim.,* **7**(1969), 198–212.

[5] Li Tatsien, Yu Wenci & Shen Weixi, Second initial-boundary value problems for quasi-linear hyperbolic-parabolic coupled systems (in Chinese), *Chin. Ann. Math.,* **2:**1(1981), 65–91.

[6] Li Tatsien & Jin Yi, Semi-global C^1 solutions to mixed initial-boundary value problem for quasilinear hyperbolic systems, *Chin. Ann. Math.,* **22B:**3(2001), 325–336.

[7] Yu Lixin, Semi-global C^1 solutions for a general class of quasilinear hyperbolic systems, to appear.

[8] Li Tatsien & Yu Wenci, Boundary value problems for quasilinear hyperbolic systems, Duke University Mathematics Series V, 1985.