Chin. Ann. Math.
24B:4(2003),423–432.

TRANSITION FROM A DEFLAGRATION TO A DETONATION IN GAS DYNAMIC COMBUSTION****

CHEN Shaozhong* LI Jiequan** ZHANG Tong***

Abstract

The transition from a deflagration to a detonation (DDT) in gas dynamics is investigated through the process of a deflagration with a finite width flame overtaken by a shock. The problem is formulated as a free boundary value problem in an angular domain with a strong detonation and a reflected shock as boundaries. The main difficulty lies in the fact that the strength of reflected shock is zero at the vertex where the shock speed degenerates to be the same as the characteristic speed. The conclusion is that a strong detonation and a retonation (a reflected shock) form locally. Also the entropy satisfaction of this solution is presented.

Keywords Gas dynamic combustion, Deflagration-to-detonation transition, Retonation
2000 MR Subject Classification 35R35, 35L60, 35L65, 35L67, 76N10
Chinese Library Classification O175.27, O175.29, O175.8, O383
Document Code A **Article ID** 0252-9599(2003)04-0423-10

§1. Introduction

About 1880 some French physicists found the phenomenon of transition from a deflagration to a detonation in a shock tube filled with combustible gases ignited near a closed end[1], which is called DDT for short. This phenomenon has been one of the core problems in gas dynamic combustion and studied extensively with physically experimental and numerical tools. This complex physical process involves deflagrations, shocks and shock reflections, boundary layers, and all of their interactions with each other (see [2] and references therein). Unfortunately very few is demonstrated rigorously. The earliest theoretical interpretation was credited to Chapman (1899) and Jouguet (1905), who proposed two kinds of combustion waves, detonations and deflagrations, under the assumption that the reaction rate is infinite. Furthermore, each of them was classified to be three cases, according to the Jouguet rule[1].

With this theory, Zhang and Zheng solved the Riemann problem for compressible Euler equations of combustible gases in [3], in which a mathematical explanation for the transition from a deflagration to a detonation is given, when the deflagration is overtaken by a

Manuscript received April 18, 2003.
*Department of Mathematics, Capital Normal University, Beijing 100037, China.
Department of Mathematics, Ningbo University, Ningbo 315211, Zhejiang, China.
E-mail: cszgdj@mail.nbptt.zj.cn
**Department of Mathematics, Ningbo University, Ningbo 315211, Zhejiang, China.
E-mail: jiequan@mail.cnu.edu.cn
***Institute of Mathematics, Academy of Mathematics and System Sciences, the Chinese Academy of Sciences, Beijing 100080, China. **E-mail:** tzhang@math.ac.cn
****Project supported by the Program of Key Laboratory of Military Defenses (No.00JS75.1.1.QT1901).

shock, after noticing the deflagration is rarefactive while the detonation is compressive. The deflagration may transit to a detonation if the shock is strong enough. Just as mentioned in [1], the Chapman-Jouguet assumption is too idealized for a deflagration. Therefore in [1] a deflagration front is expanded to be a zone of finite width, and an explicit formula for the states inside the reaction zone is presented.

Taking into account this formula, we extend the result in [3] to the case of the deflagration with a finite width. This problem is then reduced to a free boundary value problem in an angular region for the compressible Euler equations. The boundaries comprise of a forward detonation and a backward retonation (shock). The main purpose in the present paper is to establish the local existence of solutions in the angular region.

In [4], the local existence results are available for the boundary value problems for quite general quasilinear hyperbolic systems (Theorem 6.1 in Chapter 2). However some degenerate cases are exceptional, and are just what we encountered in the present study. That is, the strength of retonation is zero at the vertex of the angular domain so that the Lax entropy condition cannot be validated on the boundaries in the theorem. Our approach is to introduce an auxiliary boundary (2.14) so that critical Lax entropy condition is satisfied and prove that the conclusion still makes sense. At this moment we can only prove the modified boundary is of $C^{1,1}$, instead of C^2 that is necessary in [4]. Therefore, we have to modify the assumption of associated theorems and prove the local existence of solutions to our problem with the auxiliary boundaries. Then we prove that the entropy condition is satisfied a posteriori. Thus we complete the proof of theorem.

We organize this paper in 6 sections. Section 2 formulates the DDT problem. In Section 3, we use the partial hodograph transform technique to fix the free boundaries. The crucial element is that the specific volume can be decoupled from the governing equations. In Section 4, we prove two inequalities, which are important in the estimate on the minimal number of characterizing matrix. This is necessary in the proof of local existence of main theorem in Section 5. The entropy satisfaction of this solution is shown in Section 6.

§2. Formulation of Deflagration Detonation Transition (DDT) Problem

In this section we formulate the deflagration detonation transition (DDT) problem mathematically. Let state variables τ, p and u be the specific volume, pressure and velocity of a fluid respectively, $q>0$ is the specific binding energy. Consider a strip of deflagration $\overrightarrow{DF}$ separating an unburnt gas (τ_4, p_4, u_4, q_4) from the burnt gas $(\tau_3, p_3, u_3, 0)$ and travelling forward with a speed λ_3, and denote by $\vec{b}_2=(\tau_2, p_2, u_2, q_2)$ the state inside the strip. Then, as in [1], the state $\vec{b}_2=(\tau_2, p_2, u_2, q_2)$ should satisfy

$$\begin{cases} p_2(x,t)=p_3+\lambda_3(u_2(x,t)-u_3), \\ \tau_2(x,t)=\tau_3-(u_2(x,t)-u_3)/\lambda_3, \\ q_2(x,t)=((p_3+\mu^2p_2)(\tau_3-\mu^2\tau_2)-(1-\mu^4)p_2\tau_2)/(2\mu^2), \\ u_2(x,t)=u_2(x-\lambda_3 t), \end{cases} \tag{2.1}$$

where $\lambda_3=\sqrt{\gamma p_3/\tau_3}$, $\mu^2=\frac{\gamma-1}{\gamma+1}$, and $\gamma>$ is the adiabatic exponent. This deflagration $\overrightarrow{DF}$ is overtaken, at time $t=0$, by a shock $\overrightarrow{S}$ separating states $(\tau_3, p_3, u_3, 0)$ and $(\tau_1, p_1, u_1, 0)$, and then a transition process occurs simultaneously. Experiments show that the deflagration $\overrightarrow{DF}$ transits to be a strong detonation $\overrightarrow{DT}: x=g_2(t)$, and in the meantime a reflected shock $x=g_1(t)$ forms from the interaction point $x=0$ of $\overrightarrow{DF}$ and $\overrightarrow{S}$, $g_1(0)=g_2(0)=0$. The reflected shock is called a retonation. This is the famous DDT problem.

Thus the combustible gas is burnt in the angular domain

$$R(T_0) = \{(x,t);\ g_1(t) \le x \le g_2(t),\ 0 \le t \le T_0\} \tag{2.2}$$

for some time $T_0 > 0$, and inside this domain, it is acceptable to assume that the fluid is inviscid and the flame is sharp. So the classical inviscid Euler equation

$$\begin{cases} \tau_t - u_x = 0, \\ u_t + p_x = 0, \\ \Big(\dfrac{1}{2}u^2 + e\Big)_t + (pu)_x = 0 \end{cases} \tag{2.3}$$

is applied to approximately governing the flow, where $e = \frac{p\tau}{\gamma-1}$.

As is well known, the retonation $x = g_1(t)$ and the detonation $\overrightarrow{DT} : x = g_2(t)$ are determined with the Rankine-Hugoniot conditions. Then on the $x = g_i(t)$, $i = 1, 2$, we have

$$\tau = K_i(p, \vec{b}_i) := \mu^2\tau_i + ((1-\mu^4)p_i\tau_i + 2\mu^2 q_i)/(p + \mu^2 p_i), \tag{2.4}$$

$$u = U_i(p, \vec{b}_i) := u_i + (-1)^i\sqrt{-(p-p_i)(K_i(p,\vec{b}_i) - \tau_i)}, \tag{2.5}$$

$$g_i'(t) = G_i(p, \vec{b}_i) := (-1)^i\sqrt{-(p-p_i)/(K_i(p,\vec{b}_i) - \tau_i)}, \tag{2.6}$$

where $\vec{b}_1 = (\tau_1, p_1, u_1, 0)$ is the state on the reflected shock front $x = g_1(t)$ and $\vec{b}_2(g_2(t), t) = (\tau_2, p_2, u_2, q_2)(g_2(t), t)$ is the state of the detonation front $x = g_2(t)$. Besides, the entropy condition should be satisfied

$$p \ge p_1. \tag{2.7}$$

We remark here that (τ_i, p_i, u_i, q_i) are related through the following equalities

$$\begin{aligned} (p_1 + \mu^2 p_3)(\tau_1 - \mu^2\tau_3) &= (1-\mu^4)p_3\tau_3, \\ (u_1 - u_3)^2 &= -(p_1 - p_3)(\tau_1 - \tau_3), \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} (p_3 + \mu^2 p_4)(\tau_3 - \mu^2\tau_4) &= (1-\mu^4)p_4\tau_4 + 2\mu^2 q_4, \\ (u_3 - u_4)^2 &= -(p_3 - p_4)(\tau_3 - \tau_4). \end{aligned} \tag{2.9}$$

Thus this DDT problem is formulated to be a free boundary value problem (2.3), (2.4)–(2.7). For this problem, we have the following theorem about the local existence of piecewise smooth solutions and $g_i(t)$, $i = 1, 2$.

Theorem 2.1. *Assume that* $u_2(s) \in C^1[0,\infty)$, *and*

$$u_2'(s) > 0, \qquad \forall s \in [0,\infty) \tag{2.10}$$

with $u_2(0) = u_3$ *and* $u_2(\infty) = u_4$. *Then there is a constant* $\delta > 0$ *such that the solution* (τ, p, u) *to* (2.3), (2.4)–(2.7) *exists piecewise smoothly in* $R(\delta)$ *if* $\mu^2 \in [1/35, 1/4]$. *Moreover, the retonation front is in* $C^{1,1}$.

We remark that the restriction $\mu^2 \in [1/35, 1/4]$ is technical, but this covers most cases of combustible gases. For air, $\mu^2 = 1/6$.

The crucial point of this theorem is that we need to prove the entropy condition (2.7) is satisfied, which is equivalent to the following geometric inequality

$$g_1'(t) \le -\lambda(p,\tau), \tag{2.11}$$

where $\lambda(p,\tau) = \sqrt{\gamma p/\tau}$. However, at the vertex $(x,t) = (0,0)$ of the angular region $R(T_0)$, only is the critical case satisfied,

$$g_1'(0) = -\lambda(p,\tau)|_{(x,t)=(0,0)}, \tag{2.12}$$

which is just the exceptional degenerate case in [4], and causes difficulties in the proof. In [4],

$$g_1'(0) < -\lambda(0,0) \tag{2.13}$$

is required. To overcome this difficulty, we introduce a cutoff function

$$g_1'(t) = G_1(p, \vec{b}_1) = \begin{cases} -\sqrt{-\dfrac{p-p_1}{\tau-\tau_1}} & \text{for } p > p_1, \\ -\sqrt{\dfrac{\gamma p}{\tau}} & \text{for } p \le p_1 \end{cases} \tag{2.14}$$

to replace that in (2.6). Then the critical entropy condition is always satisfied. But this yields another difficulty again. Here, $G_1(p, \vec{b}_1)$ is only Lipschitzian continuity in terms of p, and does not satisfy the requirement in [4]. With the almost same strategy as in [4], we can prove that C^1 solution still exists in $R(T_0)$ for a $T_0 > 0$ with the modified boundary $g_1(t)$ in (2.14), which is only in $C^{1,1}$. This is completed in Sections 3–5 through transforming the above problem into a problem with fixed boundaries and checking that all hypotheses are satisfied corresponding to those in [4, Theorem 6.1, Chapter 2]. In Section 6 we will check that the entropy condition (2.7) is satisfied. This shows Theorem 2.1 a posterior.

§3. Reformulation of the Problem (2.3), (2.4)–(2.6)

The problem (2.3), (2.4)–(2.6) is a free boundary problem. To prove Theorem 2.1, we need to fix the free boundaries $x = g_i(t)$, $i = 1, 2$, for which the partial hodograph transform is used, and then an equivalent problem with fixed boundaries is reformulated.

We divide the angular domain $R(T_0)$ into two parts:

$$R_i(T_0) = \{(x,t) \in R(T_0); (-1)^i x \ge 0\}, \qquad i = 1, 2, \tag{3.1}$$

and write the Euler equations in a characteristic form

$$\begin{cases} -\lambda(u_t - \lambda u_x) + (p_t - \lambda p_x) = 0, \\ \lambda^2 \tau_t + p_t = 0, \\ \lambda(u_t + \lambda u_x) + (p_t + \lambda p_x) = 0, \end{cases} \tag{3.2}$$

where $\lambda = \sqrt{\gamma p/\tau}$. In $R_i(T_0)$ the entropy is constant along a streamline, $s(x) := p(x,t) \cdot \tau^\gamma(x,t)$. Eliminating τ from the boundary condition (2.4) yields

$$\tau(x,t) = (p(x, g_i^{-1}(x))/p(x,t))^{1/\gamma} K_i(p(x, g_i^{-1}(x)), \vec{b}(x, g_i^{-1}(x))). \tag{3.3}$$

Denote $a_i(t) = (-1)^i t/g_i(t)$, $i = 1, 2$, and define a partial hodograph transform $F_i : (x,t) \to (\xi, \eta)$,

$$(\xi, \eta) = F_i(x,t) := (a_i(t)x, t). \tag{3.4}$$

Then the domain $R_i(T_0)$ is transformed into

$$\overline{R}_i(T_0) := \{(\xi,\eta);\ \eta \in [0, T_0],\ \xi \in [-\eta, \eta],\ (-1)^i \xi \ge 0\}, \tag{3.5}$$

and (3.2) is transformed into the following form

$$\begin{cases} -\overline{\lambda}_i\Big(\dfrac{\partial \bar{u}_i}{\partial \eta} + \overline{\lambda}_{i1}\dfrac{\partial \bar{u}_i}{\partial \xi}\Big) + \Big(\dfrac{\partial \overline{p}_i}{\partial \eta} + \overline{\lambda}_{i1}\dfrac{\partial \overline{p}_i}{\partial \xi}\Big) = 0, \\ \overline{\lambda}_i\Big(\dfrac{\partial \bar{u}_i}{\partial \eta} + \overline{\lambda}_{i2}\dfrac{\partial \bar{u}_i}{\partial \xi}\Big) + \Big(\dfrac{\partial \overline{p}_i}{\partial \eta} + \overline{\lambda}_{i2}\dfrac{\partial \overline{p}_i}{\partial \xi}\Big) = 0 \end{cases} \tag{3.6}$$

with the boundaries $\xi = \pm\eta$ and $\xi = 0$. The boundary values are

$$\begin{cases} -\lambda_1 \bar{u}_2(\eta,\eta) + \overline{p}_2(\eta,\eta) = -\lambda_1 U_2(\overline{p}_2(\eta,\eta), \vec{b}_2(g_2(\eta),\eta)) + \overline{p}_2(\eta,\eta), \\ \lambda_1 \bar{u}_2(0,\eta) + \overline{p}_2(0,\eta) = \lambda_1 \bar{u}_1(0,\eta) + \overline{p}_1(0,\eta), \end{cases} \tag{3.7}$$

and

$$\begin{cases} -\lambda_1\bar{u}_1(0,\eta)+\overline{p}_1(0,\eta)=-\lambda_1\bar{u}_2(0,\eta)+\overline{p}_2(0,\eta), \\ \lambda_1\bar{u}_1(-\eta,\eta)+\overline{p}_1(-\eta,\eta)=\lambda_1 U_1(\overline{p}_1(-\eta,\eta),\vec{b}_1)+\overline{p}_1(-\eta,\eta), \end{cases} \tag{3.8}$$

where $\lambda_1=\sqrt{\gamma p_1/\tau_1}$, and $\bar{u}_i$, $\overline{p}_i$, $\overline{\tau}_i$ and $\overline{\lambda}_i$ are defined as

$$(\bar{u}_i,\overline{p}_i,\overline{\tau}_i,\overline{\lambda}_i):=(u,p,\tau,\lambda)\circ F_i^{-1}, \tag{3.9}$$

$$\overline{\lambda}_{il}:=a_i'(\eta)\xi/a_i(\eta)+(-1)^l\overline{\lambda}_i(\xi,\eta)a_i(\eta), \qquad i,l=1,2. \tag{3.10}$$

The function $g_i(\eta)$ is

$$g_i(\eta)=\int_0^\eta G_i(\overline{p}_i((-1)^i\eta,\eta),\vec{b}_i(g_i(\eta),\eta))d\eta. \tag{3.11}$$

Then Theorem 2.1 with the modified boundary (2.14) (except the entropy satisfaction) is stated in the following.

Theorem 3.1. *Under the same assumption as in Theorem* 2.1, *there is a constant* $\delta>0$ *such that the solution to the problem* (3.6)–(3.8) *exists smoothly in* $\overline{R}_i(\delta)$, $i=1,2$.

§4. Two Inequalities

To prove Theorem 3.1, we need two inequalities to provide the properties of DDT near the origin. The first is used in the estimate on the minimal number of characterizing matrix of our problem (3.6) and (3.8). The second is used in the proof of entropy satisfaction.

Lemma 4.1. *If* $\mu^2\in[1/35,1/4]$, *then for* $p>p_2$, $u_2=0$, $q_2=0$, *we have*

$$\frac{1}{2}\Big|-\lambda\frac{\partial U_2(p,\vec{b}_2)}{\partial p}+1\Big|<1, \tag{4.1}$$

where $\lambda=\sqrt{\gamma p/K_2(p,\vec{b}_2)}$.

Proof. Without loss of generality, we assume $p_2=1$, $\tau_2=1$. Let

$$f(p)=\Big(\lambda\frac{\partial U_2(p,\vec{b}_2)}{\partial p}\Big)^2. \tag{4.2}$$

It is easily checked that

$$\begin{aligned} f(p)&=\frac{1}{4}(1+\mu^2)\cdot\frac{p}{1+\mu^2p}\Big(1+\frac{1+\mu^2}{p+\mu^2}\Big)^2, \\ f'(p)&=(p-1)((1-2\mu^2-2\mu^4)p-(\mu^2+2\mu^4))d, \end{aligned} \tag{4.3}$$

where

$$d=\frac{1}{4}(1+\mu^2)\Big(1+\frac{1+\mu^2}{p+\mu^2}\Big)\frac{1}{(1+\mu^2p)^2(p+\mu^2)^2}>0. \tag{4.4}$$

With $\mu^2\in[1/35,1/4]$, we have $0<\mu^2+2\mu^4<1-2\mu^2-2\mu^4$. It follows that $f'(p)>0$ for $p>1$. Since $f(1)=1$ and $f(\infty)=(1+\mu^2)/(4\mu^2)\le 9$, (4.1) is proved.

Since p_2, τ_2 and q_2 are functions of u_2 as in (2.1), we use the notation

$$U_2(p,u_2):=U_2(p,\vec{b}_2). \tag{4.5}$$

Then we have the following lemma.

Lemma 4.2. *If* $\mu^2\le 1/4$ *and* $p=p_1$, $u_2=u_3=0$, *then*

$$\frac{\partial U_2(p,u_2)}{\partial u_2}<0. \tag{4.6}$$

Proof. From the Rankine-Hugoniot condition (2.4)–(2.6), we have

$$(p+\mu^2p_2)(K_2(p,\vec{b}_2)-\mu^2\tau_2)=(p_3+\mu^2p_2)(\tau_3-\mu^2\tau_2), \tag{4.7}$$

which is differentiated with respect to u_2 to get

$$\frac{\partial K_2}{\partial u_2}=\mu^2\Big(\frac{\tau_3-K_2(p,\vec{b}_2)}{p+\mu^2p_2}\lambda_3+\frac{p_3-p}{p+\mu^2p_2}\cdot\frac{1}{\lambda_3}\Big), \tag{4.8}$$

$$\frac{\partial U_2}{\partial u_2}=1+\frac{1}{2}\Big[-\lambda_3\sqrt{\frac{\tau_2-K_2(p,\vec{b}_2)}{p-p_2}}-\sqrt{\frac{p-p_2}{\tau_2-K_2(p,\vec{b}_2)}}\Big(\frac{\partial K_2}{\partial u_2}+\frac{1}{\lambda_3}\Big)\Big]. \tag{4.9}$$

When $p=p_1$ and $u_2=u_3=0$, we have $p_2=p_3$, $\tau_2=\tau_3$ and $K_2(p,\vec{b}_2)=\tau_1$. Substituting (4.8) into (4.9) yields

$$\frac{\partial U_2}{\partial u_2}=\frac{1}{2}(\xi_{31}/\lambda_3-1)\Big(\mu^2(1+\lambda_3/\xi_{31})\Big(1-\frac{p_3+\mu^2p_3}{p_1+\mu^2p_3}\Big)-(1-\lambda_3/\xi_{31})\Big), \tag{4.10}$$

where $\xi_{31}=\sqrt{-(p_1-p_3)/(\tau_1-\tau_3)}$. Note $\lambda_3<\xi_{31}$, $p_3<p_1$ and $4\mu^2\le 1$ as well as $\lambda_3/\xi_{31}=\sqrt{(p_3+\mu^2p_3)/(p_1+\mu^2p_3)}$. Then this lemma follows immediately.

§5. The Proof of Main Theorems

In this section, we will prove Theorem 3.1, and then Theorem 2.1. First, some facts are listed, which are necessary for the proof of Theorem 3.1. We introduce some notations as follows:

$$\|\overline{p}_i\|_\delta:=\sup\{|\overline{p}_i(\xi,\eta)|:\ (\xi,\eta)\in\overline{R}_i(T_0),\ \eta\le\delta\}, \tag{5.1}$$

$$\|\overline{p}\|_{1,\delta}:=\sup\Big\{\Big|\frac{\partial\overline{p}_i}{\partial\eta}\pm\frac{\partial\overline{p}_i}{\partial\xi}\Big|:\ (\xi,\eta)\in\overline{R}_i(T_0),\eta\le\delta\Big\}, \tag{5.2}$$

$$P_i(\epsilon,T_0):=C^1(\overline{R}_i(T_0))\cap\{\overline{p}_i(0,0)=p_1,\|\overline{p}_i-p_1\|\le\varepsilon\}. \tag{5.3}$$

Let $f(\xi,\eta)$ be a function in $\overline{R}_i(T_0)$. Define two kinds of continuity moduli,

$$\omega(\epsilon|f):=\sup\{|f(\xi_1,\eta_1)-f(\xi_2,\eta_2)|:(\xi_i,\eta_i)\in\overline{R}_i(T_0),i=1,2,\ |\xi_1-\xi_2|^2+|\eta_1-\eta_2|^2\le\epsilon^2\}, \tag{5.4}$$

$$\omega_\xi(\epsilon|f):=\sup\{|f(\xi_1,\eta)-f(\xi_2,\eta)|:(\xi_i,\eta)\in\overline{R}_i(T_0),i=1,2,|\xi_1-\xi_2|\le\epsilon\}, \tag{5.5}$$

$$\omega(\epsilon|\dot{f})=\sup\Big\{\omega\Big(\epsilon\Big|\frac{\partial f}{\partial\eta}+\frac{\partial f}{\partial\xi}\Big),\omega\Big(\epsilon\Big|\frac{\partial f}{\partial\eta}-\frac{\partial f}{\partial\xi}\Big)\Big\}. \tag{5.6}$$

Lemma 5.1. *From the boundary conditions* (3.7) *and* (3.8), *we find at* $(\xi,\eta)=(0,0)$,

$$\bar{u}_1(0,0)=\bar{u}_2(0,0)=u_1,\qquad \overline{p}_1(0,0)=\overline{p}_2(0,0)=p_1. \tag{5.7}$$

Since the boundaries $g_i(\eta)$ are determined together with the solutions of (3.6), they are free boundaries. Write in a clear form about the dependent relation,

$$g_i(\eta)=\int_0^\eta G_i(\overline{p}_i((-1)^i\eta,\eta),\vec{b}_2(g_i(\eta),\eta))d\eta,\qquad \eta\in[0,T_0]. \tag{5.8}$$

Then we have

Lemma 5.2. *There are* $\epsilon_0>0$ *and* $T_0>0$ *such that for all* $0<\epsilon<\epsilon_0$ *and* $\overline{p}_i\in P_i(\epsilon,T_0)$, *the integral equation* (5.8) *has a unique solution denoted by* $g_i(\eta|\overline{p}_i)$, *for* $\eta\in[0,T_0]$. *Moreover,* $g_i(\eta|\overline{p}_i)$ *has properties:*

(i) *There exists* $K>0$ *such that* $K^{-1}\le|g_i'(\eta|\overline{p}_i)|\le K$ *for* $\eta\in[0,T_0]$ *and* $\overline{p}_i\in P_i(\epsilon,T_0)$.

(ii) *There exists* $K>0$ *such that*

$$\|g_i(\eta|\overline{p}_{i1})-g_i(\eta|\overline{p}_{i2})\|_\delta\le K\delta\|\overline{p}_{i1}-\overline{p}_{i2}\|_\delta \tag{5.9}$$

for all $\overline{p}_{i1},\overline{p}_{i2}\in P_i(\epsilon,T_0)$, $\delta\in(0,T]$, $\eta\le\delta$.

Remark 5.1. Since $G_1(p,\vec{b}_1)$ is only a Lipschitzian continuous function of p but not in C^1, $g_1(\eta)$ is in $C^{1,1}$ but may not be a C^2 function. This is inconsistent with that in [4], which results in many difficulties so that we cannot use theorems there in a straightforward way.

By (3.9) and (3.10), we can define the functional $\overline{\lambda}_i(\cdot,\cdot|\overline{p}_i)$ and $\overline{\lambda}_{il}(\cdot,\cdot|\overline{p}_i)$ for $\overline{p}_i \in P_i$.

Lemma 5.3. *There exist $T_0>0$ and $K>0$ such that*

(i) *for all $\delta\in(0,T_0]$, $\overline{p}_{i1},\overline{p}_{i2}\in P_i(\epsilon,T_0)$, $i=1,2$, if $\|\overline{p}_{i1}-\overline{p}_{i2}\|_\delta=0$, we have*

$$\|\overline{\lambda}_i(\cdot,\cdot|\overline{p}_{i1})-\overline{\lambda}_i(\cdot,\cdot|\overline{p}_{i2})\|_\delta=0. \tag{5.10}$$

(ii) *For all $\delta\in(0,T_0]$, $\overline{p}_{i1},\overline{p}_{i2}\in P_i(\epsilon,T_0)$, $i=1,2$,*

$$\Big\|\frac{\partial\overline{\lambda}_i(\cdot,\cdot|\overline{p}_i)}{\partial\xi}\Big\|_\delta\le K(1+\|\overline{p}_i\|_{1,\delta})(1+\delta\|\overline{p}_i\|_{1,\delta}), \tag{5.11}$$

$$\Big\|\frac{\partial\overline{\lambda}_i(\cdot,\cdot|\overline{p}_i)}{\partial\eta}\Big\|_\delta\le K(1+\|\overline{p}_i\|_{1,\delta})(1+\delta\|\overline{p}_i\|_{1,\delta}), \tag{5.12}$$

$$\|\overline{\lambda}_i(\cdot,\cdot|\overline{p}_i)-\overline{\lambda}_i(0,0|\overline{p}_i)\|_\delta\le K(1+\|\overline{p}_i\|_\delta)(1+\delta\|\overline{p}_i\|_{1,\delta})\delta, \tag{5.13}$$

$$\|\overline{\lambda}_i(\cdot,\cdot|\overline{p}_{i1})-\overline{\lambda}_i(\cdot,\cdot|\overline{p}_{i2})\|_\delta\le K(1+\delta\|\overline{p}_{i1}\|_{1,\delta}+\delta\|\overline{p}_{i2}\|_{1,\delta})\|\overline{p}_{i1}-\overline{p}_{i2}\|_\delta. \tag{5.14}$$

Lemma 5.4. *For the functional $\overline{\lambda}_{il}(\cdot,\cdot|\overline{p}_i)$, $\overline{p}_i\in P_i(\epsilon,T_0)$, $i,l=1,2$, defined by* (3.10), *the analogues of* (5.10), (5.11), (5.13) *and* (5.14) *are satisfied. Furthermore, there is $T_0>0$, such that*

(iii)

$$\overline{\lambda}_{i1}(\xi,\eta|\overline{p}_i)<\overline{\lambda}_{i2}(\xi,\eta|\overline{p}_i),\qquad \forall\overline{p}_i\in P_i,\ (\xi,\eta)\in\overline{R}_i(T_0),\ i=1,2; \tag{5.15}$$

$$\overline{\lambda}_{11}(-\eta,\eta|\overline{p}_1)\le-1,\qquad \forall\overline{p}_1\in P_1,\ \eta\in[0,T_0], \tag{5.16}$$

$$\overline{\lambda}_{12}(0,0|\overline{p}_1)>0,\quad \overline{\lambda}_{21}(0,0|\overline{p}_2)<0,\quad \overline{\lambda}_{22}(0,0|\overline{p}_2)>1,\qquad \forall\overline{p}_i\in P_i,\ i=1,2. \tag{5.17}$$

Lemma 5.5. *There is $K>0$ and an infinitesimal quantity $\rho(\epsilon)$ in terms of ϵ, so that*

$$\omega_\xi\Big(\epsilon\Big|\frac{\partial\overline{\lambda}_i}{\partial\xi}\Big)+\omega_\xi\Big(\epsilon\Big|\frac{\partial\overline{\lambda}_i}{\partial\eta}\Big)+\omega_\xi\Big(\epsilon\Big|\frac{\partial\overline{\lambda}_{il}}{\partial\xi}\Big)+\omega\Big(\epsilon\Big|\frac{d\overline{\lambda}_{il}((-1)^i\eta,\eta)}{d\eta}\Big)\le K\omega(\epsilon|\overline{p}_i)+\rho(\epsilon) \tag{5.18}$$

for all $\overline{p}_i\in P_i(\epsilon,T_0)$.

Lemma 5.6. *$\overline{\lambda}_{il}(\xi,\eta|\overline{p}_i)$ and $\overline{g}_i'(\eta|\overline{p}_i)$ is equi-continuous and uniformly continuous with respect to all $(\xi,\eta)\in\overline{R}_i(T_0)$ and $\overline{p}_i\in P_i(\epsilon,T_0)$.*

The proof of Lemmas 5.1–5.6 is not difficult. We omit the details.

With complicated calculations, we obtain that the characterizing matrix for the problem (3.6)–(3.8) is

$$H=\begin{pmatrix} \frac12\big(-\lambda_1\frac{\partial U_2}{\partial p}+1\big) & \frac12\big(-\lambda_1\frac{\partial U_2}{\partial p}+1\big) & 0 & 0\\ 0 & 0 & 0 & 1\\ 1 & 0 & 0 & 0\\ 0 & 0 & \frac12\big(\lambda_1\frac{\partial U_1}{\partial p}+1\big) & \frac12\big(\lambda_1\frac{\partial U_1}{\partial p}+1\big)\end{pmatrix}. \tag{5.19}$$

The definition of characterizing matrix is referred to [4]. Then we have the following lemma.

Lemma 5.7. *The minimal characterizing number of H is*

$$\|H\|_{\min}=\frac12\Big|-\lambda_1\frac{\partial U_2(p_1,\vec{b}_2(0,0))}{\partial p}+1\Big|<1, \tag{5.20}$$

where $\lambda_1=\sqrt{\frac{\gamma p_1}{\tau_1}}$.

Proof. Recall that $\vec{b}_1(0,0) = (\tau_1, p_1, u_1, 0)$. It is straightforward to get

$$\frac{\partial U_1(p_1, \vec{b}_1(0,0))}{\partial p} = -\frac{1}{\lambda_1}. \tag{5.21}$$

So at $(x,t) = (0,0)$ or equivalently $(\xi, \eta) = (0,0)$, we have

$$\frac{1}{2}\Big(\lambda_1 \frac{\partial U_1}{\partial p} + 1\Big) = 0. \tag{5.22}$$

With Lemma 4.1, we have

$$\frac{1}{2}\Big| - \lambda_1 \frac{\partial U_2(p_1, \vec{b}_2(0,0))}{\partial p} + 1\Big| < 1. \tag{5.23}$$

Thus (5.20) follows.

Remark 5.2. Although the condition $\overline{\lambda}_{11}(0,0) < -1$ is not satisfied, the critical entropy condition (5.16) is true along $\xi = -\eta$.

Proof of Theorem 3.1. With Lemmas 5.1–5.7, all hypotheses are satisfied corresponding to those in [4, Theorem 6.1, Chapter 2]. Thus Theorem 3.1 holds, following the almost same strategy as in [4].

Proof of Theorem 2.1. Theorem 3.1 corresponds to the part of Theorem 2.1 with modified boundary (2.14). Then using the entropy satisfaction of solution shown in the next section, we can show Theorem 2.1 holds.

§6. Entropy Satisfaction of Local Solution to Problem (2.3), (2.4)–(2.6)

An important fact in DDT experiments is that the pressure is increasing along the detonation and retonation fronts, that is, the entropy increases. In this section, we will prove it rigorously.

Theorem 6.1. *For the local solution to* (2.3), (2.4)–(2.6)*, there exists a time interval* $[0, T_0]$*, in which*

$$\frac{dp}{dt}(g_i(t), t) \geq 0 \quad \textit{and} \quad p(g_i(t), t) \geq p_1. \tag{6.1}$$

Proof. The proof of this theorem has three parts: One is for the detonation and others for retonation. For the time being, we use (u_i, p_i, τ_i), $i = 1, 2$, to denote the local solution in $R_i(\delta)$, and the directional derivative is denoted as

$$\frac{d}{d_\alpha t} := \frac{\partial}{\partial t} + \alpha \frac{\partial}{\partial x}. \tag{6.2}$$

Part 1. For $i = 1$, (2.5) can be written as

$$u_1(g_1(t), t) = U_1(p_1(g_1(t), t), \vec{b}_1), \tag{6.3}$$

which is differentiated along the retonation front at the origin, so we have

$$\frac{\partial u_1}{\partial t} + g_1'(0)\frac{\partial u_1}{\partial x} = \frac{\partial U_1}{\partial p}\Big(\frac{\partial p_1}{\partial t} + g_1'(0)\frac{\partial p_1}{\partial x}\Big). \tag{6.4}$$

For the time being, let $\lambda := \lambda_1$. Then $g_1'(0) = -\lambda_1 = -\lambda$ and $\frac{\partial U_1}{\partial p} = \frac{-1}{\lambda_1} = \frac{-1}{\lambda}$. Then, the above expression can be written as

$$\lambda \frac{du_1}{d_{-\lambda} t} + \frac{dp_1}{d_{-\lambda} t} = 0. \tag{6.5}$$

Using (6.5), we write from (3.2) to derive

$$\lambda\frac{du_1}{d_0t}+\frac{dp_1}{d_0t}=\frac{1}{2}\Big(\lambda\frac{du_1}{d_{-\lambda}t}+\frac{dp_1}{d_{-\lambda}t}\Big)+\frac{1}{2}\Big(\lambda\frac{du_1}{d_\lambda t}+\frac{dp_1}{d_\lambda t}\Big)=0, \tag{6.6}$$

$$\lambda\frac{du_2}{d_0t}+\frac{dp_2}{d_0t}=\lambda\frac{\partial u_2}{\partial t}+\frac{\partial p_2}{\partial t}=\lambda\frac{\partial u_1}{\partial t}+\frac{\partial p_1}{\partial t}=\lambda\frac{du_1}{d_0t}+\frac{dp_1}{d_0t}=0. \tag{6.7}$$

On the other hand, for $i=2$, by (4.5), (2.5) can be written as

$$u_2(g_2(t),t)=U_2(p_2(g_2(t),t),u_2(g_2(t)-\lambda_3t)). \tag{6.8}$$

Denote $\alpha:=g_2'(0)$. Differentiating (6.8) at $t=0$ yields

$$\frac{du_2}{d_\alpha t}=\frac{\partial U_2}{\partial p}\cdot\frac{dp_2}{d_\alpha t}+\frac{\partial U_2}{\partial u_2}u_2'(0)(\alpha-\lambda_3). \tag{6.9}$$

By using (6.7), (3.2) can be written as

$$\lambda\frac{du_2}{d_\alpha t}+\frac{dp_2}{d_\alpha t}=\frac{\lambda-\alpha}{\lambda}\Big(\lambda\frac{du_2}{d_0t}+\frac{dp_2}{d_0t}\Big)+\frac{\alpha}{\lambda}\Big(\lambda\frac{du_2}{d_\lambda t}+\frac{dp_2}{d_\lambda t}\Big)=0. \tag{6.10}$$

Note that $\frac{\partial U_2}{\partial p}>0$ and $\frac{\partial U_2}{\partial u_2}<0$ (this is Lemma 4.2) as well as

$$0<\lambda_3<\alpha=g_2'(0)=\sqrt{-(p_1-p_3)(\tau_1-\tau_3)}<\lambda_1. \tag{6.11}$$

It follows that at $t=0$,

$$\frac{dp_2(g_2(t),t)}{dt}\Big|_{t=0}=\frac{dp_2}{d_\alpha t}=\frac{-\frac{\partial U_2}{\partial u_2}u_2'(0)(\alpha-\lambda_3)}{\frac{1}{\lambda_1}+\frac{\partial U_2}{\partial p}}>0, \tag{6.12}$$

which implies that there exists a time interval $[0,T_0]$ and for $i=2$, (6.1) satisfies on it.

Part 2. We want to prove for all $t\in[0,\delta]$, $p_1(g_1(t),t)\ge p_1$. Assume on the contrary that there is $t_0\in(0,\delta]$ such that $p_1(g_1(t_0),t_0)<p_1$. Denote $p(t):=p_1(g_1(t),t)$, $t_1:=\sup\{t\in[0,t_0]:p(t)\ge p_1\}$. Then for $0\le t_1<t_0$, we use the mean value theorem to get that there is $t_2\in(t_1,t_0)$ such that

$$p'(t_2)=\frac{dp_1(g_1(t),t)}{dt}\Big|_{t=t_2}=\frac{p(t_1)-p(t_0)}{t_1-t_0}<0. \tag{6.13}$$

Now denote $\lambda:=\lambda_1(g_1(t_2),t_2)$. Since $p(t_2)<p_1$, we have

$$g_1'(t_2)=G_1(p_1(g_1(t_2),t_2),\vec{b}_1)=-\sqrt{\frac{\gamma p_1(g_1(t_2),t_2)}{\tau_1(g_1(t_2),t_2)}}=-\lambda_1(g_1(t_2),t_2)=-\lambda. \tag{6.14}$$

Differentiate (6.3) at $t=t_2$. Then we get

$$\frac{du_1}{d_{-\lambda}t}-\frac{\partial U_1}{\partial p}\cdot\frac{dp_1}{d_{-\lambda}t}=0, \tag{6.15}$$

where $\frac{\partial U_1}{\partial p}<0$. On the other hand, the first equation of (3.2) can be written as

$$-\lambda\frac{du_1}{d_{-\lambda}t}+\frac{dp_1}{d_{-\lambda}t}=0. \tag{6.16}$$

Notice that at $t=t_2$,

$$\begin{vmatrix}1 & -\frac{\partial U_1}{\partial p}\\ -\lambda & 1\end{vmatrix}=1-\lambda\frac{\partial U_1}{\partial p}>0. \tag{6.17}$$

It follows that

$$\frac{dp_1}{d_{-\lambda}t}=\frac{dp_1(g_1(t_2),t_2)}{dt}=0, \tag{6.18}$$

which is a contradiction.

Part 3. We are going to prove that there exists $T_0 > 0$ such that along $x = g_1(t)$, $\frac{dp(g_1(t),t)}{dt} \geq 0$ for all $t \in [0, T_0]$. At $(x,t) = (0,0)$, we have

$$-\lambda \frac{du_2}{d_0 t} + \frac{dp_2}{d_0 t} = \frac{\alpha}{\alpha+\lambda}\Big(- \lambda \frac{du_2}{d_{-\lambda} t} + \frac{dp_2}{d_{-\lambda} t}\Big) + \frac{\alpha}{\alpha+\lambda}\Big(- \lambda \frac{du_2}{d_\alpha t} + \frac{dp_2}{d_\alpha t}\Big), \tag{6.19}$$

where $\alpha = g_2'(0)$, $\lambda = \lambda_1$. It follows that

$$-\lambda \frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial t} = 0 + \frac{\alpha}{\alpha+\lambda}\Big(2 \frac{dp_2}{d_\alpha t}\Big) > 0. \tag{6.20}$$

With the continuity of u and p, there exists $T_0 > 0$ such that

$$-\lambda \frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial t} > 0 \tag{6.21}$$

for all $(x,t) \in R(T_0)$. Then for any $t_0 \in [0, T_0]$, differentiating (6.3) at $t = t_0$ yields

$$\frac{du_1}{d_\beta t} = \frac{\partial U_1}{\partial p} \cdot \frac{dp_1}{d_\beta t}, \tag{6.22}$$

where $\beta = g_1'(t_0)$.

Similarly at $(x,t) = (g_1(t_0), t_0) \in R(T_0)$, we have

$$-\lambda \frac{du_1}{d_\beta t} + \frac{dp_1}{d_\beta t} = \frac{-\beta}{\lambda}\Big(- \lambda \frac{du_1}{d_{-\lambda} t} + \frac{dp_1}{d_{-\lambda} t}\Big) + \frac{\lambda+\beta}{\lambda}\Big(- \lambda \frac{du_1}{d_0 t} + \frac{dp_1}{d_0 t}\Big), \tag{6.23}$$

where $\lambda = \lambda_1(g_1(t_0), t_0)$. It follows from (2.6) that

$$\beta = g_1'(t_0) = G_1(p_1(g_1(t_0), t_0), \vec{b}_1) \geq -\lambda_1(g_1(t_0), t_0) = -\lambda. \tag{6.24}$$

Notice that $\frac{\partial U_1}{\partial p} < 0$. Then we obtain

$$\frac{dp_1(g_1(t_0), t_0)}{dt} = \frac{dp_1}{d_\beta t} = \frac{\frac{\lambda+\beta}{\lambda}\big(- \lambda \frac{\partial u_1}{\partial t} + \frac{\partial p_1}{\partial t}\big)}{-\lambda \frac{\partial U_1}{\partial p} + 1} \geq 0. \tag{6.25}$$

References

[1] Courant, R. & Friedrichs, K., Supersonic flow and shock waves, Springer-Verlag, 1948.

[2] Oran, E. S. & Khokhlov, A. M., Deflagrations, hot spots, and the transition to detonation, Combustion science at the end of the millennium (Cambridge, 1997), *R. Soc. Lond. Philos. Trans. Ser. A Math. Phys. Eng. Sci.,* **357:**1764(1999), 3539–3551.

[3] Zhang, T. & Zheng, Y., Riemann problem for gas dynamic combustion, *J. Differential Equations,* **77**(1989), 203–228.

[4] Li, T. & Yu, W., Boundary value problems for quasilinear hyperbolic systems, Duke University, Mathematical Series V, 1985.