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A RANDOM FUNCTIONAL CENTRAL LIMIT THEOREM FOR PROCESSES OF PRODUCT SUMS OF LINEAR PROCESSES GENERATED BY MARTINGALE DIFFERENCES***

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Abstract

A random functional central limit theorem is obtained for processes of partial sums and product sums of linear processes generated by non-stationary martingale differences. It develops and improves some corresponding results on processes of partial sums of linear processes generated by strictly stationary martingale differences, which can be found in [5].

Keywords Random functional central limit theorem, Processes of product sums, Wiener measure, Linear processes, Martingale difference
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§1. Introduction

In this article, let $\{\varepsilon_t : t \in \mathbf{Z}\}$ be a sequence of r.v.'s defined on an identical probability space $(\Omega, \mathcal{F}, P)$. $\mathcal{F}_t = \sigma(\varepsilon_s : s \le t)$, $t \in \mathbf{Z}$. $\{a_j : j \in \mathbf{Z}\}$ is a sequence of constants, satisfying

$$\sum_{j=-\infty}^{\infty} |a_j| < \infty. \tag{1.1}$$

We say that $X_t = \sum\limits_{j=-\infty}^{\infty} a_j \varepsilon_{t-j}$, $t \in \mathbf{N}$ is a linear process generated by $\{\varepsilon_t : t \in \mathbf{Z}\}$. Linear processes are of special importance in time-series analysis (see [6]), as well as in many fields such as mathematical statistics, insurance and finance and so on (see [8, 3], etc.). Therefore many show great interests in limit properties of all kinds of linear processes, for example, Fakhre-Zakeri and Farshidi[4] got the random central limit theorem for linear processes generated by i.i.d. r.v.'s, Fakhre-Zakeri and Lee[5] achieved the random functional central limit theorem for strictly stationary linear processes generated by martingale differences. Recently Kim and Baek[7] obtained the functional central limit theorem for linear processes generated by strictly stationary LPQD r.v.'s, etc.

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In above studies, Fakhre-Zakeri and Lee[5] required that $\{\varepsilon_t : t \in \mathbf{Z}\}$ should be a sequence of strictly stationary martingale differences and satisfy the condition (1.1) and

$$E(\varepsilon_t|\mathcal{F}_{t-1}) = 0, \quad E({\varepsilon_t}^2|\mathcal{F}_{t-1}) = \sigma^2, \quad \text{a.s.} \tag{1.2}$$

This article has two objects. First substitute the strictly stationary condition with the condition that there exists an r.v. ε defined on $(\Omega, \mathcal{F}, P)$ and a positive constant C, such that

$$\sup_{t\in\mathbf{Z}} P(|\varepsilon_t| > x) \le CP(|\varepsilon| > x) \quad \text{for any} \quad x \ge 0, \tag{1.3}$$

$$\begin{cases} E(\varepsilon^2) < \infty, & \text{if } \{\varepsilon_t : t \in \mathbf{Z}\} \text{ is independent or strictly stationary;} \\ E(\varepsilon^2 \log^+ |\varepsilon|) < \infty, & \text{else.} \end{cases} \tag{1.4}$$

Second, on the base of obtaining the random functional central limit theorem for processes of partial sums of linear processes generated by non-stationary martingale differences, prove the theorem for processes of product sums of linear processes above mentioned. For this, we firstly give some signs.

For any $n, m \in \mathbf{N}$, set

$$V_n(m) = \sum_{1\le i_1<\cdots<i_m\le n} \prod_{j=1}^{m} X_{i_j}, \quad \text{if } n \ge m; \qquad V_n(m) = 0, \quad \text{if } n < m.$$

$$\xi_n(u) = n^{-1/2}\tau^{-1}(S_{[nu]} + (nu - [nu])X_{[nu]+1}), \quad u \in [0,1],\ n \in \mathbf{N}.$$

$$U_n(m,u) = n^{-m/2}\tau^{-m}(V_{[nu]}(m) + V_{[nu]}(m-1)(nu - [nu])X_{[nu]+1}), \qquad u \in [0,1],\ n \in \mathbf{N},$$

where $S_n = V_n(1)$, $\tau^2 = \sigma^2\Big(\sum\limits_{j=-\infty}^{\infty} a_j\Big)^2$ and $[nu]$ is the integral part of nu. Denote the random elements of $\xi_n(u)$ and $U_n(m,u)$ by ξ_n and $U_n(m)$ respectively. ξ_n and $U_n(m)$ are respectively called the process of partial sum and the process of product sum of the linear process $\{\varepsilon_t : t \in \mathbf{Z}\}$. Besides, denote by W the Wiener measure on $C[0,1]$.

We shall give some lemmas in Section 2, including the functional central limit theorem for processes of partial sums of linear processes generated by non-stationary martingale differences. Its corresponding results on processes of product sums will be discussed in Section 3.

§2. Some Lemmas

First, we extend the lemma 3 in [5] to the case of non-stationary.

Lemma 2.1. *Let $\{\varepsilon_t, \mathcal{F}_t : t \in \mathbf{Z}\}$ be a sequence of martingale differences, satisfying the conditions* (1.2), (1.3) *and*

$$E(\varepsilon^2) < \infty. \tag{2.1}$$

$\{X_t : t \in \mathbf{N}\}$ is the linear process generated by $\{\varepsilon_t : t \in \mathbf{Z}\}$ and $\{a_j : j \in \mathbf{Z}\}$, satisfying the condition (1.1). *And set $\widetilde{X}_t = \Big(\sum\limits_{j=-\infty}^{\infty} a_j\Big)\varepsilon_t$, $t \in \mathbf{N}$, $\widetilde{S}_k = \sum\limits_{t=1}^{k} \widetilde{X}_t$, $S_k = \sum\limits_{t=1}^{k} X_t$, $k \in \mathbf{N}$. Then*

$$\max_{1\le k\le n} n^{-1/2}|\widetilde{S}_k - S_k| \to 0, \quad n \to \infty, \quad \textit{in probability.} \tag{2.2}$$

Proof. By the proof of Lemma 3 in [5], in order to prove (2.2), it suffices to prove that for any $l \in \mathbf{N}$,

$$Z_{n,l} = n^{-1/2} \max_{1\le k\le n} \Big| \sum_{j=1}^{l} a_j \sum_{i=1}^{j} \varepsilon_{k-j+1} \Big| \to 0, \quad n \to \infty, \quad \text{in probability.} \tag{2.3}$$

For this, we point out that with the help of the condition (1.3) and $E(\varepsilon^2) < \infty$, it is clear that $\{{\varepsilon_t}^2 : t \in \mathbf{Z}\}$ is uniformly integrable. Hence for any fixed $l \in \mathbf{N}$, $\Big\{ \sum\limits_{i=1}^{l} \varepsilon_{t+i}^2 : t \in \mathbf{Z} \Big\}$ is uniformly integrable. Thus, for any $\delta > 0$, set $d = l^{-1} \Big(\sum\limits_{j=-\infty}^{\infty} |a_j| \Big)^{-2} \delta^2$, then

$$\begin{aligned}
P(Z_{n,l} \ge \delta) &\le P\Big(\max_{1\le k\le n} \sum_{i=1}^{l} |\varepsilon_{k-i+1}| \sum_{j=i}^{l} |a_j| \ge n^{1/2}\delta \Big) \\
&\le P\Big(\max_{1\le k\le n} \sum_{i=1}^{l} \varepsilon_{k-i+1}^2 \ge dn \Big) \\
&= \sum_{k=1}^{n} P\Big(\max_{1\le s\le k-1} \sum_{i=1}^{l} \varepsilon_{s-i+1}^2 < dn, \sum_{i=1}^{l} \varepsilon_{k-i+1}^2 \ge dn \Big) \\
&\le d^{-1} n^{-1} \sum_{k=1}^{n} E\Big(\sum_{i=1}^{l} \varepsilon_{k-i+1}^2 \Big) I\Big(\sum_{i=1}^{l} \varepsilon_{k-i+1}^2 \ge dn \Big) \\
&\le d^{-1} \sup_{t\in\mathbf{Z}} E\Big(\sum_{i=1}^{l} \varepsilon_{t+i}^2 \Big) I\Big(\sum_{i=1}^{l} \varepsilon_{t+i}^2 \ge dn \Big) \to 0, \quad n \to \infty.
\end{aligned}$$

Denote by $\widetilde{\xi}_n$ the random element defined on $C[0,1]$ of $\{\widetilde{X}_t : t \in N\}, n \in \mathbf{N}$. Then by Lemma 2.1, we have $\xi_n - \widetilde{\xi}_n \to 0$, $n \to \infty$, in probability. And it is easy to verify the conditions in [1]

$$\sum_{i=1}^{n} E(\widetilde{X_i}^2 | \mathcal{F}_{i-1}) \Big(E\Big(\sum_{i=1}^{n} \widetilde{X_i} \Big)^2 \Big)^{-1} \to 1, \quad n \to \infty, \quad \text{in probability,} \tag{2.4}$$

$$\Big(E\Big(\sum_{i=1}^{n} \widetilde{X_i} \Big)^2 \Big)^{-1} \sum_{i=1}^{n} E\widetilde{X_i}^2 I\Big(|\widetilde{X_i}| \ge \delta \Big(E\Big(\sum_{i=1}^{n} \widetilde{X_i} \Big)^2 \Big)^{1/2} \Big) \to 0, \quad n \to \infty \quad \text{for any } \delta > 0. \tag{2.5}$$

So quite similarly to the proofs of Theorem 1 and Theorem 2 in [5], we can get the Renyi's central limit theorem and random functional central limit theorem for processes of partial sums of linear processes generated by non-stationary martingale differences.

Lemma 2.2. *Let $\{\varepsilon_t, \mathcal{F}_t : t \in \mathbf{Z}\}$ be a sequence of martingale differences, satisfying* (1.2), (1.3) *and* (2.1). *$\{X_t : t \in \mathbf{N}\}$ is a linear process generated by $\{\varepsilon_t : t \in \mathbf{Z}\}$ and $\{a_j : j \in \mathbf{Z}\}$, satisfying the condition* (1.1). *Then for any $k \in \mathbf{N}$ and $B \in \mathcal{F}_k, P(B) > 0$, we have*

$$\lim_{n\to\infty} P(n^{-1/2} t S_n \le x | B) = (2\pi)^{-1/2} \int_{-\infty}^{x} e^{-y^2/2} dy, \quad x \in \mathbf{R}. \tag{2.6}$$

Further, all the finite-dimensional distributions of ξ_n converge weakly under the probability measure $P(\cdot|B)$ to the finite-dimensional distribution of the Wiener measure.

Lemma 2.3. *Let the condition in Lemma* 2.2 *be satisfied, and $\{N_n : n \in \mathbf{N}\}$ be a sequence of positive integer-valued random variables defined on the probability space $(\Omega, \mathcal{F}, P)$. If*

$$n^{-1} N_n \to N, \quad n \to \infty, \quad \textit{in probability,} \tag{2.7}$$

where N is a real-valued random variable with

$$P(0 < N < \infty) = 1, \tag{2.8}$$

then the process $\{\xi_{N_n}(u) : u \in [0,1],\ n \in \mathbf{N}\}$ converges weakly to the Wiener measure. Denote $\xi_{N_n} \Rightarrow W,\ n \to \infty$.

Lemma 2.4. *Let $\{x_i : i \in \mathbf{N}\}$ be a sequence of real numbers. Then for any $n, m \in \mathbf{N}$ and $n \geq m$, we have*

$$v_n(m) = \sum_{1 \leq i_1 < \cdots < i_m \leq n} \prod_{j=1}^{m} x_{i_j} = \sum_{\sum\limits_{j=1}^{m} r_j s_j = m} A(m, r_j, s_j : 1 \leq j \leq m) \prod_{j=1}^{m} \Big(\sum_{i=1}^{n} {x_i}^{r_j} \Big)^{s_j}, \tag{2.9}$$

where $A(m, r_j, s_j : 1 \leq j \leq m)$, $\sum\limits_{j=1}^{m} r_j s_j = m$ are constants irrelevant to n and $\{x_i : i \in \mathbf{N}\}$.

Remark 2.1. Now we take the example of $m = 4$ to explain the scope of the sum and the product in (2.9). In view of

$$m = 4 = 1 \times 4 = 1 \times 2 + 2 \times 1 = 2 \times 2 = 3 \times 1 + 1 \times 1 = 4 \times 1,$$

we have

$$\begin{aligned} v_n(4) = {} & A(4,1,4)\Big(\sum_{i=1}^{n} x_i\Big)^4 + A(4,1,2,2,1)\Big(\sum_{i=1}^{n} x_i\Big)^2\Big(\sum_{i=1}^{n} x_i^2\Big) \\ & + A(4,2,2)\Big(\sum_{i=1}^{n} x_i^2\Big)^2 + A(4,3,1,1,1)\Big(\sum_{i=1}^{n} x_i^3\Big)\Big(\sum_{i=1}^{n} x_i\Big) + A(4,4,1)\Big(\sum_{i=1}^{n} x_i^4\Big). \end{aligned}$$

Proof. For m we use the mathematical induction. When $m = 1, 2$, (2.9) is obvious. Assume when $m \leq k-1$, (2.9) holds. Then $m = k$. Set

$$f(k, x_i : 1 \leq i \leq n) = \Big(\sum_{i=1}^{n} x_i\Big)^k - k! v_n(k).$$

We know that $f(k, x_i : 1 \leq i \leq n)$ is a k-homogeneous symmetric polynomial. By using the principal theorem of symmetric polynomial, it follows that there exists a unique polynomial $g(k, y_i : 1 \leq i \leq k-1)$ such that

$$f(k, x_i : 1 \leq i \leq n) = g(k, v_n(k) : 1 \leq i \leq k-1). \tag{2.10}$$

In view of the inductive assumption, put $v_n(k), 1 \leq i \leq k-1$ into the right-hand side of (2.10). After arrangement, we immediately obtain that (2.9) holds when $m = k$ and the coefficients are still irrelevant to n and $\{x_i : i \in \mathbf{N}\}$.

Lemma 2.5. *Let $\{Y_i : i \in \mathbf{N}\}$ be a sequence of real-valued r.v.'s, $\{N_n : n \in \mathbf{N}\}$ be a sequence of positive integer-valued r.v.'s and N be a real-valued r.v. They are all defined on the same probability space $(\Omega, \mathcal{F}, P)$. If the conditions (2.7), (2.8) are satisfied and*

$$b_n^{-1} \sum_{i=1}^{n} Y_i \to 0, \quad 0 < b_n \uparrow \infty, \quad n \to \infty, \quad a.s. \tag{2.11}$$

Then

$$b_{N_n}^{-1} \max_{1 \leq k \leq N_n} \Big| \sum_{i=1}^{k} Y_i \Big| \to 0, \quad n \to \infty, \quad \textit{in probability}. \tag{2.12}$$

Proof. For any $\eta > 0$, (2.8) yields that there exist $M_2 > M_1 > 0$, such that

$$P(N \leq M_1) < \eta/4 \quad \text{and} \quad P(N \geq M_2) < \eta/4. \tag{2.13}$$

And by (2.7), for any $M_1 > \delta > 0$, there exists $n_1 \in \mathbf{N}$ such that when $n \geq n_1$,

$$P(|n^{-1} N_n - N| \geq \delta) < \eta/4. \tag{2.14}$$

And by (2.11), we know that

$$b_n^{-1}\max_{1\le k\le n}\Big|\sum_{i=1}^k Y_i\Big|\to 0,\quad n\to\infty,\quad \text{a.s.}$$

Then for any $\theta>0$ and M_1, M_2, δ above-mentioned, there exists $n_2\in\mathbf{N}$ such that when $n\ge n_2$,

$$P\Big(b^{-1}_{n(M_2+\delta)}\max_{1\le k\le n(M_2+\delta)}\Big|\sum_{i=1}^k Y_i\Big|\ge (M_1-\delta)(M_2+\delta)^{-1}\theta\Big)<\eta/4. \tag{2.15}$$

Thus, when $n\ge n_0=\max\{n_1,n_2\}$, in view of (2.13)–(2.15), we conclude

$$\begin{aligned}
&P\Big(b^{-1}_{N_n}\max_{1\le k\le N_n}\Big|\sum_{i=1}^k Y_i\Big|\ge\theta\Big)\\
&\le P\Big(b^{-1}_{N_n}\max_{1\le k\le N_n}\Big|\sum_{i=1}^k Y_i\Big|\ge\theta, |n^{-1}N_n-N|<\delta, M_1<N<M_2\Big)\\
&\quad + P(|n^{-1}N_n-N|\ge\delta)+P(N\le M_1)+P(N\ge M_2)\\
&< P\Big(b^{-1}_{n(M_2+\delta)}\max_{1\le k\le n(M_2+\delta)}\Big|\sum_{i=1}^k Y_i\Big|\ge (M_1-\delta)(M_2+\delta)^{-1}\theta\Big)+(3\delta)/4<\eta.
\end{aligned}$$

§3. The Main Results and Their Proofs

Now we give a non-random functional central limit theorem for processes of product sums. For this we introduce some other notations first. Denote the coefficient of

$$\Big(\sum_{i=1}^n x_i\Big)^{m-2s}\Big(\sum_{i=1}^n {x_i}^2\Big)^s$$

in (2.9) by $A(m,s), s=0,\cdots,[\frac{m}{2}]$. And for any $m\in\mathbf{N}$, set

$$U(m,u)=\sum_{s=0}^{[\frac{m}{2}]}A(m,s)(W(u))^{m-2s}u^s,\quad u\in[0,1].$$

Its corresponding random element defined on $C[0,1]$ is $U(m)$.

Theorem 3.1. *Let the conditions in Lemma* 2.2 *and* (1.4) *be satisfied. Then for any* $m\in\mathbf{N}$,

$$U_n(m)\Rightarrow U(m),\quad n\to\infty. \tag{3.1}$$

On the base of Theorem 3.1, we obtain the random functional central limit theorem for processes of product sums.

Theorem 3.2. *Let the conditions in Lemma* 2.3 *and* (1.4) *be satisfied. Then for any* $m\in\mathbf{N}$,

$$U_{N_n}(m)\Rightarrow U(m),\quad n\to\infty. \tag{3.2}$$

Proof of Theorem 3.1. Set $\widetilde{U}_n(m,u)=\sum\limits_{s=0}^{[\frac{m}{2}]}A(m,s)(\xi_n(u))^{m-2s}u^s$, $u\in[0,1]$. Its corresponding random element on $C[0,1]$ is $\widetilde{U}_n(m)$, $n\in\mathbf{N}$. First we prove $U_n(m)-\widetilde{U}_n(m)\to 0$, $n\to\infty$, in probability. It suffices to prove that for any $\eta>0$,

$$P\Big(\sup_{u\in[0,1]}|U_n(m,u)-\widetilde{U}_n(m,u)|\ge\eta\Big)\to 0,\quad n\to\infty. \tag{3.3}$$

By Lemma 2.4, we see that

$$U_n(m,u) = \sum_{\sum_{j=1}^{m} r_j s_j = m} A(m, r_j, s_j : 1 \le j \le m)$$
$$\cdot \prod_{j=1}^{m} \Big(\tau^{-r_j} n^{-r_j/2} \Big(\sum_{i=1}^{[nu]} X_i^{r_j} + (nu-[nu])^{r_j} + X_{[nu]+1}^{r_j}\Big)\Big)^{s_j}, \quad u \in [0,1].$$

It will be discussed in the following according to different r_j, $1 \le j \le m$ respectively. To be concise, denote $r_j, 1 \le j \le m$ by r.

When $3 \le r \le m$, it follows from Jensen's inequality, the condition (1.1), Fubini theorem and Hölder's inequality that

$$\begin{aligned}
&\sup_{u\in[0,1]} n^{-r/2} \Big| \sum_{i=1}^{[nu]} X_i^r + (nu-[nu])^r X_{[nu]+1}^r \Big| \le n^{-r/2} \sum_{i=1}^{n} |X_i|^r \\
&\le \Big(n^{-3/2} \sum_{i=1}^{n} |X_i|^3 \Big)^{r/3} \\
&\le \Big(n^{-3/2} \sum_{i=1}^{n} \Big(\sum_{j=-\infty}^{\infty} |a_j \varepsilon_{i-j}| \Big)^3 \Big)^{r/3} \\
&= \Big(\sum_{-\infty<j,k,l<\infty} |a_j a_k a_l| n^{-3/2} \sum_{i=1}^{n} |\varepsilon_{i-j}\varepsilon_{i-k}\varepsilon_{i-l}| \Big)^{r/3} \\
&\le \Big(\sum_{-\infty<j,k,l<\infty} |a_j a_k a_l| \Big(n^{-3/2} \sum_{i=1}^{n} |\varepsilon_{i-j}|^3 \Big)^{1/3} \\
&\quad \cdot \Big(n^{-3/2} \sum_{i=1}^{n} |\varepsilon_{i-k}|^3 \Big)^{1/3} \Big(n^{-3/2} \sum_{i=1}^{n} |\varepsilon_{i-l}|^3 \Big)^{1/3} \Big)^{r/3}.
\end{aligned} \tag{3.4}$$

And in view of (1.3), (1.4), and the standard Marcinkiewicz strong law of large numbers, it is easy to prove

$$n^{-3/2} \sum_{i=1}^{n} |\varepsilon_{i-j}|^3 \to 0, \quad n \to \infty, \quad \text{a.s.} \tag{3.5}$$

By (3.5), (3.4), (1.1) and the dominated convergence theorem,

$$\sup_{u\in[0,1]} n^{-r/2} \Big| \sum_{i=1}^{[nu]} X_i^r + (nu-[nu])^r X_{[nu]+1}^r \Big| \to 0, \quad n \to \infty, \quad \text{a.s.} \tag{3.6}$$

When $r = 2$, we discuss it in two cases. First we prove

$$\begin{aligned}
&\sup_{u\in[0,1]} n^{-1} \Big| \sum_{i=1}^{[nu]} X_i^2 + (nu-[nu])^2 X_{[nu]+1}^2 \Big| = n^{-1} \sum_{i=1}^{n} X_i^2 \\
&= \sum_{j=-\infty}^{\infty} a_j^2 n^{-1} \sum_{i=1}^{n} \varepsilon_{i-j}^2 + 2 \sum_{-\infty<j<k<\infty} a_j a_k n^{-1} \sum_{i=1}^{n} \varepsilon_{i-j}\varepsilon_{i-k} \to \tau^2, \quad n \to \infty, \quad \text{a.s.}
\end{aligned} \tag{3.7}$$

And by the conditions (1.2)–(1.4) and Theorem 2.19 in Hall and Heyde (1980), we have

$$n^{-1}\sum_{i=1}^{n}\varepsilon_{i-j}^2 \to \sigma^2, \quad n\to\infty, \quad \text{a.s.} \tag{3.8}$$

Observe that $j<k$, then it is easy to know that $\{\varepsilon_{i-j}\varepsilon_{i-k}, \mathcal{F}_{i-j} : i\in \mathbf{N}\}$ is still a sequence of martingale differences. And by (1.2),

$$E\Big(\sum_{i=1}^{\infty} i^{-2}\varepsilon_{i-k}^2\Big) = \sigma^2\sum_{i=1}^{\infty} i^{-2} < \infty.$$

Then it yields that

$$\sigma^2\sum_{i=1}^{\infty} i^{-2}\varepsilon_{i-k}^2 = \sum_{i=1}^{\infty} i^{-2}E(\varepsilon_{i-j}^2\varepsilon_{i-k}^2|\mathcal{F}_{i-j-1}) < \infty, \quad \text{a.s.}$$

Thus taking into account of Theorem 2.15 in Hall and Heyde (1980) and Kronecker lemma, we see that

$$n^{-1}\sum_{i=1}^{n}\varepsilon_{i-j}\varepsilon_{i-k} \to 0, \quad n\to\infty, \quad \text{a.s.} \tag{3.9}$$

With the help of (3.8), (3.9), the condition (1.1) and the dominated convergence theorem, we arrive at (3.7) immediately.

On the other hand, by (3.7) we have

$$\begin{aligned}
&\sup_{u\in[0,1]} n^{-1}\Big|\sum_{i=1}^{[nu]}X_i^2 + (nu-[nu])^2X_{[nu]+1}^2 - nu\tau^2\Big| \\
\le& \sup_{u\in[0,1]} n^{-1}\Big|\sum_{i=1}^{[nu]}(X_i^2-\tau^2)\Big| + \sup_{u\in[0,1]} n^{-1}|X_{[nu]+1}^2-\tau^2| \\
&+ \sup_{u\in[0,1]} \tau^2n^{-1}(nu-[nu]-(nu-[nu])^2) \\
\le& n^{-1}\max_{1\le k\le n}\Big|\sum_{i=1}^{k}(X_i^2-\tau^2)\Big| + n^{-1}\max_{1\le i\le n+1}|X_i^2-\tau^2| + n^{-1}\tau^2 \\
\le& 4(n+1)^{-1}\max_{1\le k\le n+1}\Big|\sum_{i=1}^{k}(X_i^2-\tau^2)\Big| + n^{-1}\tau^2 \to 0, \quad n\to\infty, \quad \text{a.s.}
\end{aligned} \tag{3.10}$$

When $r=1$, in Lemma 2.3, take $N_n=n$, $n\in\mathbf{N}$. Then we have $\xi_n \Rightarrow W$, $n\to\infty$. And $h_1(X)=\sup\limits_{u\in[0,1]}(X(u))^{m-2s}$, $s=0,\cdots,[\frac{m}{2}]$ is a continuous functional in X. Hence by Theorem 5.1 in [2], we have

$$\begin{aligned}
&\sup_{u\in[0,1]}(\tau^2n)^{-(m-2s)/2}\Big(\sum_{i=1}^{[nu]}X_i + (nu-[nu])X_{[nu]+1}\Big)^{m-2s} \\
=& \sup_{u\in[0,1]}(\xi_n(u))^{m-2s} \Rightarrow \sup_{u\in[0,1]}(W(u))^{m-2s}, \quad n\to\infty, \quad s=0,\cdots,\Big[\frac{m}{2}\Big].
\end{aligned} \tag{3.11}$$

By (3.4), (3.6), (3.7), (3.10) and (3.11), we get that (3.3) holds. And by (3.3) we know that to prove (3.1), we shall only prove

$$\widetilde{U}_n(m) \Rightarrow U(m), \quad n\to\infty. \tag{3.12}$$

For this, set $Y(u) = Y_n(u) = u$, $u \in [0,1]$. Their corresponding random elements on $C[0,1]$ are Y and Y_n respectively, $n \in \mathbf{N}$. Lemma 2.2 tells us that $(\xi_n, Y_n) \Rightarrow (W, Y)$, $n \to \infty$. And $\widetilde{U}_n(m)$ and $U(m)$ are the same continuous functionals of (ξ_n, Y_n) and (W, Y) respectively, thus (3.12) holds.

Proof of Theorem 3.2. Continuing to use the notations and proof idea in Theorem 3.1, we shall only prove

$$U_{N_n}(m) - \widetilde{U}_{N_n}(m) \to 0, \quad n \to \infty, \quad \text{in probability.} \tag{3.13}$$

When $3 \le r \le m$, by (3.6) and Lemma 2.5,

$$N_n{}^{-r/2} \sup_{u\in[0,1]} \Big| \sum_{j=1}^{[N_n u]} X_i^r + (N_n u - [N_n u])^r X^r_{[N_n u]+1} \Big| \to 0, \quad n \to \infty, \quad \text{in probability.} \tag{3.14}$$

When $r = 2$, on one hand, by (3.7) and Lemma 2.5,

$$N_n^{-1} \sup_{u\in[0,1]} \Big| \sum_{j=1}^{[N_n u]} X_i^2 + (N_n u - [N_n u])^2 X^2_{[N_n u]+1} \Big| \to \tau^2, \quad n \to \infty, \quad \text{in probability.} \tag{3.15}$$

On the other hand, in view of (3.10) and Lemma 2.5,

$$\begin{aligned} &N_n^{-1} \sup_{u\in[0,1]} \Big| \sum_{j=1}^{[N_n u]} X_i^2 + (N_n u - [N_n u])^2 X^2_{[N_n u]+1} - N_n u \tau^2 \Big| \\ &\to 0, \quad n \to \infty, \quad \text{in probability.} \end{aligned} \tag{3.16}$$

When $r = 1$, Lemma 2.3 yields $\xi_{N_n} \Rightarrow W$, thus

$$\begin{aligned} &(\tau^2 N_n)^{-(m-2s)/2} \sup_{u\in[0,1]} \Big(\sum_{i=1}^{[N_n u]} X_i + (N_n u - [N_n u]) X_{[N_n u]+1} \Big)^{m-2s} \\ &= \sup_{u\in[0,1]} (\xi_{N_n}(u))^{m-2s} \Rightarrow \sup_{u\in[0,1]} (W(u))^{m-2s}, \quad n \to \infty, \quad s = 0, \cdots, \Big[\frac{m}{2}\Big]. \end{aligned} \tag{3.17}$$

It follows from (3.14), (3.17) and Slutsky theorem that (3.13) holds.

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