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POINTWISE ESTIMATES OF SOLUTIONS TO CAUCHY PROBLEM FOR QUASILINEAR HYPERBOLIC SYSTEMS***

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Abstract

This paper considers the pointwise estimate of the solutions to Cauchy problem for quasilinear hyperbolic systems, which bases on the existence of the solutions by using the fundamental solutions. It gives a sharp pointwise estimates of the solutions on domain under consideration. Specially, the estimate is precise near each characteristic direction.

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§1. Introduction

In this paper, we consider the following first order quasilinear strictly hyperbolic systems of balance laws

$$u_t + A(u)u_x = B(u), \tag{1.1}$$

where $u(x,t) = (u^1, \cdots, u^n)^t(x,t)$ is the unknown vector function, $(x,t) \in R \times R_+$, $A(u) = (a_{ij})$ is an $n \times n$ matrix with suitably smooth elements $a_{ij}\,(i,j = 1,2,\cdots,n)$, and $B(u)$ is a given smooth vector function. By strictly hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n real eigenvalues $\lambda_1(u), \cdots, \lambda_n(u)$ and a complete system of left (resp. right) eigenvectors. For $i = 1, \cdots, n$, let $l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{i1}(u), \cdots, r_{in}(u))^t$) be a left (resp. right) eigenvector corresponding to $\lambda_i(u)$,

$$l_i(u)A(u) = \lambda_i(u)l_i(u) \qquad (\text{resp. } A(u)r_i(u) = \lambda_i(u)r_i(u)). \tag{1.2}$$

We have

$$\det l_{ij} \neq 0 \qquad (\text{equivalently } \det r_{ij} \neq 0). \tag{1.3}$$

Here $\lambda_i(u), l_{ij}(u), r_{ij}(u)$ are supposed to have the same regularity as $a_{ij}\,(i,j = 1,\cdots,n)$. And $u(x,t)$ is considered in a neighborhood of $u = 0$. Without loss of generality, we suppose

$$0 < \lambda_1(u) < \cdots < \lambda_n(u), \tag{1.4}$$

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in the neighborhood of $u = 0$, and furthermore we suppose that on the domain under consideration

$$l_i(u) \cdot r_j(u) \equiv \delta_{ij}, \qquad i, j = 1, \cdots, n, \tag{1.5}$$

$$r_i(u)^t \cdot r_i(u) \equiv 1, \qquad i, j = 1, \cdots, n, \tag{1.6}$$

where δ_{ij} stands for the Kronecker's symbol.

Throughout this paper, we always assume that the term $B(u)$ satisfies

$$B(0) = 0, \qquad \nabla B(0) = 0, \tag{1.7}$$

namely $B(u)$ is a nonlinear term of higher order. The Cauchy problem is (1.1) with the following initial data

$$u(x, 0) = \phi(x), \tag{1.8}$$

where $\phi(x)$ is a "small" C^2 vector function of x with suitably decay properties as $|x| \to \infty$. We will investigate the global existence and the precise pointwise estimates of the solutions to the Cauchy problem (1.1) and (1.8).

In the case that $B(u) = 0$, (1.1) is homogenous system as follows

$$u_t + A(u)u_x = 0. \tag{1.9}$$

Suppose that $\phi(x)$ possesses a compact support and $A(u)$ and $\phi(x)$ are C^2 functions. In a neighborhood of $u = 0$, if the systems (1.1) is genuinely nonlinear in the sense of Lax, or a part of the characteristics are genuinely nonlinear while the other part of the characteristics are linearly degenerate in the same sense, John[2] and Liu[6] studied the blow-up phenomenon of C^2 solutions to the Cauchy problem (1.8) and (1.9) for small initial data. Hörmander[1] reproved the result given in [2] and obtained the sharp estimate for the life- span of C^2 solutions. By introducing the concept of weakly linear degenerate, Li et al[8] gave a complete result on the global existence of C^1 solutions to the Cauchy problem (1.8) and (1.9) for small initial data with compact support. And Li et al[3, 9] generalized the result to the case that $\phi(x)$ has no compact support but possesses suitably decay properties as $|x| \to \infty$ and for inhomogeneous system (1.1). They obtained that the solutions of the systems have some decay property except for the neighborhood around the characteristics.

In this paper, we try to obtain the whole pointwise estimates of the solutions for the above case in [3, 9]. As we know, the estimate near the characteristics is important to hyperbolic system, but it is more difficult than outside the neighborhood around the characteristics. In general, one only can prove that solutions are bounded near the characteristics. One of the key point in our study is how to get the pointwise estimates of the solutions near the characteristics. In fact, by using the fundamental solution (i.e. Green function) in dealing with the hyperbolic system with diffuse structure[5, 7, 10], we obtain a sharp pointwise estimates of the solution on whole domain under consideration. Specially, we obtain the explicit expressions of the time-asymptotic behavior of solution near each characteristic direction. Since they are not of diffuse structure, the Green function is not decay for time, so we need more precise estimate.

On the other hand, since it is difficult to deal with the above problem in multi-dimension space by the characteristics method. We hope to extend the notion of weakly degenerate and study the time asymptotic behavior of solutions in multi-dimensions by using the fundamental solution method. But, we consider the case in one-dimension.

We suppose $\phi(x) \in C^4$, and denote

$$\theta = \sup_{x \in R}\{(1 + |x|)^{1+\mu}(|\phi| + |\phi'| + |\phi''| + |\phi'''| + |\phi''''|)\}, \tag{1.10}$$

where $\mu > 1$. Set

$$v_i(u) = l_i(u) \cdot u, \quad w_i(u) = l_i(u) \cdot u_x, \quad b_i(u) = l_i(u) \cdot B(u). \tag{1.11}$$

The definitions of weakly linear degenerate, matching condition and normalized coordinate can be seen in [3], here we omit it. Throughout this paper we denote the generic constants by C, and denote $\lambda_i = \lambda_i(0)$ in the sequel. Our main result is as follows:

Theorem 1.1. *Under the hypotheses mentioned above, suppose that $A(u)$ and $B(u)$ are C^4 in a neighborhood of $u = 0$. Furthermore, suppose that the system (1.1) is weakly linearly degenerate and $B(u)$ satisfies the matching condition. Then there exists $\theta_0 > 0$ which is so small that for any given $\theta \in [0, \theta_0]$, the Cauchy problem (1.1) and (1.9) admits a unique global C^1 solution $u = u(x, t)$ on $t \geq 0$, and there exists C which is independent of θ, such that*

$$|v_i(x,t)| \leq C\theta(1 + |x - \lambda_i t|)^{1+\mu}, \tag{1.12}$$

$$|w_i(x,t)| \leq C\theta(1 + |x - \lambda_i t|)^{1+\mu}. \tag{1.13}$$

Finally in this section, we enclose a lemma and some formulas given in [3, 8].

Let

$$\frac{d}{d_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x}$$

be the directional derivative along the i-th characteristic. From (1.11)–(1.13), for $i = 1, \cdots, n$,

$$\begin{cases} \dfrac{dv_i}{d_i t} = \displaystyle\sum_{j,k=1}^{n} \beta_{ijk}(u) v_j w_k + \sum_{j,k=1}^{n} \upsilon_{ijk}(u) v_j b_k(u) + b_i(u), \\ \dfrac{dw_i}{d_i t} = \displaystyle\sum_{j,k=1}^{n} \gamma_{ijk}(u) w_j w_k + \sum_{j,k=1}^{n} \xi_{ijk}(u) w_j b_k + (b_i(u))_x, \end{cases} \tag{1.14}$$

where

$$\beta_{ijk}(u) = (\lambda_k(u) - \lambda_i(u)) l_i(u) \nabla r_j(u) r_k(u), \tag{1.15}$$

$$\upsilon_{ijk}(u) = -l_i(u) \nabla r_j(u) r_k(u), \tag{1.16}$$

$$\xi_{ijk}(u) = l_i(\triangledown r_k r_j - \triangledown r_j r_k), \tag{1.17}$$

$$\begin{aligned} \gamma_{ijk}(u) = \frac{1}{2}\{&(\lambda_j(u) - \lambda_k(u)) l_i(u) \nabla r_k(u) r_j(u) \\ &- \nabla\lambda_k(u) r_j(u) \delta_{ik} + (j|k)\}, \end{aligned} \tag{1.18}$$

where δ_{ij} is Kronecker's symbol, and $(j|k)$ stands for all terms obtained by changing j and k in the previous terms. Hence, we have

$$\beta_{iji}(u) \equiv 0 \qquad \text{for all} \quad j. \tag{1.19}$$

We get some relations in the normalized coordinates similar in [3]:

$$r_i(u_ie_i) \equiv e_i = (0, \cdots, \overset{i}{1}, \cdots, 0)^t, \tag{1.20}$$

$$\beta_{ijj}(u_je_j) \equiv 0 \qquad \text{for all} \quad |u_j| \quad \text{small and} \quad j = 1, \cdots, n, \tag{1.21}$$

$$\gamma_{ijj}(u_je_j) \equiv 0 \qquad \text{for all} \quad |u_j| \quad \text{small and} \quad j = 1, \cdots, n, \tag{1.22}$$

$$b_i(u) = \sum_{j \neq k} b_{ijk}(u)u_ju_k \qquad \text{for all} \quad i = 1, \cdots, n, \tag{1.23}$$

$$(b_i(u))_x = \sum_{k=1}^{n} \tilde{b}_{ik}(u)w_k \qquad \text{for all} \quad i = 1, \cdots, n, \tag{1.24}$$

$$\tilde{b}_{ik}(u) = \sum_{j \neq k} \Big[\int_0^1 \frac{\partial \tilde{b}_{ik}}{\partial u_i}(su_1, \cdots, u_k, \cdots, su_n)ds \Big] u_j \qquad \text{for} \quad \tilde{b}_{ik}(u_ke_k) = 0. \tag{1.25}$$

§2. Preliminaries

In [3] for all $T > 0$, in the normalized coordinate, we know that there exists $C > 0$ independent of θ and T, such that

$$|u(x,t)|_{C^1(R \times [0,T])} \leq C\theta \tag{2.1}$$

for θ being small. We can see that the solutions $u(x,t) \in C^2(R \times [0,T])$ if we employ the same method as in [3] to the system (1.1) and (1.8). In the following, we will prove our results in the normalized coordinate. Firstly, it is easy to prove the following two lemmas as follows:

Lemma 2.1. *There exists a positive constant C independent of (x,t), such that the following inequality holds:*

$$|v_i(x,t)| = |l_i(u)u| \leq C\theta, \tag{2.2}$$

$$|w_i(x,t)| = |l_i(u)u| \leq C\theta. \tag{2.3}$$

Lemma 2.2. *There exists a positive constant C independent of (x,t) and θ, such that*

$$\int_0^t (1 + |x - \lambda_i t - (\lambda_j - \lambda_i)s|)^{-(1+\mu)} ds \leq C \qquad \textit{for all} \quad j \neq i. \tag{2.4}$$

Denote

$$F_{i,j,k} = (1 + |x - \lambda_i t - (\lambda_j - \lambda_i)s|)^{-(1+\mu)}(1 + |x - \lambda_i t - (\lambda_k - \lambda_i)s|)^{-(1+\mu)}. \tag{2.5}$$

Lemma 2.3. *There exists a positive constant C independent of (x,t) and θ, such that*

$$\int_0^t F_{i,j,k} ds \leq C(1 + |x - \lambda_i t|)^{-(1+\mu)} \qquad \textit{for all} \quad j \neq k. \tag{2.6}$$

Proof. It is easy to prove for $k = i$ or $j = i$. Without loss of generality, we suppose $j > i$. We rewrite the left term in (2.4), for all $(x,t) \in R \times R_+$, in the form

$$\begin{aligned} \int_0^t F_{i,j,k} ds &= \int_{[0,t] \cap I_1} F_{i,j,k} ds + \int_{[0,t] \cap I_2} F_{i,j,k} ds + \int_{[0,t] \cap I_3} F_{i,j,k} ds \\ &\doteq A_1 + A_2 + A_3, \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} I_1 &= \{s \mid |x - \lambda_i t| \geq (1 + \eta)(\lambda_j - \lambda_i)s\}, \\ I_2 &= \{s \mid |x - \lambda_i t| \leq (1 - \eta)(\lambda_j - \lambda_i)s\}, \\ I_3 &= \{s \mid (1 - \eta)(\lambda_j - \lambda_i)s \leq |x - \lambda_i t| \leq (1 + \eta)(\lambda_j - \lambda_i)s\} \end{aligned}$$

for η being small and defined later.

For $s \in I_1$ and $s \in I_2$ respectively,

$$|x-\lambda_i t-(\lambda_j-\lambda_i)s| \geq |x-\lambda_i t|-(\lambda_j-\lambda_i)s \geq \frac{\eta}{1+\eta}|x-\lambda_i t|,$$
$$|x-\lambda_i t-(\lambda_j-\lambda_i)s| \geq (\lambda_j-\lambda_i)s-|x-\lambda_i t| \geq \frac{\eta}{1-\eta}|x-\lambda_i t|.$$

Thus from Lemma 2.2 and direct calculation, we have

$$A_1, A_2 \leq C(1+|x-\lambda_i t|)^{-(1+\mu)}. \tag{2.8}$$

For $s \in I_3 = I_{31} \cup I_{32}$ with

$$I_{31} = \{s \,|\, (1-\eta)(\lambda_j-\lambda_i)s < x-\lambda_i t < (1+\eta)(\lambda_j-\lambda_i)s\}, \tag{2.9}$$
$$I_{32} = \{s \,|\, (1-\eta)(\lambda_j-\lambda_i)s < -(x-\lambda_i t) < (1+\eta)(\lambda_j-\lambda_i)s\}. \tag{2.10}$$

If $s \in I_{32}$, then

$$x-\lambda_i t-(\lambda_j-\lambda_i)s \leq -(2-\eta)(\lambda_j-\lambda_i)s \leq -\frac{2-\eta}{1+\eta}|x-\lambda_i t|.$$

It is easy to see that

$$A_{32} = \int_{[0,t]\cap I_{32}} F_{ijk}ds \leq C(1+|x-\lambda_i t|)^{-(1+\mu)}. \tag{2.11}$$

If $s \in I_{31}$, we choose $\eta \leq \min\limits_{m\neq l} \frac{|\lambda_m-\lambda_l|}{2(\lambda_n-\lambda_1)}$, then

$$\begin{aligned}|x-\lambda_i t-(\lambda_k-\lambda_i)s| &\geq |(\lambda_k-\lambda_j)s|-|x-\lambda_i t-(\lambda_j-\lambda_i)s|\\ &\geq \eta(\lambda_j-\lambda_i)s \geq \frac{\eta}{1+\eta}|x-\lambda_i t|.\end{aligned}$$

Thus we have

$$A_{31} = \int_{[0,t]\cap I_{31}} F_{ijk}ds \leq C(1+|x-\lambda_i t|)^{-(1+\mu)}. \tag{2.12}$$

From (2.7)–(2.12), the inequality (2.6) is true, and we complete the proof of the lemma.

Choosing some δ_0 which is small and satisfies $\lambda_{i+1}-\lambda_i \geq 4\delta_0, \lambda_1 \geq 4\delta_0$, we denote

$$\begin{aligned}D &= \{(x,t)|\ (\lambda_1-\delta_0)t < x < (\lambda_n+\delta_0)t\},\\ D_i &= \{(x,t)|\ |x-\lambda_i t| < \delta_0 t\},\\ D_+ &= \{(x,t)|\ x-\lambda_n t \leq \delta_0 t\},\\ D_- &= \{(x,t)|\ x-\lambda_1 t \geq \delta_0 t\}.\end{aligned}$$

By the definition of D_i, it is easy to obtain the following lemma.

Lemma 2.4. *There exist positive constants C independent of (x,t), such that*

$$C^{-1}t \leq |x-\lambda_i t| \leq Ct \qquad \textit{for all} \quad (x,t) \in D\setminus D_i,\ i=1,\cdots,n. \tag{2.13}$$

Set

$$M(t) = \max\Big\{\max_i \sup_{(x,t)}(1+|x-\lambda_i t|)^{1+\mu}|v_i(x,t)|, \max_i \sup_{(x,t)}(1+|x-\lambda_i t|)^{1+\mu}|w_i(x,t)|\Big\}. \tag{2.14}$$

Now, we will prove the following lemma.

Lemma 2.5. *There exists a positive constant C, for all $i=1,\cdots,n$, such that*

$$(1+|x-\lambda_i t|)^{1+\mu}|u_i(x,t)| \leq CM(t). \tag{2.15}$$

Proof. In fact, for any $(x,t)\in D\setminus D_i,\ i=1,\cdots,n,$

$$u_i(x,t)=u^T(x,t)e_i=\sum_{k=1}^n v_k r_k^T(u)e_i, \tag{2.16}$$

where e_i is defined by (1.20). Then from (2.13) we have

$$\begin{aligned}(1+|x-\lambda_i t|)^{1+\mu}|u_i(x,t)|&=(1+|x-\lambda_i t|)^{1+\mu}\Big|\sum_{k=1}^n v_k r_k^T(u)e_i\Big|\\ \le C\sum_{k=1}^n(1+|x-\lambda_k t|)^{1+\mu}|v_k r_k^T(u)e_i|&\le CM(t).\end{aligned} \tag{2.17}$$

If there exists i, such that $(x,t)\in D_i$, then for all $j\neq i$ from (1.20),

$$\begin{aligned}(1+|x-\lambda_j t|)^{1+\mu}|u_j(x,t)|&\le(1+|x-\lambda_j t|)^{1+\mu}\sum_{k\neq i}|v_k r_k^T(u)e_j|\\ &\quad+(1+|x-\lambda_j t|)^{1+\mu}|v_i[r_i^T(u)-r_i^T(u_ie_i)]e_j|.\end{aligned} \tag{2.18}$$

By Hadamard's formula, we have

$$r_j(u)-r_j(u_ie_i)=\int_0^1\sum_{k\neq j}\frac{\partial r_j}{\partial u_k}(su_1,\cdots,u_j,\cdots,su_n)dsu_k.$$

Then, from (2.1), (2.2) we have

$$\begin{aligned}&(1+|x-\lambda_j t|)^{1+\mu}|u_j(x,t)|\\ &\le C\sum_{k\neq i}(1+|x-\lambda_k t|)^{1+\mu}|v_k r_k^T(u)e_j|+C(1+|x-\lambda_j t|)^{1+\mu}|v_i|\sum_{k\neq i}|u_k|\\ &\le CM(t)+C\theta\sum_{k\neq i}(1+|x-\lambda_k t|)^{1+\mu}|u_k(x,t)|.\end{aligned} \tag{2.19}$$

If θ is small enough, we can get the following inequality

$$\sum_{k\neq j}(1+|x-\lambda_k t|)^{1+\mu}|u_k(x,t)|\le CM(t). \tag{2.20}$$

For $j=i$, we can obtain

$$\begin{aligned}&(1+|x-\lambda_i t|)^{1+\mu}|u_i(x,t)|\\ &\le(1+|x-\lambda_i t|)^{1+\mu}\sum_{k\neq i}|v_k r_{ki}|+(1+|x-\lambda_i t|)^{1+\mu}|v_i r_{ii}|\\ &\le CM(t).\end{aligned} \tag{2.21}$$

For any $(x,t)\notin D$, $x\le(\lambda_1-\delta_0)t$, or $x\ge(\lambda_n+\delta_0)t$. We only consider the case of $x\le(\lambda_1-\delta_0)t$. Then $x-\lambda_i t\le(\lambda_1-\lambda_i-\delta_0)t$ for all $i=1,\cdots,n$. Thus

$$x-\lambda_i t\le x-\lambda_1 t\le x-\lambda_i t+(\lambda_i-\lambda_1)\frac{x-\lambda_i t}{\lambda_1-\lambda_i-\delta_0}\le\frac{\delta_0}{\lambda_i-\lambda_1-\delta_0}(x-\lambda_i t),$$

then

$$\begin{aligned}(1+|x-\lambda_i t|)^{1+\mu}|u_i(x,t)|&=(1+|x-\lambda_i t|)^{1+\mu}\sum_{k=1}^n|v_k r_k^T(u)e_i|\\ &\le C\sum_{k=1}^n(1+|x-\lambda_k t|)^{1+\mu}|v_k(u)|\le CM(t).\end{aligned} \tag{2.22}$$

By (2.20) to (2.22), the lemma is proved.

§3. Proof of the Theorem

In [3] Kong proved the existence and uniqueness of the solutions for the Cauchy problem (1.1) and (1.8). Here we need to obtain the estimates (1.12) and (1.13). We know that $v_i(x,t)$ and $w_i(x,t)$ satisfy (1.11), where λ_i, l_i, r_i are functions in the neighborhood of $u=0$, and they are bounded by some constants.

We will prove the main theorem from (1.14). Our proof includes the two order derivatives of the solutions which will depend on the existence of C^4 solutions. And the proof of the existence of the solutions is similar to that in [3], so we will not repeat it. In fact, by differentiating two sides of the second equation of (1.14) with respect to x, and denoting $\rho_i = \frac{\partial w_i}{\partial x}$ for all $i = 1, \cdots, n$, we have

$$\begin{aligned}\frac{d\rho_i}{d_i t} &= \sum_{j\neq k} \frac{\partial \gamma_{ijk}}{\partial x}(u) w_j w_k + 2\sum_{j\neq k} \gamma_{ijk}(u)\rho_j w_k - \bigtriangledown\lambda_i u_x \rho_i \\ &\quad + (b_i(u))_{xx} + \sum_{j,k=1}^{n} (\xi_{ijk}(u) w_i b_i)_x. \end{aligned} \tag{3.1}$$

Similarly we differentiate two sides of (3.1) with respect to x, and denote $\alpha_i = \frac{\partial \rho_i}{\partial x}$, then we obtain that, for all $i = 1, \cdots, n$,

$$\begin{aligned}\frac{d\alpha_i}{d_i t} &= \sum_{j,k=1}^{n} \Big(\frac{\partial^2 \gamma_{ijk}}{\partial x^2}\Big)(u) w_j w_k + 4\sum_{j,k=1}^{n} \Big(\frac{\partial \gamma_{ijk}}{\partial x}\Big)(u)\rho_j w_k \\ &\quad + 2\sum_{j,k=1}^{n} \gamma_{ijk}\rho_j\rho_k + 2\sum_{j,k=1}^{n} \gamma_{ijk} w_j \alpha_k + \frac{\partial^2}{\partial x^2}\lambda_i \cdot \rho_i + \bigtriangledown\lambda_i u_x \alpha_i \\ &\quad + (b_i(u))_{xxx} + \sum_{j,k=1}^{n} (\xi_{ijk}(u) w_j b_k)_{xx}. \end{aligned} \tag{3.2}$$

Firstly, we have the following lemma.

Lemma 3.1. *Under the hypotheses mentioned above, if $A(u)$ and $B(u)$ are C^2 in a neighborhood of $u=0$, suppose furthermore that the system (1.1) is weakly linearly degenerate and $B(u)$ satisfies the matching condition. Then there exists $\theta_0 > 0$ so small that for any given $\theta \in [0, \theta_0]$, the Cauchy problem (1.1) and (1.8) admits a unique global C^4 solution $u = u(x,t)$ on $t \geq 0$. For all $i = 1, \cdots, n$,*

$$|v_i(x,t)| \le C\theta(1+|x-\lambda_i t|)^{-(1+\mu)}, \qquad (x,t) \in R^2 \setminus D_i, \tag{3.3}$$

$$|w_i(x,t)| \le C\theta(1+|x-\lambda_i t|)^{-(1+\mu)}, \qquad (x,t) \in R^2 \setminus D_i, \tag{3.4}$$

$$|\rho_i(x,t)| \le C\theta(1+|x-\lambda_i t|)^{-(1+\mu)}, \qquad (x,t) \in R^2 \setminus D_i, \tag{3.5}$$

$$|\alpha_i(x,t)| \le C\theta(1+|x-\lambda_i t|)^{-(1+\mu)}, \qquad (x,t) \in R^2 \setminus D_i. \tag{3.6}$$

Furthermore, we have

$$|\rho_i(x,t)| \le C\theta, \quad |\alpha_i(x,t)| \le C\theta, \qquad (x,t) \in R^2. \tag{3.7}$$

Proof. It is easy by using the method in [3].

Secondly, We will give the estimates of v_i, w_i. But we need a rough estimate for ρ_i if we want to reduce the estimate of w_i. We prove the theorem in three steps.

Step 1. The rough estimate of ρ_i

We linearized (3.1) at the neighborhood of $u=0$,

$$\begin{aligned}\Big(\frac{\partial}{\partial t}+\lambda_i\frac{\partial}{\partial x}\Big)\rho_i&=\sum_{j,k=1}^{n}\frac{\partial\gamma_{ijk}}{\partial x}(u)w_jw_k+2\sum_{j,k=1}^{n}\gamma_{ijk}(u)w_j\rho_k-\bigtriangledown\lambda_i(u)u_x\rho_i\\&\quad+(b_i(u))_{xx}+\sum_{j,k=1}^{n}(\xi_{ijk}(u)w_jb_k)_x-(\lambda_i(u)-\lambda_i)\alpha_i\\&\doteq f(x,t).\end{aligned}\tag{3.8}$$

For fixed $(x,t)\in D_i$, suppose that the line $Y-x=\lambda_i(S-t)$ intersects the line $Y=(\lambda_i+\delta_0)S$ (resp. $Y=(\lambda_i-\delta_0)S$) at the point $(y,\frac{y}{\lambda_i+\delta_0})$ (resp. $(y,\frac{y}{\lambda_i-\delta_0})$). Without loss of generality, we only consider the first case, i.e. $y=\frac{\lambda_i+\delta_0}{\delta_0}(x-\lambda_it)$. As we know that the line between (x,t) and $(y,\frac{y}{\lambda_i+\delta_0})$ included in D_i, so we integrate (3.7) from $\frac{y}{\lambda_i+\delta_0}$ to t,

$$\begin{aligned}\rho_i(x,t)&=\rho_i\Big(y,\frac{y}{\lambda_i+\delta_0}\Big)+\int_{\frac{y}{\lambda_i+\delta_0}}^{t}\int_R\delta(x-y-\lambda_i(t-s))f(y,s)dyds\\&=\rho_i\Big(y,\frac{y}{\lambda_i+\delta_0}\Big)+\int_{\frac{y}{\lambda_i+\delta_0}}^{t}f(x-\lambda_i(t-s),s)ds.\end{aligned}\tag{3.9}$$

We now estimate the right side terms of (3.8). From (3.5),

$$\Big|\rho_i\Big(y,\frac{y}{\lambda_i+\delta_0}\Big)\Big|\le C\theta\Big(1+\Big|y-\frac{\lambda_iy}{\lambda_i+\delta_0}\Big|\Big)^{-(1+\mu)}\le C\theta(1+|x-\lambda_it|)^{-(1+\mu)}.\tag{3.10}$$

From Lemma 2.3 and (2.14),

$$\begin{aligned}&\int_{\frac{y}{\lambda_i+\delta_0}}^{t}\Big|\sum_{j\ne k}\frac{\partial\gamma_{ijk}}{\partial x}w_jw_k\Big|(x-\lambda_i(t-s),s)ds\\&\le CM^2(t)\int_{\frac{y}{\lambda_i+\delta_0}}^{t}\sum_{j\ne k}F_{i,j,k}ds\le CM^2(t)(1+|x-\lambda_it|)^{-(1+\mu)}.\end{aligned}\tag{3.11}$$

From (1.22), using Hadamard's formula and (2.1), we have

$$\int_{\frac{y}{\lambda_i+\delta_0}}^{t}\Big|\sum_{j\ne i}\frac{\partial\gamma_{ijj}}{\partial x}w_j^2(x-\lambda_i(t-s),s)\Big|ds\le C\theta M(t)(1+|x-\lambda_it|)^{-(1+\mu)}.\tag{3.12}$$

From the definition of $M(t)$ and the relation (1.22) for $j=i$,

$$\int_{\frac{y}{\lambda_i+\delta_0}}^{t}\Big|\frac{\partial\gamma_{iii}}{\partial x}w_i^2(x-\lambda_i(t-s),s)\Big|ds\le C\theta M(t)(1+|x-\lambda_it|)^{-(1+\mu)}.\tag{3.13}$$

From (3.5) in Lemma 3.1,

$$|\rho_k(x-\lambda_i(t-s),s)|\le C\theta(1+|x-\lambda_i(t-s)-\lambda_ks|)^{-(1+\mu)}\qquad\text{for all}\quad k\ne i.\tag{3.14}$$

From Lemma 2.3 and (3.5),

$$\int_{\frac{y}{\lambda_i+\delta_0}}^{t}\Big|\sum_{j\ne i,k\ne i}\gamma_{ijk}w_j\rho_k(x-\lambda_i(t-s),s)\Big|ds\le C\theta M(t)(1+|x-\lambda_it|)^{-(1+\mu)},$$

$$\int_{\frac{y}{\lambda_i+\delta_0}}^{t}\Big|\sum_{k=1}^{n}\gamma_{iik}w_i\rho_k(x-\lambda_i(t-s),s)\Big|ds\le C\theta M(t)(1+|x-\lambda_it|)^{-(1+\mu)}.$$

Since $(x-\lambda_i(t-s),s)\in D_i$, for $l\ne i$,

$$|x-\lambda_i(t-s)-\lambda_ls|\ge|\lambda_l-\lambda_i|s-|x-\lambda_it|\ge\delta_0s\ge|x-\lambda_it|.\tag{3.15}$$

From (3.7) in Lemma 3.1,

$$\begin{aligned}&\int_{\frac{y}{\lambda_i+\delta_0}}^{t}\Big|\sum_{j\neq i}\gamma_{iji}w_j\rho_i(x-\lambda_i(t-s),s)\Big|ds\\ &\leq C\theta M(t)\int_{\frac{y}{\lambda_i+\delta_0}}^{t}(1+|x-\lambda_i(t-s)-\lambda_j s|)^{-(1+\mu)}ds\\ &\leq C\theta M(t)(1+|x-\lambda_i t|)^{-(1+\mu)}t. \qquad (3.16)\end{aligned}$$

Because in the normalized coordinate, $\nabla\lambda_i\cdot r_i(u_ie_i)=0$, from Lemma 2.2, (3.4)–(3.6), and using Hadamard's formula for $\nabla\lambda_i\cdot r_i(u)$, we have

$$\begin{aligned}&\int_{\frac{y}{\lambda_i+\delta_0}}^{t}\Big|\Big[\sum_{k\neq i}w_k\bigtriangledown\lambda_i r_k\rho_i(x-\lambda_i(t-s),s)+w_i\bigtriangledown\lambda_i r_i\rho_i(x-\lambda_i(t-s),s)\Big]\Big|ds\\ &\leq C\theta M(t)\int_{\frac{y}{\lambda_i+\delta_0}}^{t}\sum_{k\neq i}(1+|x-\lambda_i t-(\lambda_j-\lambda_i)s|)^{-(1+\mu)}ds\\ &\leq C\theta M(t)(1+|x-\lambda_i t|)^{-(1+\mu)}t. \qquad (3.17)\end{aligned}$$

We can calculate directly

$$(b_i(u))_{xx}=\sum_{k,j=1}^{n}\tilde b_{ijk}(u)w_jw_k+\sum_{k=1}^{n}\tilde b_{ik}(u)\rho_k,$$

where $\tilde b_{ijk}(u_je_j)=\frac{\partial\tilde b_{ik}}{\partial u_j}(u_je_j)=0$. The estimate of the term $(b_i(u))_{xx}$ is similar to the above terms. For the estimate of $\sum\limits_{j,k=1}^{n}(\xi_{ijk}(u)w_jb_k)_x$, we only need to consider the term $b_i(u)$ and $b_i(u)_x$. From (1.23)–(1.25), Lemmas 2.3 and 2.5,

$$\begin{aligned}&\Big|\int_0^t b_i(u)(x-\lambda_i s,t-s)ds\Big|\\ &\leq C\int_0^t\sum_{j\neq k}|u_ju_k(x-\lambda_i s,t-s)|ds\\ &\leq CM^2(t)\int_0^t\sum_{j\neq k}F_{i,j,k}ds\leq C(1+|x-\lambda_i t|)^{-(1+\mu)}M^2(t). \qquad (3.18)\end{aligned}$$

It is similar to the estimate of $b_i(u)_x$, and

$$\Big|\int_0^t(b_i(u))_x(x-\lambda_i s,t-s)ds\Big|\leq C(1+|x-\lambda_i t|)^{-(1+\mu)}M^2(t).$$

Now, we need to estimate the last term. When $(x-\lambda_i(t-s),s)\in D_i$, from (3.5), (3.7) and $\lambda_i=\lambda_i(u_ie_i)=0$,

$$|\lambda_i-\lambda_i(u)|=\Big|\int_0^t\sum_{k\neq i}\frac{\partial}{\partial u_k}\lambda_i(su_1,su_2,\cdots,su_n)dsu_k\Big|, \qquad (3.19)$$

$$\int_{\frac{y}{\lambda_i+\delta_0}}^{t}|(\lambda_i-\lambda_i(u))\alpha_i(x-\lambda_i(t-s),s)|ds\leq C\theta M(t)t(1+|x-\lambda_i t|)^{-(1+\mu)}. \qquad (3.20)$$

Combining these inequalities above, we obtain that for $(x,t)\in D_i$,

$$|\rho_i(x,t)|\leq C\theta(1+|x-\lambda_i t|)^{-(1+\mu)}+C\theta M(t)t(1+|x-\lambda_i t|)^{-(1+\mu)}. \qquad (3.21)$$

Step 2. The estimate of v_i

Now, we will get estimates of v_i from the first equation of (1.14). First of all, we linearize this equation at the neighborhood of $u=0$,

$$\Big(\frac{\partial}{\partial t}+\lambda_i\frac{\partial}{\partial x}\Big)v_i=\sum_{j,k=1}^n\beta_{ijk}(u)v_jw_k+\sum_{j,k=1}^n v_{ijk}(u)v_jb_k(u)+b_i(u)+(\lambda_i-\lambda_i(u))\frac{\partial v_i}{\partial x}$$
$$\doteq f_1(x,t). \tag{3.22}$$

Then we get the expression of solutions for the systems, where δ is the Dirac function,

$$\begin{aligned}v_i(x,t)&=\int_R\delta(x-y-\lambda_it)v_i(y,0)dy+\int_0^t\int_R\delta(x-y-\lambda_i(t-s))f_1(y,s)dsdy\\&=v_i(x-\lambda_it,0)+\int_0^t\Big[\sum_{j,k=1}^n\beta_{ijk}(u)v_jw_k+\sum_{j,k=1}^n v_{ijk}(u)v_jb_k(u)+b_i(u)\\&\quad+(\lambda_i-\lambda_i(u))\Big(w_i-\sum_{j,k=1}^n l_i\bigtriangledown r_jr_kv_jw_k\Big)\Big](x-\lambda_is,t-s)ds\\&\doteq B_1+B_2+B_3+B_4+B_5.\end{aligned}\tag{3.23}$$

From (1.8), we know that

$$\begin{aligned}|B_1|&=|v_i(x-\lambda_it,0)|=|l_i(u)\cdot u(x-\lambda_it,0)|\\&\le C\theta(1+|x-\lambda_it|)^{-(1+\mu)}.\end{aligned}$$

Rewrite B_2 as

$$\begin{aligned}B_2&=\int_0^t\sum_{j\neq k,k\neq i}\beta_{ijk}(u)v_jw_kds+\int_0^t\sum_{k\neq i}\beta_{ikk}(u)v_kw_kds\\&\doteq B_{21}+B_{22}.\end{aligned}\tag{3.24}$$

From (2.14), (2.4) and Lemma 2.3,

$$|B_{21}|\le C\int_0^t\sum_{j\neq k}F_{i,j,k}ds\le C(1+|x-\lambda_it|)^{-(1+\mu)}M^2(t).\tag{3.25}$$

From (1.21), in the normalized coordinate, we get

$$\beta_{iii}(u)=\beta_{iii}(u)-\beta_{iii}(u_ie_i)=\int_0^1\sum_{k\neq i}\frac{\partial\beta_{iii}}{\partial u_k}(su_1,\cdots,u_i,\cdots,su_n)dsu_k.\tag{3.26}$$

From (2.2), (2.14), (3.26) and (2.15), we have the following inequality

$$\begin{aligned}|B_{22}|&\le C\int_0^t\sum_{j\neq k,k\neq i}|u_jv_kw_k(x-\lambda_is,t-s)|ds\\&\le C(1+|x-\lambda_it|)^{-(1+\mu)}M^2(t).\end{aligned}\tag{3.27}$$

From (3.24)–(3.27), we have

$$|B_2|\le C(1+|x-\lambda_it|)^{-(1+\mu)}M^2(t).\tag{3.28}$$

Noting (2.2), (3.17), thus

$$|B_4|\le C(1+|x-\lambda_it|)^{-(1+\mu)}M^2(t).\tag{3.29}$$

Furthermore, it is easy to check the following inequality

$$\begin{aligned}|B_3| &= \Big|\int_0^t \sum_{j,k=1}^n \gamma_{ijk}(u)v_j b_k(x-\lambda_i s, t-s)ds\Big| \\ &\le C\Big|\int_0^t \sum_k b_k(u)(x-\lambda_i s, t-s)ds\Big| \\ &\le C(1+|x-\lambda_i t|)^{-(1+\mu)}M^2(t). \end{aligned} \tag{3.30}$$

Now we estimate the term B_5,

$$\begin{aligned}B_5 &= \int_0^t (\lambda_i - \lambda_i(u))w_i(x-\lambda_i s, t-s)ds \\ &\quad - \int_0^t (\lambda_i - \lambda_i(u)) \sum_{j,k=1}^n l_i \bigtriangledown r_j r_k v_j w_k(x-\lambda_i s, t-s)ds \\ &\doteq B_{51} + B_{52}. \end{aligned} \tag{3.31}$$

Notice that from (2.15), (3.19) and (2.6),

$$\begin{aligned}|B_{51}| &\le \int_0^t |(\lambda_i - \lambda_i(u))w_i(x-\lambda_i(t-s), s)|ds \\ &\le CM^2(t)(1+|x-\lambda_i t|)^{-(1+\mu)}. \end{aligned} \tag{3.32}$$

For the estimate of B_{52}, because of (1.21), (3.19) and (3.28), we have

$$\begin{aligned}|B_{52}| &\le \int_0^t \Big|\sum_{k\neq i} \frac{\lambda_i - \lambda_i(u)}{\lambda_k(u) - \lambda_i(u)} \beta_{ijk} v_j w_k\Big|(x-\lambda_i(t-s), s)ds \\ &\le CM^2(t)(1+|x-\lambda_i t|)^{-(1+\mu)}. \end{aligned} \tag{3.33}$$

Now we can get that

$$|v_i(x,t)| \le C\theta(1+|x-\lambda_i t|)^{-(1+\mu)} + CM^2(t)(1+|x-\lambda_i t|)^{-(1+\mu)}. \tag{3.34}$$

Step 3. The estimate of w_i

We linearize the second equation of (1.14). Then

$$\begin{aligned}\Big(\frac{\partial}{\partial t} + \lambda_i \frac{\partial}{\partial x}\Big)w_i &= \sum_{j,k=1}^n \gamma_{ijk}(u)w_j w_k + \sum_{j,k=1}^n (\xi_{ijk}(u)w_j b_k)_x \\ &\quad + (b_i(u))_x + (\lambda_i - \lambda_i(u))\rho_i \\ &\doteq f_2(x,t). \end{aligned} \tag{3.35}$$

We just need to estimate $w_i(x,t)$ for $(x,t)\in D_i$ because of (3.4) for $(x,t)\in R^2\setminus D_i$, here y has the same meaning as in Step 1. Similar to Step 2, it follows that

$$w_i(x,t) = w_i\Big(y, \frac{y}{\lambda_i+\delta_0}\Big) + \int_{\frac{y}{\lambda_i+\delta_0}}^t f_2(x-\lambda_i(t-s), s)ds. \tag{3.36}$$

From (3.4) we know that

$$|w_i(y, \frac{y}{\lambda_i+\delta_0})| \le C\theta(1+|x-\lambda_i t|)^{-(1+\mu)}.$$

The estimate of $f_2(x,t)$ is similar to that of $v_i(x,t)$ except for the last term.

We will use (3.19), (3.21), Lemmas 2.5 and 2.2, and furthermore we need the condition $\mu > 1$.

$$\begin{aligned}&(1+|x-\lambda_i t|)^{1+\mu}\int_{\frac{y}{\lambda_i+\delta_0}}^{t} |[\lambda_i-\lambda_i(u)]\rho_i(x-\lambda_i(t-s),s)|ds\\ &\le C\theta M(t)+C\theta M^2(t)\int_{\frac{y}{\lambda_i+\delta_0}}^{t}\sum_{j\neq i}(1+|x-\lambda_i(t-s)-\lambda_j s|)^{-(1+\mu)}sds.\end{aligned} \tag{3.37}$$

Because $(x,t)\in D_i$, we have

$$|x-\lambda_i(t-s)-\lambda_j s|\ge|\lambda_j-\lambda_i|s-|x-\lambda_i t|\ge\delta_0 s \qquad \text{for} \quad \frac{y}{\lambda_i+\delta_0}\le s\le t.$$

Furthermore since $\mu > 1$,

$$\int_{\frac{y}{\lambda_i+\delta_0}}^{t}\sum_{j\neq i}(1+|x-\lambda_i(t-s)-\lambda_j s|sds\le C\int_{\frac{y}{\lambda_i+\delta_0}}^{t}(1+s)^{-(1+\mu)}sds\le C. \tag{3.38}$$

From (2.14) and the above estimates,

$$(1+|x-\lambda_i t|)^{1+\mu}|w_i(x,t)|\le C\theta+C\theta M(t)+CM^2(t). \tag{3.39}$$

Combining the inequalities (3.34), (3.39), we see that there is a positive constant C such that

$$M(t)\le C\theta+C\theta M(t)+CM^2(t). \tag{3.40}$$

Since θ is sufficiently small, by continuity we have

$$M(t)\le C\theta. \tag{3.41}$$

This completes the proof of the main theorem.

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