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ON THE FUNDAMENTAL GROUP OF OPEN MANIFOLDS WITH NONNEGATIVE RICCI CURVATURE****

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Abstract

The authors establish some uniform estimates for the distance to halfway points of minimal geodesics in terms of the distantce to end points on some types of Riemannian manifolds, and then prove some theorems about the finite generation of fundamental group of Riemannian manifold with nonnegative Ricci curvature, which support the famous Milnor conjecture.

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§1. Introduction

Let (M, g) be a complete noncompact n-manifold with nonnegative Ricci curvature or even Ricci curvature bounded from below. The topological properties of (M, g) have received much attention recently. Numerous works have been done under the same title "curvature and topology". We can see, for example, [1, 7, 8]. In this paper, we study the fundamental group of complete Riemannian manifolds M of nonnegative Ricci curvature $\mathrm{Ric}_M \geq 0$.

Recall that if M is compact (without boundary), then its fundamental group $\pi_1(M)$ is finitely generated, and if M is compact with $\mathrm{Ric}_M > 0$, then by Myers Theorem $\pi_1(M)$ is finite. If M is noncompact, then the most basic question is whether $\pi_1(M)$ is finitely generated. In 1968, Milnor[6] conjectured that every complete noncompact manifold M^n with nonnegative Ricci curvature must have finitely generated fundamental group. There have been many works supporting this conjecture, see [2, 3, 4, 6] etc.

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Recently, in C. Sormani's paper [9], she studied the relation between fundamental group $\pi_1(M)$ and the diameter growth of a complete Riemannian manifold M with nonnegative Ricci curvature $\mathrm{Ric}_M \geq 0$, which supports Milnor conjecture strongly. In this paper, we will show that the universal constant obtained in [9] can be improved greatly. We also get a kth-Ricci version of the estimate and give a similar theorem. Furthermore, we show that nonnegative Ricci curvature is not a necessary condition for such an estimate. For example, we have a similar estimate for complete Riemannian manifold with Ricci curvature bounded below by a negative constant, which improves Sormani's correspondent lemma in [10].

§2. Basic Estimates

In this section we will establish some uniform estimates for the distance to halfway points in terms of the distance to end points on some types of Riemannian manifolds.

First let us recall the concept of the excess function. The excess function $e_{pq}(x)$ is defined as

$$e_{pq}(x) = d(x,p) + d(x,q) - d(p,q), \quad \forall x \in M,$$

where $p, q \in M$ and d denotes the distance function on M induced from the metric.

In their pioneering work[1], Abresh and Gromoll give an important excess estimate for "thin" triangular, which was used by them and many others to prove many topological finite results on many types of Riemannian manifolds. To be precise, they proved

Lemma 2.1.[1, 9, 10] *Let M be a complete Riemannian n-manifold with Ricci curvature* $\mathrm{Ric}_M \geq -(n-1)k$, $n \geq 3$. *Denote $r_0 = d(x, \gamma(0))$ and $r_1 = d(x, \gamma(D))$, where γ is a minimal geodesic with length $L(\gamma) = D$. Suppose $l = d(x, \gamma) \leq \min\{r_0, r_1\}$. Then*

$$e_{\gamma(0),\gamma(D)}(x) = r_0 + r_1 - D \leq 2\Big(\frac{n-1}{n-2}\Big)\Big(\frac{1}{2}C_3 l^n\Big)^{1/n-1},$$

where

(1) *if $k = 0$, then*

$$C_3 = \frac{n-1}{n}\Big(\frac{1}{r_0 - l} + \frac{1}{r_1 - l}\Big),$$

(2) *if $k > 0$, then*

$$C_3 = \frac{n-1}{n}\Big(\frac{\sinh\sqrt{k}l}{\sqrt{k}l}\Big)^{n-1}\sqrt{k}[\coth\sqrt{k}(r_0 - l) + \coth\sqrt{k}(r_1 - l)].$$

Now we can give our main estimate.

Lemma 2.2. *Let M^n be a complete Riemannian n-manifold with Ricci curvature* $\mathrm{Ric}_M \geq -(n-1)k$, *where $n \geq 3$. Let γ be a minimal geodesic from $\gamma(0)$ to $\gamma(D)$ with length $L(\gamma) = D$. If $x \in M$ is a point with $d(x, \gamma(0)) \geq (\frac{1}{2} + \varepsilon_1)D$ and $d(x, \gamma(D)) \geq (\frac{1}{2} + \varepsilon_2)D$, then*

$$d(x, \gamma(D/2)) \geq \alpha(\varepsilon_1, \varepsilon_2)D,$$

where

(1) *if $k = 0$, then*

$$\alpha(\varepsilon_1, \varepsilon_2) = \min\Big\{\frac{1}{4}, \Big(\frac{\varepsilon_1 + \varepsilon_2}{2}\Big)^{\frac{n-1}{n}}\Big(\frac{1}{4}\frac{n}{n-1}\Big(\frac{n-2}{n-1}\Big)^{n-1}\Big)^{\frac{1}{n}}\Big\},$$

(2) *if* $k>0$ *and* $D\le 1$, *then*

$$\alpha(\varepsilon_1,\varepsilon_2)=\min\Big\{\frac{1}{4},\Big(\frac{\varepsilon_1+\varepsilon_2}{2}\Big)^{\frac{n-1}{n}}\Big(\frac{n-2}{n-1}\Big)^{\frac{n-1}{n}}\Big(\frac{n}{n-1}\frac{\sqrt{k}}{\coth\frac{\sqrt{k}}{4}}\Big(\frac{\frac{\sqrt{k}}{4}}{\sinh\frac{\sqrt{k}}{4}}\Big)^{n-1}\Big)^{\frac{1}{n}}\Big\}.$$

Proof. Suppose on the contrary, $d(x,\gamma(D/2))\le\alpha(\varepsilon_1,\varepsilon_2)D$.

First, by definition, we can get

$$e_{\gamma(0),\gamma(D)}(x)=d(x,\gamma(0))+d(x,\gamma(D))-D\ge(\varepsilon_1+\varepsilon_2)D.$$

On the other hand, we have

$$l=d(x,\gamma)\le d(x,\gamma(D/2))\le\alpha(\varepsilon_1,\varepsilon_2)D\le\min\{r_0,r_1\}.$$

Thus by Lemma 2.1, we have

$$e_{\gamma(0),\gamma(D)}(x)\le 2\Big(\frac{n-1}{n-2}\Big)\Big(\frac{1}{2}C_3l^n\Big)^{1/n-1}.$$

Note that $r_0-l\ge D/4$.

(1) If $\mathrm{Ric}_M\ge 0$, together with $\alpha(\varepsilon_1,\varepsilon_2)\le\frac{1}{4}$ and Lemma 2.1, we have

$$\begin{aligned}C_3&\le\frac{n-1}{n}\Big(\frac{1}{(\frac{1}{2}+\varepsilon_1-\alpha(\varepsilon_1,\varepsilon_2))D}+\frac{1}{(\frac{1}{2}+\varepsilon_1-\alpha(\varepsilon_1,\varepsilon_2))D}\Big)\\&<\frac{n-1}{n}\frac{8}{D}.\end{aligned}$$

Thus by Lemma 2.1,

$$\begin{aligned}(\varepsilon_1+\varepsilon_2)D&<2\Big(\frac{n-1}{n-2}\Big)\Big(\frac{1}{2}\frac{n-1}{n}\frac{8}{D}(\alpha(\varepsilon_1,\varepsilon_2))^nD^n\Big)^{\frac{1}{n-1}}\\&=2\Big(\frac{n-1}{n-2}\Big)\Big(4\frac{n-1}{n}\Big)^{\frac{1}{n-1}}(\alpha(\varepsilon_1,\varepsilon_2))^{\frac{n}{n-1}}D.\end{aligned}$$

So we get

$$\begin{aligned}\alpha(\varepsilon_1,\varepsilon_2)&>\Big(\frac{\varepsilon_1+\varepsilon_2}{2}\frac{n-2}{n-1}\Big(\frac{n}{4(n-1)}\Big)^{\frac{1}{n-1}}\Big)^{\frac{n-1}{n}}\\&=\Big(\frac{\varepsilon_1+\varepsilon_2}{2}\Big)^{\frac{n-1}{n}}\Big(\frac{1}{4}\frac{n}{n-1}\Big(\frac{n-2}{n-1}\Big)^{n-1}\Big)^{\frac{1}{n}},\end{aligned}$$

which is a contradiction to the definition of $\alpha(\varepsilon_1,\varepsilon_2)$.

(2) Now suppose $\mathrm{Ric}_M\ge-(n-1)k$. Noting that $\frac{\sinh l}{l}$ and $\coth(r-l)$ are both increasing functions of l and that $D<1$, from Lemma 2.1 we know that

$$\begin{aligned}C_3&=\frac{n-1}{n}\Big(\frac{\sinh\sqrt{k}l}{\sqrt{k}l}\Big)^{n-1}\sqrt{k}[\coth\sqrt{k}(r_0-l)+\coth\sqrt{k}(r_1-l)]\\&<\frac{n-1}{n}\Big(\frac{\sinh\frac{\sqrt{k}}{4}}{\frac{\sqrt{k}}{4}}\Big)^{n-1}\sqrt{k}\Big(2\coth\frac{\sqrt{k}}{4}\Big).\end{aligned}$$

Thus

$$\begin{aligned}(\varepsilon_1+\varepsilon_2)D&<2\Big(\frac{n-1}{n-2}\Big)\Big(\frac{1}{2}\frac{n-1}{n}\Big(\frac{\sinh\frac{\sqrt{k}}{4}}{\frac{\sqrt{k}}{4}}\Big)^{n-1}\sqrt{k}\Big(2\coth\frac{\sqrt{k}}{4}\Big)(\alpha(\varepsilon_1,\varepsilon_2))^nD^n\Big)^{\frac{1}{n-1}}\\&\le 2\Big(\frac{n-1}{n-2}\Big)\Big(\frac{n-1}{n}\Big(\frac{\sinh\frac{\sqrt{k}}{4}}{\frac{\sqrt{k}}{4}}\Big)^{n-1}\sqrt{k}\Big(\coth\frac{\sqrt{k}}{4}\Big)(\alpha(\varepsilon_1,\varepsilon_2))^n\Big)^{\frac{1}{n-1}}D\end{aligned}$$

and

$$\alpha(\varepsilon_1,\varepsilon_2) > \Big(\frac{\varepsilon_1+\varepsilon_2}{2}\Big)^{\frac{n-1}{n}}\Big(\frac{n-2}{n-1}\Big)^{\frac{n-1}{n}}\Big(\frac{n}{n-1}\frac{\sqrt{k}}{\coth\frac{\sqrt{k}}{4}}\Big(\frac{\frac{\sqrt{k}}{4}}{\sinh\frac{\sqrt{k}}{4}}\Big)^{n-1}\Big)^{\frac{1}{n}},$$

which is a contradiction.

We can see that the excess estimate is crucial to our lemma. Note that Shen[8] gives a kth-Ricci version excess estimate, thus in the same way, we have a similar estimate for complete Riemannian n-manifold with k-th Ricci curvature $\mathrm{Ric}_M^k \ge 0$.

Lemma 2.3. *Let M^n be a complete Riemannian n-manifold with k-th Ricci curvature $\mathrm{Ric}_M^k \ge 0$. Let γ be a minimal geodesic from $\gamma(0)$ to $\gamma(D)$ with length $L(\gamma)=D$. If $x\in M$ is a point with $d(x,\gamma(0)) \ge (\frac12+\varepsilon)D$ and $d(x,\gamma(D)) \ge (\frac12+\varepsilon)D$, then*

$$d(x,\gamma(D/2)) \ge \alpha_k(\varepsilon)D,$$

where

$$\alpha_k(\varepsilon)=\frac14\varepsilon^{\frac{k}{k+1}}.$$

Proof. Suppose on the contrary, $d(x,\gamma(D/2)) < \alpha_k(\varepsilon)D$.

Obviously we have

$$e_{\gamma(0),\gamma(D)}(x) \ge 2\varepsilon D.$$

On the other hand,

$$\begin{aligned} &s(x)=\min\{d(x,\gamma(0)),d(x,\gamma(D))\} \ge \Big(\frac12+\varepsilon\Big)D,\\ &h(x)=d(x,\gamma)<\alpha_k(\varepsilon)D. \end{aligned}$$

Thus by the excess estimate for Riemannian manifold with $\mathrm{Ric}_k \ge 0$ (cf. [8]), we can get

$$2\varepsilon D \le e_{\gamma(0),\gamma(D)}(x) < 8\alpha_k(\varepsilon)D\Big(\frac{\alpha_k(\varepsilon)D}{(\frac12+\varepsilon)D}\Big)^{1/k},$$

and

$$\alpha_k(\varepsilon) > \Big(\frac12+\varepsilon\Big)^{\frac{1}{k+1}}\varepsilon^{\frac{k}{k+1}}4^{-\frac{k}{k+1}} > \frac14\varepsilon^{\frac{k}{k+1}},$$

which is a contradiction.

§3. Fundamental Group and Nonnegative Ricci Curvature

Now we can give a constant better than Somani's in [9], and thus support Milnor conjecture more strongly. First we need a lemma.

Lemma 3.1.[9, 5] *Let M be a complete Riemannian n-manifold with fundamental group $\pi_1(M,x_0)$, where $x_0\in M$. Then there exists an ordered set of independent generators $\{g_1,g_2,g_3,\cdots\}$ of $\pi_1(M,x_0)$ with minimal representative geodesic loops γ_k of length d_k, such that*

$$d_M(\gamma_k(0),\gamma_k(d_k/2))=d_k/2.$$

Now we give a uniform cut lemma which is superior than Somani's correspondent one.

Lemma 3.2. *Let M^n be a complete manifold with nonnegative Ricci curvature, $n\ge 3$ and γ be a noncontractible geodesic loop based at a point $x_0\in M^n$ with length $L(\gamma)=D$, such that the following conditions hold:*

(1) *If σ based at x_0 is a loop homotopic to γ, then $L(\sigma) \geq D$.*

(2) *The loop γ is minimal on $[0, D/2]$ and is also minimal on $[D/2, D]$.*

Then for any $\varepsilon > 0$, there is a universal constant $\alpha(\varepsilon) = \alpha(\varepsilon, \varepsilon)$, such that if $x \in \partial B_{x_0}(RD)$, where $R \geq (\frac{1}{2} + \varepsilon)$, then

$$d_M(x, \gamma(D/2)) \geq \Big(R - \frac{1}{2}\Big)D + (\alpha(\varepsilon) - \varepsilon)D.$$

Similarly, if M has nonnegative k-Ricci curvature $\mathrm{Ric}_k \geq 0$*, then*

$$d_M(x, \gamma(D/2)) \geq \Big(R - \frac{1}{2}\Big)D + (\alpha_k(\varepsilon) - \varepsilon)D.$$

Proof. Note that by Lemma 3.1, such loop γ exists.

First we suppose that $R = (\frac{1}{2} + \varepsilon)$.

Let $\widetilde{M}$ be the universal covering of M, $\tilde{x}_0 \in \widetilde{M}$ be a lift of x_0, and $g \in \pi_1(M, x_0)$ be the element represented by the given loop γ. By conditions of γ, its lift $\tilde{\gamma}$ is a minimal geodesic running from $\tilde{x}_0$ to $g\tilde{x}_0$. Thus $d_{\widetilde{M}}(\tilde{x}_0, g\tilde{x}_0) = D$. Obviously we have

$$r_0 = d_{\widetilde{M}}(\tilde{x}, \tilde{x}_0) \geq d_M(x, x_0) = (\varepsilon + 1/2)D,$$
$$r_1 = d_{\widetilde{M}}(g\tilde{x}_0, \tilde{x}) \geq d_M(x, x_0) = (\varepsilon + 1/2)D.$$

Now we lift the minimal geodesic $C : [0, H] \to M^n$ from $\gamma(D/2)$ to x to a curve $\widetilde{C}$ in the universal cover, which runs from $\tilde{\gamma}(D/2)$ to a point $\tilde{x} \in \widetilde{M}$. Note that $L(\widetilde{C}) = L(C) = H$. Thus by applying Lemma 2.2 to $\tilde{x}$ and $\tilde{\gamma}$ in $\widetilde{M}$, we have

$$d(x, \gamma(D/2)) = L(C) = L(\widetilde{C}) = d(\tilde{x}, \tilde{\gamma}(D/2)) \geq \alpha(\varepsilon)D.$$

For $R \geq \frac{1}{2} + \varepsilon$, take $x \in \partial B_{x_0}(RD)$ and suppose that $y \in \partial B_{x_0}((\frac{1}{2} + \varepsilon)D)$ lies in the minimal geodesic from x to $\gamma(D/2)$. Now we have

$$\begin{aligned} d_M(x, \gamma(D/2)) &= d_M(x, y) + d_M(y, \gamma(D/2)) \\ &\geq \Big(RD - \Big(\frac{1}{2} + \varepsilon\Big)D\Big) + \alpha(\varepsilon)D \\ &= \Big(R - \frac{1}{2}\Big)D + (\alpha(\varepsilon) - \varepsilon)D. \end{aligned}$$

The last assertion can be proved in the same way by using Lemma 2.3 and we omit the proof here.

Recall that the ray density function $D(r)$ is defined as

$$D(r) = \sup_{x \in \partial B_{x_0}(r)} \inf_{\text{rays } \gamma, \gamma(0) = x_0} d(x, \gamma(r)).$$

Now we can give our main theorems which improve Sormani's theorems and support Milnor conjecture strongly.

Theorem 3.1. *There exists a universal constant*

$$S_n = \frac{1}{4}\frac{1}{n-1}\Big(\frac{n-2}{n}\Big)^{n-1},$$

such that if M^n is complete and noncompact with nonnegative Ricci curvature and has small linear diameter growth,

$$\lim_{r \to \infty} \sup \frac{D(r)}{r} < 2S_n,$$

then it has a finitely generated fundamental group.

Proof. First let $f(\varepsilon)=\alpha(\varepsilon)-\varepsilon$. By elementary calculus we can see that the maximum point of f is $\varepsilon_0=(\frac{n-1}{n})^n(\frac{1}{4}\frac{n}{n-1}(\frac{n-2}{n-1})^{n-1})$ and the maximum value of f is $f(\varepsilon_0)=S_n=\frac{1}{4}\frac{1}{n-1}(\frac{n-2}{n})^{n-1}$.

Now we assume that M^n has infinitely generated fundamental group. Thus by Lemma 3.1, there is a sequence of generators g_k, whose minimal representative geodesic loops γ_k based at x_0, satisfying the hypothesis of Lemma 3.2. Let $d_k=L(\gamma_k)$. Note that d_k diverges to infinity.

Now for any γ_k we choose a ray γ such that

$$d(\gamma_k(d_k/2),\gamma(d_k/2))=\inf_{\text{rays }\gamma,\gamma(0)=x_0} d(\gamma_k(d_k/2),\gamma(d_k/2)).$$

Take $x_k=\gamma((1/2+\varepsilon_0))d_k$ and $y_k=\gamma((1/2)d_k)$. Then we have

$$d(y_k,\gamma_k(d_k/2))\geq d(x_k,\gamma_k(d_k/2))-d(x_k,y_k)\geq(\alpha(\varepsilon_0)-\varepsilon_0)d_k=S_nd_k.$$

So

$$\lim_{r\to\infty}\sup\frac{D(r)}{r}\geq\lim_{k\to\infty}\sup\frac{d(y_k,\gamma_k(d_k/2))}{d_k/2}\geq\lim_{k\to\infty}\sup\frac{S_nd_k}{d_k/2}=2S_n,$$

which is a contradiction.

Similarly, for $\mathrm{Ric}_k\geq 0$ we can get

Theorem 3.2. *There exists a universal constant*

$$S_k=\frac{1}{4(k+1)}\Big(\frac{k}{4(k+1)}\Big)^k,$$

such that if M^n is complete and noncompact with nonnegative Ricci curvature and has small linear diameter growth,

$$\lim_{r\to\infty}\sup\frac{\mathrm{diam}(\partial B_p(r))}{r}<2S_k,$$

then it has a finitely generated fundamental group.

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