Chin. Ann. Math.
24B:4(2003),475–484.

ON A HYPER HILBERT TRANSFORM****

CHEN Jiecheng*   DING Yong**   FAN Dashan***

Abstract

The authors define the directional hyper Hilbert transform and give its mixed norm estimate. The similar conclusions for the directional fractional integral of one dimension are also obtained in this paper. As an application of the above results, the authors give the L^p-boundedness for a class of the hyper singular integrals and the fractional integrals with variable kernel. Moreover, as another application of the above results, the authors prove the dimension free estimate for the hyper Riesz transform. This is an extension of the related result obtained by Stein.

Keywords Hyper Hilbert transform, Sobolev spaces, Dimension free estimate, Singular integral, Fractional integral
2000 MR Subject Classification 42B25, 42B99
Chinese Library Classification O177.8   **Document Code** A
Article ID 0252-9599(2003)04-0475-10

§1. Introduction

For $n \geq 2$, let S^{n-1} be the unit sphere in $\mathbb{R}^n$ with normalized Lebesgue measure $d\theta$. The classical directional Hilbert transform is defined, initially on the test space $\mathcal{S}(\mathbb{R}^n)$, by

$$Hf(x,\theta) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \text{sgn}(t)|t|^{-1} f(x-t\theta)dt, \tag{1.1}$$

where $(x,\theta) \in \mathbb{R}^n \times S^{n-1}$. Since the Hilbert transform plays a significant role in studying several different fields in mathematics, it, as well as its related maximal operators, has been well studied. Among many features of the Hilbert transform, one of the important properties is the boundedness of its mixed norm

$$\|Hf\|_{L^p(L^q)} = \Big\{\int_{\mathbb{R}^n}\Big(\int_{S^{n-1}}|Hf(x,\theta)|^q d\theta\Big)^{p/q} dx\Big\}^{1/p} \tag{1.2}$$

Manuscript received June 17, 2002.

*Department of Mathematics, Zhejiang University, Hangzhou 310028, China.
E-mail: jcchen@mail.hz.zj.cn

**Department of Mathematics, Beijing Normal University, Beijing 100875, China.
E-mail: dingy@bnu.edu.cn

***Department of Mathematics, University of Wisconsin, Milwaukee, WI 53201, USA.
E-mail: fan@uwm.edu

****Project supported by the 973 Project of China (No.G1999075105), the National Natural Science Foundation of China (No.19631080, No.10271016) and the Zhejiang Provincial Natural Science Foundation of China (No.RC97017, No.197042).

(see [6] for more details). For $0 \le \alpha < 1$, we can formally define a hyper Hilbert transform H_α,

$$H_\alpha f(x,\theta) = \text{p.v.}\frac{1}{\pi}\int_{\mathbb{R}} \operatorname{sgn}(t)|t|^{-1-\alpha} f(x-t\theta)dt. \tag{1.3}$$

However, for the practice reason, we will study a modified operator $\mathcal{H}_\alpha$ defined by

$$\mathcal{H}_\alpha f(x,\theta) = \frac{1}{\pi}\int_0^\infty t^{-1-\alpha}\{f(x-t\theta)-f(x)\}dt. \tag{1.4}$$

Similarly to the case of $\alpha = 0$, first we can easily obtain the following lemma.

Lemma 1.1.

$$\|\mathcal{H}_\alpha f(\cdot,\theta)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p_\alpha(\mathbb{R}^n)}, \quad 1<p<\infty,$$

where $\|f\|_{L^p_\alpha}$ *is the homogeneous Sobolev* L^p *norm of* f*, and the constant* C *is independent of* θ, f *and* n*.*

Proof. For fixed θ, pick a rotation R such that $R\theta = \mathbf{1} = (1,0,\cdots,0)$. Let R^{-1} be the inverse of R. For any function f, we denote the function f_R by $f_R = f(Rx)$ so that

$$f(x-t\theta)-f(x) = f_{R^{-1}}(Rx - t\mathbf{1}) - f_{R^{-1}}(Rx).$$

Let

$$x = (x_1,\overline{x}) \in \mathbb{R}\times\mathbb{R}^{n-1}.$$

Then

$$\begin{aligned}
\|\mathcal{H}_\alpha f(\cdot,\theta)\|^p_{L^p(\mathbb{R}^n)} &\sim \int_{\mathbb{R}^n}\Big|\int_0^\infty \{f_{R^{-1}}(Rx-t\mathbf{1}) - f_{R^{-1}}(Rx)\}t^{-1-\alpha}dt\Big|^p dx \\
&= \int_{\mathbb{R}^n}\Big|\int_0^\infty \{f_{R^{-1}}(x-t\mathbf{1}) - f_{R^{-1}}(x)\}t^{-1-\alpha}dt\Big|^p dx \\
&= \int_{\mathbb{R}^n}\Big|\int_0^\infty \{f_{R^{-1}}(x_1-t,\overline{x}) - f_{R^{-1}}(x_1,\overline{x})\}t^{-1-\alpha}dt\Big|^p dx \\
&= \int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}^1}\Big|\int_0^\infty \{f_{R^{-1}}(x_1-t,\overline{x}) - f_{R^{-1}}(x_1,\overline{x})\}t^{-1-\alpha}dt\Big|^p dx_1 d\overline{x}.
\end{aligned}$$

Let

$$h(x_1) = f_{R^{-1}}(x_1,\overline{x}).$$

Then it is easy to see that

$$\int_0^\infty \{h(u-t)-h(u)\}t^{-1-\alpha}dt = -\int_0^1\int_0^\infty h'(u-st)t^{-\alpha}dtds = C\mathcal{J}_{1-\alpha}(h')(u),$$

where

$$\mathcal{J}_\alpha(h)(u) = \int_0^\infty h(u-t)t^{-1+\alpha}dt.$$

By checking the Fourier transform, we can see that, up to a constant, $\mathcal{J}_\alpha$ is the one-dimension Riesz potential (see [13]). Thus

$$\Big\|\int_0^\infty \{h(\cdot-t)-h(\cdot)\}t^{-1-\alpha}dt\Big\|_{L^p(\mathbb{R}^1)} \le C\|h'\|_{L^p_{-1+\alpha}(\mathbb{R}^1)} \cong \|h\|_{L^p_\alpha(\mathbb{R}^1)}.$$

Therefore, we have

$$\|\mathcal{H}_\alpha f(\cdot,\theta)\|^p_{L^p(\mathbb{R}^n)} \le C\int_{\mathbb{R}^{n-1}}\|h\|^p_{L^p_\alpha(\mathbb{R}^1)}d\overline{x}.$$

Now by [17], we can obtain

$$\|\mathcal{H}_\alpha f(\cdot,\theta)\|^p_{L^p(\mathbb{R}^n)} \le C\|f_{R^{-1}}\|^p_{L^p_\alpha(\mathbb{R}^n)} = C\|f\|^p_{L^p_\alpha(\mathbb{R}^n)}, \tag{1.5}$$

and obviously, the constant C is independent of n and θ, and f.

In the second section, we will study the mixed norms of $\mathcal{H}_\alpha f$, as well as the mixed norm of the directional fractional integral $\mathcal{I}_\alpha f$. Some applications will be obtained in Sections 3 and 4. Parts of our arguments will be based on the ideas in [6], with some non-trivial computations.

§2. Mixed Norm Estimates

First we establish the following L^2 estimate.

Lemma 2.1. *For $\beta < \frac{1}{2} + \alpha$, we have*

$$\|\mathcal{H}_\alpha f\|_{L^2(L^2_\beta)} \le C\|f\|_{L^2_\alpha(\mathbb{R}^n)}.$$

Proof. Following the idea in [6], since $L^2_\beta(S^{n-1})$ is a Hilbert space, we have

$$\|\mathcal{H}_\alpha f\|_{L^2(L^2_\beta)} = \|\widehat{\mathcal{H}_\alpha f}\|_{L^2(L^2_\beta)}$$

with

$$\widehat{\mathcal{H}_\alpha f}(\xi,\theta) = \int_{\mathbb{R}^n} e^{-2\pi i(x,\xi)}\mathcal{H}_\alpha f(x,\theta)dx = \hat{f}(\xi)\int_0^\infty t^{-1-\alpha}\{e^{2\pi it(\xi,\theta)} - 1\}dt.$$

Thus we have

$$\widehat{\mathcal{H}_\alpha f}(\xi,\theta) = C\hat{f}(\xi)|\xi|^\alpha|\langle\xi',\theta\rangle|^\alpha \mathrm{sgn}\langle\xi',\theta\rangle,$$

where $\xi' = \xi/|\xi|$ and

$$C = C(\langle\xi',\theta\rangle) = \begin{cases} \displaystyle\int_0^\infty t^{-1-\alpha}\{e^{2\pi it} - 1\}dt, & \text{if } \ \langle\xi',\theta\rangle > 0, \\ \displaystyle\int_0^\infty t^{-1-\alpha}\{e^{-2\pi it} - 1\}dt, & \text{if } \ \langle\xi',\theta\rangle < 0. \end{cases}$$

Now to prove the lemma, it suffices to show that

$$\sup_{\xi'} \|C\mathrm{sgn}\langle\xi',\theta\rangle|\langle\xi',\theta\rangle|^\alpha\|_{L^2_\beta(S^{n-1})} < \infty. \tag{2.1}$$

For each ξ' fixed, we need to find the spherical harmonic development of $C\mathrm{sgn}\langle\xi',\theta\rangle|\langle\xi',\theta\rangle|^\alpha$. Let $\{Y_{m,j}\}$ be the spherical harmonic polynomials. By the Funk-Hecke formula[11], for large m,

$$\int_{S^{n-1}} C(\langle\xi',\theta\rangle)\mathrm{sgn}\langle\xi',\theta\rangle|\langle\xi',\theta\rangle|^\alpha Y_{m,j}(\theta)d\theta \cong \lambda_m Y_{m,j}(\xi') \tag{2.2}$$

with

$$\lambda_m = \int_{-1}^1 C(t)\mathrm{sgn}(t)P_m(n,t)|t|^\alpha(1-t^2)^{(n-3)/2}\,dt, \tag{2.3}$$

where

$$C(t) = \begin{cases} \displaystyle\int_0^\infty s^{-1-\alpha}\{e^{2\pi is} - 1\}ds, & \text{if } \ t > 0, \\ \displaystyle\int_0^\infty s^{-1-\alpha}\{e^{-2\pi is} - 1\}ds, & \text{if } \ t < 0. \end{cases}$$

Moreover, by the Rodrigues representation,

$$P_m(n,t) = (-1)^n R_m(n)(1-t^2)^{(3-n)/2}\,(d/dt)^m(1-t^2)^{m+(n-3)/2} \tag{2.4}$$

with the Rodrigues constant

$$R_m(n) = 2^{-m}\Gamma((n-1)/2)/\Gamma(m+(n-1)/2).$$

Now we need to estimate λ_m. Write $\lambda_m = C(\int_0^1 + \int_{-1}^0)$ in (2.3). Clearly we only need to estimate

$$\int_0^1 P_m(n,t)t^\alpha(1-t^2)^{(n-3)/2}\,dt,$$

since the estimate of another term is the same. By the definition of $P_m(n,t)$ and integration by parts,

$$\begin{aligned}
&\Big|\int_0^1 P_m(n,t)t^\alpha(1-t^2)^{(n-3)/2}\,dt\Big|\\
&=\Big|R_m(n)\int_0^1 t^\alpha(d\,/dt)^m(1-t^2)^{m+(n-3)/3}\,dt\Big|\\
&\le CR_m(n)\Big|\int_0^1 t^{\alpha-1}(d\,/dt)^{m-1}(1-t^2)^{m-1+(n-1)/2}\,dt\Big|\\
&= CR_m(n)\,/R_{m-1}(n+2)\,\Big|\int_0^1 t^{\alpha-1}(1-t^2)^{(n-1)/2}\,P_{m-1}(n+2,t)dt\Big|.
\end{aligned}$$

By the definition of $R_m(n)$, it is easy to check that $R_m(n)\,/R_{m-1}(n+2)\ \le C$, and C is independent of m. Therefore, let

$$N=n+2,\quad \gamma=1-\alpha.$$

Without loss of generality, we may write

$$\begin{aligned}
|\lambda_m| &\le \Big|\int_{\frac{1}{m}}^1 P_m(N,t)t^{-\gamma}(1-t^2)^{(N-3)/2}dt\Big| + \int_0^{\frac{1}{m}} |P_m(N,t)|t^{-\gamma}(1-t^2)^{(N-3)/2}dt\\
&:= I_1+I_2.
\end{aligned}$$

From [11, Chapter 8], we know that

$$|P_m(n,t)|\le Cm^{(2-n)/2}(1-t^2)^{(2-n)/2}. \tag{2.5}$$

Thus we have a constant C independent of m such that

$$I_2\le Cm^{-N/2+\gamma}=Cm^{-n/2-\alpha}.$$

By (2.4), using integration by parts, we have

$$\begin{aligned}
I_1 &\le CR_m(N)m^\gamma\Big|\Big\{\Big(\frac{d}{dt}\Big)^{m-1}(1-t^2)^{m+(N-3)/2}\Big\}_{t=1/m}\Big|\\
&\quad + CR_m(N)\Big|\int_{\frac{1}{m}}^1 t^{-\gamma-1}\Big(\frac{d}{dt}\Big)^{m-1}(1-t^2)^{m-1+(N-1)/2}dt\Big|\\
&:= J_1+J_2.
\end{aligned}$$

By the definition of P_m,

$$J_2\le CR_m(N)\,/R_{m-1}(N+2)\,\Big|\int_{\frac{1}{m}}^1 t^{-\gamma-1}(1-t^2)^{(n-1)/2}\,P_{m-1}(N+2,t)dt\Big|$$

So using (2.5), we have

$$J_2\le Cm^{-N/2}\int_{\frac{1}{m}}^1 t^{-\gamma-1}(1-t^2)^{-1/2}dt\le Cm^{-N/2+\gamma}=Cm^{-n/2-\alpha}. \tag{2.6}$$

By the definition of $P_m(n,t)$ and (2.5),

$$\begin{aligned}
J_1 &= C\big[R_m(N)\,/R_{m-1}(N+2)\ \big]\,m^\gamma\big|P_{m-1}(N+2,1/m)(1-1/m^2)^{(N-1)/2}\big|\\
&\le Cm^{\gamma-N/2}=Cm^{-n/2-\alpha}.
\end{aligned}$$

From the estimates for I_1, J_1 and J_2, we now have the estimate

$$|\lambda_m| \le C m^{-n/2-\alpha}. \tag{2.7}$$

By the spherical harmonic development and (2.2), we have for each fixed ξ',

$$\operatorname{sgn}\langle \xi', \theta\rangle |\langle \xi', \theta\rangle|^\alpha = \sum_{m=1}^{\infty} \sum_{j=1}^{N(m)} \lambda_m Y_{m,j}(\xi') Y_{m,j}(\theta),$$

where $N(m) = O(m^{n-2})$, and $\sum\limits_{j=1}^{N(m)} |Y_{m,j}(\xi')|^2 = O(m^{n-2})$ (see [11] or [16]). Thus

$$\begin{aligned}\|\operatorname{sgn}\langle \xi', \theta\rangle |\langle \xi', \theta\rangle|^\alpha\|^2_{L^2_\beta(S^{n-1})} &\cong \sum_{m=1}^{\infty} \sum_{j=1}^{N(m)} |\lambda_m|^2 |m|^{2\beta} |Y_{m,j}(\xi')|^2 \\ &\cong \sum_{m=1}^{\infty} m^{-n-2\alpha+2\beta} m^{n-2} \\ &= \sum_{m=1}^{\infty} m^{-2-2\alpha+2\beta} < \infty,\end{aligned}$$

since $\beta < \frac{1}{2} + \alpha$. The lemma is proved.

Next, we define the directional fractional integral $\mathcal{I}_\alpha f$ by

$$\mathcal{I}_\alpha f(x, \theta) = \int_0^\infty t^{\alpha-1} f(x - t\theta) dt, \quad 0 < \alpha < 1.$$

Then following the same proof of Lemma 2.1, we have

Lemma 2.2. *For* $0 < \beta < \frac{1}{2} - \alpha, 0 < \alpha < \frac{1}{2}$,

$$\|\mathcal{I}_\alpha(f)\|_{L^2(L^2_\beta)} \le C\|f\|_{L^2_{-\alpha}(\mathbb{R}^n)}.$$

Theorem 2.1. *Let* $1 < p \le 2$.

(i) *For* $0 < \alpha < 1/2$ *and* $\sigma = 2n/(n + 2\alpha)$,

$$\|\mathcal{I}_\alpha(f)\|_{L^2(L^r)} \le C\|f\|_{L^\sigma} \tag{2.8}$$

for any r *with* $1 \le r < 2(n-1)/(n-2+2\alpha)$.

(ii) *For* $0 < \alpha < 1/2, 1 < q < p \le 2$ *with* $1/p = 1/q - \alpha/n$,

$$\|\mathcal{I}_\alpha(f)\|_{L^p(L^r)} \le C\|f\|_{L^q} \tag{2.9}$$

for any r *with*

$$1 \le r < \frac{2(n-1)}{2(n-1)(1-\beta) + \beta(n-2+2\alpha)},$$

where

$$\beta = \frac{1/p' - \alpha/n}{1/2 - \alpha/n}$$

(iii) *For* $0 \le \alpha < 1$ *and* $\gamma < (1 + 2\alpha)/p'$,

$$\|\mathcal{H}_\alpha(f)\|_{L^p(L^p_\gamma)} \le C\|f\|_{L^p_\alpha(\mathbb{R}^n)}.$$

Proof. (i) is obvious from Lemma 2.2, by using the Sobolev theorem. For any $s >$

$n/(n-\alpha)$, we have

$$\begin{aligned}\|\mathcal{I}_\alpha(f)\|_{L^s(L^1)} &= \Big\{\int_{\mathbb{R}^n}\Big(\int_{S^{n-1}}\Big|\int_0^\infty t^{-1+\alpha}f(x-t\theta)dt\Big|d\theta\Big)^s dx\Big\}^{1/s}\\ &\le \Big\{\int_{\mathbb{R}^n}\Big(\int_0^\infty t^{-1+\alpha}\int_{S^{n-1}}|f(x-t\theta)|d\theta dt\Big)^s dx\Big\}^{1/s}\\ &= C\|J_\alpha(|f|)\|_{L^s(\mathbb{R}^n)},\end{aligned}$$

where $J_\alpha(f)=\int_{\mathbb{R}^n} f(x-y)|y|^{-n+\alpha}dy$ is the classical fraction integral with order α and it is known that

$$\|J_\alpha f\|_{L^s(\mathbb{R}^n)} \le C\|f\|_{L^r(\mathbb{R}^n)}$$

with $1/s=1/r-\alpha/n$. This shows for any $s>n/(n-\alpha)$ that

$$\|\mathcal{I}_\alpha(f)\|_{L^s(L^1)} \le C\|f\|_{L^r}. \tag{2.10}$$

Thus (ii) comes from an interpolation between (2.8) and (2.10). By Lemma 1.1, it is easy to see that for any $1<q<\infty$,

$$\|\mathcal{H}_\alpha(f)\|_{L^q(L^q)} \le \|f\|_{L^q_\alpha(\mathbb{R}^n)}. \tag{2.11}$$

Thus (iii) follows by an interpolation between (2.11) and Lemma 2.1.

Two applications of Theorem 2.1 on the singular and fraction integrals will be studied in the following section.

§3. Singular and Fractional Integrals

A function $\Omega(x,z)$ defined on $\mathbb{R}^n\times\mathbb{R}^n$ is said to belong to $L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})$, $r\ge 1$, if it satisfies the following conditions:

(i) $\Omega(x,\lambda z)=\Omega(x,z)$ for all $\lambda>0$ and all $x,z\in\mathbb{R}^n$.

(ii) $\|\Omega\|_{L^\infty\times L^r} := \sup\limits_{x\in\mathbb{R}^n}(\int_{S^{n-1}}|\Omega(x,z')|^r d\sigma(z'))^{1/r}<\infty$, where $z'=z/|z|$.

For $0\le\alpha<1$, we define the operator $T_\alpha f(x)$ with variable singular kernel by

$$T_\alpha f(x)=\text{p.v.}\int_{\mathbb{R}^n}\Omega(x,x-y)|x-y|^{-n-\alpha}f(y)dy,$$

where $f\in\mathcal{S}(\mathbb{R}^n)$ and $\Omega\in L^\infty(\mathbb{R}^n)\times L^1(S^{n-1})$ satisfies

$$\int_{S^{n-1}}\Omega(x,y')d\sigma(y')=0 \quad \text{for any} \quad x\in\mathbb{R}^n. \tag{3.1}$$

Similarly, the fractional integral with variable kernel is defined by

$$\mathcal{F}_\alpha(f)(x)=\int_{\mathbb{R}^n}\Omega(x,x-y)|x-y|^{-n+\alpha}f(y)dy \quad \text{for} \quad 0<\alpha<n,$$

where Ω does not need to satisfy (3.1) (see [4, 5, 8, 10] for more information). We recall the following theorem by Calderón and Zygmund.

Theorem A.[2, 3] *Let $1<p\le 2$. If $\Omega\in L^\infty(\mathbb{R}^n)\times L^r(S^{n-1})$, $r>p'(n-1)/n$ satisfying (3.1), then there is a $C>0$ such that*

$$\|T_0 f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)} \quad \textit{for all} \quad f\in\mathcal{S}(\mathbb{R}^n).$$

As an extension of the above theorem, we will establish the following theorem.

Theorem 3.1. *Let $0\le\alpha<1, 1<p\le 2$ and $r>p'(n-1)/(n+2\alpha)$. If $\Omega\in L^\infty(\mathbb{R}^n)\times(H^r\cap L^1)(S^{n-1})$, where H^r is the Hardy space if $r\le 1$ and $H^r=L^r$ if $1<r<\infty$. Additionally, assume that Ω satisfies (3.1). Then there exists a $C>0$ such that*

$$\|T_\alpha f\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p_\alpha(\mathbb{R}^n)}.$$

Proof. The case $\alpha = 0$ is well known (see Theorem A). For $\alpha > 0$, by changing to the spherical coordinates, we have

$$\begin{aligned}|T_\alpha f(x)| &= \Big|\int_{\mathbb{R}^n} |y|^{-n-\alpha}\Omega(x,y)\{f(x-y)-f(x)\}dy\Big| \\ &= C\Big|\int_{S^{n-1}} \Omega(x,\theta)\mathcal{H}_\alpha(f)(x,\theta)d\theta\Big|.\end{aligned}$$

Thus for any $1 < p \le 2$ and $0 \le \gamma < (1+2\alpha)/p'$,

$$|T_\alpha f(x)| \le C\sup_x \|\Omega(x,\cdot)\|_{L^{p'}_{-\gamma}(S^{n-1})}\|\mathcal{H}_\alpha(f)(x,\cdot)\|_{L^p_\gamma(S^{n-1})}.$$

By (iii) of Theorem 2.1, to prove the theorem, it suffices to show

$$\sup_x \|\Omega(x,\cdot)\|_{L^{p'}_{-\gamma}(S^{n-1})} < \infty.$$

Let $\tilde{J}_\gamma$ be the Riesz potential on S^{n-1}. By [7], we know that

$$\|\Omega(x,\cdot)\|_{L^{p'}_{-\gamma}(S^{n-1})} \cong \Big(\int_{S^{n-1}} |\tilde{J}_\gamma\Omega(x,\theta)|^{p'}d\theta\Big)^{1/p'} \le C\|\Omega(x,\cdot)\|_{H^r(S^{n-1})},$$

where $r > p'(n-1)/(n+2\alpha)$. This proves Theorem 3.1.

Using the exactly same argument, by (i), (ii) of Theorem 2.1, we have the following result for the fractional integral $\mathcal{F}_\alpha$.

Theorem 3.2. *Let $0 < \alpha < 1/2$ and $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$.*

(i) *If $r > 2(n-1)/(n-2\alpha)$, then $\|\mathcal{F}_\alpha(f)\|_{L^2(\mathbb{R}^n)} \le C\|f\|_{L^\sigma(\mathbb{R}^n)}$, where $\sigma = 2n/(n+2\alpha)$.*

(ii) *If $r > 2\delta(n-1)/(n-2\alpha)$, where $\delta = (1/2-\alpha/n)/(1/p'-\alpha/n)$, then*

$$\|\mathcal{F}_\alpha(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^q(\mathbb{R}^n)} \quad \textit{with} \quad 1/q = 1/p + \alpha/n.$$

For a larger range of α, we have the following theorem which is an improvement of Theorem 4 in [10] for the unweighted $1 < p \le 2$ case.

Theorem 3.3. *Let $n/(n-\alpha) < p \le 2$, $1/q = 1/p + \alpha/n$ and $\Omega \in L^\infty(\mathbb{R}^n) \times L^r(S^{n-1})$. If $r = q' = p'n/(n-p'\alpha)$, then*

$$\|\mathcal{F}_\alpha(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^q(\mathbb{R}^n)}.$$

Remark 3.1. In Theorem 4 of [10], one needs the condition $r > q'$.

Proof. We may assume both Ω and f are non-negative. Write

$$|\mathcal{F}_\alpha(f)(x)| \le I_1(x) + I_2(x)$$

with

$$\begin{aligned}I_1(x) &= \int_{|y|<R} \Omega(x,y)|y|^{-n+\alpha}f(x-y)dy, \\ I_2(x) &= \int_{|y|>R} \Omega(x,y)|y|^{-n+\alpha}f(x-y)dy,\end{aligned}$$

where R is a number to be chosen. Choose a small positive $\epsilon < \alpha$. Then

$$I_1(x) \le R^{\alpha-\epsilon}\mathcal{F}_\epsilon(f)(x).$$

By Hölder's inequality, we have

$$I_2(x) \le \|f\|_{L^q(\mathbb{R}^n)}\Big(\int_{|y|\ge R} |\Omega(x,y)|^{q'}|y|^{q'(\alpha-n)}dy\Big)^{1/q'}.$$

It is easy to check that $q'(n-\alpha) > n$. Thus by the condition assumed on Ω, we have

$$I_2(x) \le CR^{-(n-\alpha)+n/q'}\|f\|_{L^q(\mathbb{R}^n)}.$$

Combining the estimates for $I_1(x)$ and $I_2(x)$, we obtain

$$|\mathcal{F}_\alpha(f)(x)| \le CR^{\alpha-\epsilon}\{\mathcal{F}_\epsilon(f)(x)R^{\epsilon-n+\frac{n}{q'}}\|f\|_{L^q(\mathbb{R}^n)}\}.$$

Letting $R^{\epsilon-n+n/q'} = \mathcal{F}_\epsilon(f)(x)\,/\|f\|_{L^q(\mathbb{R}^n)}$, we obtain that

$$|\mathcal{F}_\alpha(f)(x)| \le C|\mathcal{F}_\epsilon(f)(x)|^{1+\lambda}\|f\|^{-\lambda}_{L^q(\mathbb{R}^n)},$$

where

$$\lambda = (\alpha-\epsilon)/(\epsilon-n+n/q') = -(\alpha-\epsilon)/(\alpha-\epsilon+n/p),$$
$$1+\lambda = n/(n+p(\alpha-\epsilon)).$$

Now for any $n/(n-\alpha) < p \le 2$, choose small $\epsilon > 0$ such that

$$r = q' > 2\zeta(n-1)/(n-2\epsilon)$$

with $\zeta = (1/2-\epsilon/n)/(1/p'-\epsilon/n)$. Then by (ii) in Theorem 3.2, we have

$$\|\mathcal{F}_\alpha(f)\|_{L^p(\mathbb{R}^n)} \le C\|f\|^{-\lambda}_{L^q(\mathbb{R}^n)}\|\mathcal{F}_\epsilon(f)(x)\|^{(1+\lambda)}_{L^{p(1+\lambda)}(\mathbb{R}^n)} \le C\|f\|^{-\lambda}_{L^q(\mathbb{R}^n)}\|f\|^{(1+\lambda)}_{L^\sigma(\mathbb{R}^n)},$$

where $1/\sigma = 1/(p(1+\lambda)) + \epsilon/n = (n+p(\alpha-\epsilon))/pn + \epsilon/n = 1/p + \alpha/n$. Thus $\sigma = q$, which implies

$$\|\mathcal{F}_\alpha(f)\|_{L^q} \le C\|f\|_{L^p}.$$

The theorem is proved.

§4. Dimension Free Estimate

For $n \ge 2$, $j = 1, 2, \cdots, n$, the j-th Riesz transform R_j defined on the test space $\mathcal{S}(\mathbb{R}^n)$ is given by

$$R_j(f)(x) = \text{p.v. } A_n\int_{\mathbb{R}^n} y_j|y|^{-n-1}f(x-y)dy \tag{4.1}$$

with $A_n = \Gamma((n+1)/2)\pi^{-(n+1)/2}$, where $f \in \mathcal{S}(\mathbb{R}^n)$ and $y = (y_1, y_2, \cdots, y_n)$. It is well known that Riesz transforms play a very important role in studying Harmonic Analysis, as well as in the studying many different topics related to Harmonic Analysis (see [16, 13, 14] among numerous references). Consider the vector valued Riesz transform

$$R(f)(x) = (R_1(f)(x), \cdots, R_n(f)(x)). \tag{4.2}$$

Then for fixed f and x, the norm of R is

$$|R(f)|(x) = \Big(\sum_{j=1}^n |R_j(f)(x)|^2\Big)^{1/2}. \tag{4.3}$$

E. M. Stein established an interesting dimension free estimate

$$\|\,|R(f)|\,\|_{L^p(\mathbb{R}^n)} \le C\|f\|_{L^p(\mathbb{R}^n)}, \tag{4.4}$$

where $1 < p < \infty$ and $C = C(p)$ is a constant independent of n and f. For the details, readers can see [15], also [9] for an extension of (4.4) to the more general harmonic polynomials. In this section we will consider the Hyper Riesz Transform, for $0 \le \alpha < 1$ and $j = 1, 2, \cdots, n$,

$$R_{\alpha,j}(f)(x) = \text{p.v. } A_{n,\alpha}\int_{\mathbb{R}^n} y_j|y|^{-n-1-\alpha}f(x-y)dy, \tag{4.5}$$

where

$$A_{n,\alpha} = \Gamma((n+\alpha+1)/2)\pi^{-(n+1)/2}.$$

Clearly $R_{\alpha,j}(f)(x)$ exists for each x, since $f \in \mathcal{S}(\mathbb{R}^n)$ and the kernel of $R_{\alpha,j}$ satisfies the cancellation condition. The vector-valued hyper Riesz transform now is defined by

$$\mathcal{R}_\alpha(f)(x) = (R_{\alpha,1}(f)(x), \cdots, R_{\alpha,n}(f)(x)). \tag{4.6}$$

Theorem 4.1. *For $1 < p < \infty$ and $0 \le \alpha < 1$, there exists a constant $C = C(p, \alpha)$, independent of n and $f \in \mathcal{S}(\mathbb{R}^n)$, such that*

$$\| |\mathcal{R}_\alpha(f)| \|_{L^p} \le C \|f\|_{L^p_\alpha(\mathbb{R}^n)}. \tag{4.7}$$

Proof. We will adapt a method of Pisier[12] (see also [1]). For $f \in \mathcal{S}(\mathbb{R}^n)$, $0 \le \alpha < 1$ and $\theta \in S^{n-1}$, recall that the hyper Hilbert transform acting in the direction θ,

$$H^\theta_\alpha f(x) = \int_0^\infty t^{-1-\alpha}(f(x - t\theta) - f(x))dt. \tag{4.8}$$

Now following the argument in [12], also see [1], we equip $\mathbb{R}^n$ with the Gaussin probability measure $d\gamma_n(y) = (2\pi)^{-n/2} e^{-|y|^2/2} dy$ and introduce the orthogonal projection Q of $L^2(\mathbb{R}^n, \gamma_n)$ onto the space of functions of the form $\sum\limits_{j=1}^n a_j y_j$ in which the coordinate functions $\{y_j\}_{j=1}^n$ form an orthogonal basis. Then we can establish the following formula.

Lemma 4.1. *For almost all $x \in \mathbb{R}^n$,*

$$Q\{H_\alpha f(x)\}(y) = C \sum_{j=1}^n R_{\alpha,j} f(x) y_j. \tag{4.9}$$

Proof. Let $\theta = y' = y/|y|$. We need to prove for each $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$a_j = a_j(x) = \int_{\mathbb{R}^n} H^{y'}_\alpha f(x) y_j d\gamma_n(y) = (2/\pi)^{1/2} R_{\alpha,j} f(x).$$

By the definition and polar coordinates,

$$\begin{aligned}\int_{\mathbb{R}^n} H^{y'}_\alpha f(x) y_j d\gamma_n(y) &= (2\pi)^{-n/2} \int_{S^{n-1}} y'_j \int_0^\infty s^{n+\alpha} e^{-s^2/2} ds \\ &\quad \cdot \int_0^\infty t^{-1-\alpha}\{f(x - ty') - f(x)\} dt d\sigma(y') \\ &\cong C \int_{S^{n-1}} \int_0^\infty t^{-1-\alpha} y'_j \{f(x - ty') - f(x)\} dt d\sigma(y') \\ &\cong R_{\alpha,j} f(x).\end{aligned}$$

Lemma 4.1 is proved.

Let $1 < p < \infty$ and $\gamma(p) = \big((\frac{2}{\pi})^{1/2} \int_0^\infty t^p e^{-t^2/2} dt\big)^{1/p}$. We can find the following identity in [12],

$$\begin{aligned}\Big\| \sum_{j=1}^n a_j y_j \Big\|_{L^p(\mathbb{R}^n, \gamma_n)} &= \gamma(p) \Big\| \sum_{j=1}^n a_j y_j \Big\|_{L^2(\mathbb{R}^n, \gamma_n)} \\ &= \gamma(p) \Big(\sum_{j=1}^n |a_j|^2 \Big)^{1/2}.\end{aligned} \tag{4.10}$$

By Lemma 4.1 and Lemma 2.1 and following the same argument in [12], we obtain Theorem 4.1.

References

[1] Auscher, P. & Carro, M., Transference for radial multipliers and dimension free estimates, *Trans. Amer. Math. Soc.,* **342**(1994), 575–593.

[2] Calderón, A. P. & Zygmund, A., On existence of certain singular integrals, *Acta. Math.,* **88**(1952), 85–139.

[3] Calderón, A. P. & Zygmund, A., On singular integrals, *Amer. J. Math.,* **78**(1956), 289–309.

[4] Chen, J., Fan, D. & Ying, Y., Singular integrals on function spaces, *J. Math. Anal. Appl.,* **276**(2002), 691–708.

[5] Chen, J., Ding, Y. & Fan, D., A class of integral operators with variable kernels on Hardy spaces (in Chinese), *Chinese Ann. Math.,* **23A:**3(2002), 289–296.

[6] Christ, M., Duoandikoetxea, J. & Rubio de Francia, J. L., Maximal operators related to the Radon transform and the Calderón-Zygmund method of rotations, *Duke Math. J.,* **53**(1986), 189–208.

[7] Colzani, L., Hardy spaces on unit sphere, Bollettino U.M.I., Analisi Funzionale e Applicazioni, Serie VI, **IV**(1985), 219–244.

[8] Ding, Y. & Lu, S., Weighted norm inequalities for fractional integral operators with rough kernels, *Canadian J. Math.,* **50**(1998), 29–39.

[9] Duoandikoetxea, J. & Rubio de Francia, J. L., Estimations indépendates de la dimension pour les transformées de Riesz, C.R.A.S.T. 300, **I**, 1985.

[10] Muckenhoupt, B. & Wheeden, R. L., Weighted norm inequalities for singular and fractional integrals, *Trans. Amer. Math. Soc.,* **161**(1971), 249–258.

[11] Müller, D., Analysis of spherical in Euclidean spaces, Applied Math. Scie., **129**, Springer-Verlag, New York, 1998, 29–39.

[12] Pisier, G., Riesz transform: a simple analytic proof of P. A. Meyer's inequalities, Sem. Proba. XXII, Lecture Notes in Math, **1321**, Springer, Berlin, 1988, 485–501.

[13] Stein, E. M., Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, NJ, 1970.

[14] Stein, E. M., Harmonic analysis: real-variable methods, orthogonality and oscillatory integrals, Princeton University Press, Princeton, NJ, 1993.

[15] Stein, E. M. & Strömberg, J. O., Behavior of maximal functions in $\mathbb{R}^n$ for large n, *Ark. för Math,* **21**(1983), 259–269.

[16] Stein, E. M. & Weiss, G., Introduction to Fourier analysis on Euclidean spaces, Princeton University Press, Princeton, NJ, 1971.

[17] Strichartz, R. S., Multipliers on fractional Sobolev spaces, *J. Math. Mech.,* **16**(1967), 1031–1060.