Chin. Ann. Math.
24B:4(2003),485–490.

L_p-DICHOTOMY OF LINEAR DIFFERENTIAL EQUATIONS IN AN ARBITRARY BANACH SPACE

A. KOSSEVA* S. KOSTADINOV** K. SCHNEIDER***

Abstract

The notion of L_p-dichotomy for linear differential equations with possibly unbounded operator is introduced. By help of Banach fixed point theorem sufficient conditions for the existence of bounded solutions of nonlinear differential equations with an L_p-dichotomous linear part are obtained.

Keywords Ordinary differential equations, Dichotomy, Existence of bounded solutions

2000 MR Subject Classification 34A34, 47H10

Chinese Library Classification O175.14 **Document Code** A

Article ID 0252-9599(2003)04-0485-06

§1. Introduction

We will introduce the notion of L_p-dichotomy $(1 \le p < \infty)$ for linear differential equations with possibly unbounded operators of the form

$$\frac{dx}{dt} = A(t)x.$$

The L_p-dichotomy is a generalization of the usually exponential dichotomy for such equations which guarantees the existence of bounded solutions of the nonhomogeneous equation

$$\frac{dx}{dt} = A(t)x + f(t)$$

for any $f \in L_p$ or any bounded f.

By help of Banach fixed point theorem sufficient conditions for the existence of bounded solutions of nonlinear equations with a L_p-dichotomous linear part of the form

$$\frac{dx}{dt} = A(t)x + f(t,x)$$

are obtained.

An upper estimate for the difference of two solutions of these equations is obtained too.

Manuscript received May 31, 2002.

*Department of Mathematics, Technical University, 61, Sankt Petersburg blvd, 4000 Plovdiv, Bulgaria. **E-mail:** albenak@bitex.com

**Department of Mathematics and Informatics, University of Plovdiv "P. Hilendarski", 24, Tzar Assen str., 4000 Plovdiv, Bulgaria.

***WIAS Berlin, Mohrenstr., 39, 10117 Berlin, Deutschland.

§2. Problem Statement

Let X be an arbitrary Banach space with norm $|\cdot|$ and identity operator I. Let $J = (0,\infty)$. For any operator $C : X \to X$, we shall denote by $D(C)$ the domain of C.

We consider in X the following differential equation

$$\frac{dx}{dt} = A(t)x, \tag{2.1}$$

where $A(t)$ are linear (possibly unbounded) operators acting on X, with the same domain $D = D(A(t))\,(t \in J)$ and $\overline{D} = X$.

Let $U(t,s)\,(t,s \in J)$ (see [4]) be the evolutionary operator of (2.1).

Definition 2.1.[3] *The Equation* (2.1) *is said to be exponentially dichotomous if there exist constants $M, \delta > 0$ and projector family $P(t) : X \to X\,(t \in J)$ such that*

(i) $U(t,s)P(s) = P(t)U(t,s) \quad (t \geq s,\ t,s \in J)$;

(ii) *for $t \geq s$ the restriction $U(t,s)|_{(I-P(s))(X)}$ is a homeomorphism between $(I-P(s))(X)$ and $(I-P(s))(X)$. On $(I-P(s))(X)$, for $t < s$ we set $U(t,s) = U^{-1}(s,t)$;*

(iii) $\|U(t,s)P(s)\| \leq Me^{-\delta(t-s)} \quad (t \geq s,\ t,s \in J)$;

(iv) $\|U(t,s)(I-P(s))\| \leq Me^{-\delta(s-t)} \quad (t < s,\ t,s \in J)$.

By $B(J)$ we denote the space of all bounded functions $x(t)\,(t \in J)$ with the norm

$$|||x||| = \sup_{t\in J} |x(t)|.$$

The case when $A(t)$ are bounded operators is considered in detail for example in [2, 1, 5].

Let $p \in [1,\infty)$.

Definition 2.2. *The Equation* (2.1) *is said to be L_p-dichotomous if there exist a projector family $P(s) : X \to X\,(s \in J)$ and a constant K such that*

(j1) $\left[\int_0^t \|U(t,s)P(s)\|^p ds\right]^{\frac{1}{p}} \leq K \quad (t \in J)$;

(j2) *the condition* (ii) *in Definition* 2.1 *holds;*

(j3) $\left[\int_t^\infty \|U(t,s)(I-P(s))\|^p ds\right]^{\frac{1}{p}} \leq K \quad (t \in J)$.

It is easy to see that if Equation (2.1) is exponentially dichotomous it is L_p-dichotomous. The following example shows that the opposite is not always true.

Example 2.1. We consider the ordinary differential equation

$$x' = A(t)x, \quad t \in J, \quad x \in \mathbb{R},$$

where $A(t) = -\dfrac{1}{e^t + r(t)}(e^t + r'(t))$,

$$r(t) = \begin{cases} 0, & n-1 \leq t < n-2\alpha_n; \\ ne^n \sin^2\left\{\dfrac{\pi}{2\alpha_n}(t-n+2\alpha_n)\right\}, & n-2\alpha_n \leq t < n, \end{cases}$$

and $\alpha_n = \dfrac{1}{2^{n+1}ne^n}, n = 1,2,\cdots$.

The evolutionary operator of the equation is given by the formula

$$U(t,s) = e^{\int_s^t A(\tau)d\tau}, \quad \text{i.e.} \quad U(t,s) = \frac{e^s + r(s)}{e^t + r(t)}, \quad t \geq s > 0, \quad t,s \in J.$$

Let $n \geq [M]+1$, where M is the constant from Definition 2.1. Then for $t=n$ and $s=n-\alpha_n$, we obtain

$$\|U(t,s)\| = U(n, n-\alpha_n) > M,$$

i.e. the equation is nonexponentially dichotomous.

We show that the equation is L_1-dichotomous, i.e.

$$\int_0^t \|U(t,s)\| ds < 1, \qquad t \in J.$$

We have

$$\int_0^t \|U(t,s)\| ds = \int_0^t \frac{e^s + r(t)}{e^t + r(t)} ds = \frac{1}{e^t + r(t)} \int_0^t (e^s + r(s)) ds.$$

It will be checked that

$$\int_0^t (e^s + r(s)) ds < e^t.$$

Case 1. Let $n-1 \leq t < n-2\alpha_n$. Then

$$\int_0^t (e^s + r(s)) ds = e^t - 1 + \sum_{k=1}^{n-1} \int_{k-2\alpha_k}^k k e^k \sin^2\Big\{\frac{\pi}{2\alpha_k}(s-k+2\alpha_k)\Big\} ds$$
$$= e^t - 1 + \sum_{k=1}^{n-1} \frac{1}{2^{k+1}} < e^t.$$

Case 2. Let $n-2\alpha_n \leq t < n$. Then

$$\int_0^t (e^s + r(s)) ds = e^t - 1 + \sum_{k=1}^{n-1} \frac{1}{2^{k+1}} + n e^n \int_{n-2\alpha_n}^t \sin^2\Big\{\frac{\pi}{2\alpha_n}(s-n+2\alpha_n)\Big\} ds$$
$$= e^t - \frac{1}{2} - \frac{1}{\pi 2^{n+2}} \sin\Big\{\frac{\pi}{\alpha_n}(t-n+2\alpha_n)\Big\} < e^t.$$

Hence

$$\int_0^t \|U(t,s)\| ds < \frac{e^t}{e^t + r(t)} \leq 1, \qquad t \in J,$$

i.e. the equation is L_1-dichotomous.

By $L_{p'}(J)\,(p' \in [1,\infty))$ we denote the space of all functions $f: J \to X$ with

$$\int_J |f(s)|^{p'} ds < \infty$$

and the norm

$$\|f\|_{p'} = \Big(\int_J |f(s)|^{p'} ds\Big)^{\frac{1}{p'}}.$$

Remark 2.1. It is not hard to check that the notion of L_p-equivalence is stable w. r. to the following small perturbations on the spaces $P(s)(X)$ and $(I-P(s))(X)\,(s \in J)$,

$$\|(U(t,s) - V(t,s))P(s)\| \leq \epsilon(t,s) \quad \text{for } \ t,s \in J,\ 0 \leq s \leq t,$$

where

$$\int_0^t \epsilon^p(t,s) ds \leq K_1, \qquad K_1 = \text{const.},$$

and

$$\|(U(t,s)-V(t,s))(I-P(s))\| \leq \eta(t,s) \quad \text{for} \ \ t,s \in J,\ 0 \leq t \leq s,$$

where

$$\int_t^\infty \eta^p(t,s)ds \leq K_2, \qquad K_2 = \text{const.}$$

We shall consider the homogeneous and nonlinear differential equations with an L_p-dichotomous linear part.

§3. Main Results

We consider the nonhomogeneous equation

$$\frac{dx}{dt} = A(t)x + f(t), \tag{3.1}$$

where $f(t) \in X\,(t \in J)$.

Theorem 3.1. *Let Equation* (2.1) *be L_p-dichotomous. Then for any $f \in L_q(J)\,(q = \frac{p}{p-1})$, Equation* (3.1) *has a bounded solution.*

Proof. The function

$$x(t) = \int_0^t U(t,s)P(s)f(s)ds - \int_t^\infty U(t,s)(I-P(s))f(s)ds \tag{3.2}$$

is bounded. Really by means of Hölder inequality we obtain

$$\begin{aligned}|x(t)| &\leq \Big[\int_0^t \|U(t,s)P(s)\|^p ds\Big]^{\frac{1}{p}} \Big[\int_0^t |f(s)|^q ds\Big]^{\frac{1}{q}} \\ &\quad + \Big[\int_t^\infty \|U(t,s)(I-P(s))\|^p ds\Big]^{\frac{1}{p}} \Big[\int_t^\infty |f(s)|^q ds\Big]^{\frac{1}{q}} \\ &\leq 2K\|f\|_q.\end{aligned}$$

By straightforward verification it is easy to show that the function $x(t)$ is a solution of (3.1).

Remark 3.1. Let $\xi \in X$ be such that the function $U(t)\xi\,(t \in J)$ is bounded. Then the function $z(t) = U(t)\xi + x(t)$ is a bounded solution of (3.1).

Theorem 3.2. *Let Equation* (2.1) *be L_p-dichotomous. Then Equation* (3.1) *has a bounded solution for any bounded function $f(t)\,(t \in J)$.*

The proof is analogous to the proof of Theorem 3.1.

Remark 3.2. The bounded solution $x(t)$ will belong to the space $L_p(J)$ if K is not a constant but a function $K \in L_p(J)$.

Let $B_r = \{x \in X : |x| < r\}$. Further we consider the nonlinear equation

$$\frac{dx}{dt} = A(t)x + f(t,x), \tag{3.3}$$

where $f : J \times \overline{B}_r \to X$.

Definition 3.1. *The function $x(t)$ $(t \in J)$ is said to be a solution of Equation* (3.3) *if*

(i) *$x(t) \in D \cap B_r\,(t \in J)$;*

(ii) *there exists a strong derivative* $x'(t)\,(t \in J)$;

(iii) $x(t)$ *satisfies* (3.3).

Theorem 3.3. *Let the following conditions hold:*

(1) *Equation* (2.1) *is* L_p*-dichotomous.*

(2) $\|U(t,0)P(0)\| \le K_1\,(t \in J)$, K_1 *is a constant.*

(3) $|f(t,x_2) - f(t,x_1)| \le a(t)|x_2 - x_1|\,(t \in J,\ x_1, x_2 \in B_r)$, *where* $a \in L_q(J), q = \frac{p}{1-p}$.

(4) $|f(t,x)| \le b(t)\,(t \in J,\ x \in B_r)$, *where* $b \in L_q(J)$ *and* $\|b\|_q \le \frac{r}{2K}$.

Then for any y_0 *with*

$$|y_0| \le \frac{r - 2K\|b\|_q}{K_1}, \qquad y_0 \in P(0)(X)$$

and $\|a\|_q$ *small enough, Equation* (3.3) *has exactly one solution* $x(t) \in B_r\,(t \in J)$ *with* $P(0)x(0) = y_0$.

For two solutions $x_2(t)$ and $x_1(t)$ of (3.3), the following estimate is valid

$$|x_2(t) - x_1(t)| \le K_1|x_2(0) - x_1(0)|\Big[1 + 2K\frac{\int_0^t a^q(s)\varepsilon(s)ds}{1 - [1-\varepsilon(t)]^{\frac{1}{q}}}\Big], \tag{3.4}$$

where

$$\varepsilon(t) = e^{-(2K)^q \int_0^t a^q(s)ds}, \qquad t \in J.$$

Proof. Let $D_r = \{x \in B(J) : x(t) \in \overline{B}_r, t \in J\}$. Any solution of (3.3) which lies in D_r is a solution of the equation

$$\begin{aligned} x(t) = U(t,0)y_0 &+ \int_0^t U(t,s)P(s)f(s,x(s))ds \\ &- \int_t^\infty U(t,s)(I - P(s))f(s,x(s))ds, \end{aligned} \tag{3.5}$$

where $y_0 = P(0)x(0)$ and vice versa.

We will show that the operator Q defined by

$$\begin{aligned} (Qx)(t) = U(t,0)y_0 &+ \int_0^t U(t,s)P(s)f(s,x(s))ds \\ &- \int_t^\infty U(t,s)(I - P(s))f(s,x(s))ds \end{aligned} \tag{3.6}$$

satysfies the conditions of Banach fixed point theorem in the set $D_r \subset B(J)$.

In fact, because $P(0)y_0 = y_0$, we have

$$|(Qx)(t)| \le \|U(t,0)P(0)\||y_0| + 2K\|b\|_q \le K_1|y_0| + 2K\|b\|_q \le r.$$

On the other hand, for $x_1, x_2 \in D_r$, we have

$$|||Qx_2 - Qx_1||| \le 2K\|a\|_q|||x_2 - x_1|||.$$

For $\|a\|_q$ small enough, the operator Q is a contraction and has exactly one fixed point $x(t)$.

For the difference of two solutions $x_1(t)$ and $x_2(t)$, we obtain the estimate

$$\begin{aligned}|x_2(t)-x_1(t)| &\le |U(t,0)P(0)(x_2(0)-x_1(0))| \\ &\quad + \int_0^t \|U(t,s)P(s)\||f(s,x_2(s))-f(s,x_1(s))|ds \\ &\quad + \int_t^\infty \|U(t,s)(I-P(s))\||f(s,x_2(s))-f(s,x_1(s))|ds \\ &\le K_1|x_2(0)-x_1(0)| + 2K\Big[\int_0^\infty (a(s)|x_2(s)-x_1(s)|)^q ds\Big]^{\frac{1}{q}}.\end{aligned}$$

The estimate (3.4) follows from [6].

Remark 3.3. By unessentially modifications the assertions of Theorems 3.1–3.3 are still valid if the condition (j3) is replaced by the following condition

$$\Big[\int_0^t \|U(t,s)(I-P(s))\|^p ds\Big]^{\frac{1}{p}} \le K, \qquad t\in J.$$

In this case in Theorem 3.3 we can consider the function

$$x(t)=U(t,0)y_0+\int_0^t U(t,s)P(s)f(s,x(s))ds+\int_0^t U(t,s)(I-P(s))f(s,x(s))ds.$$

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