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# EXCURSIONS AND LÉVY SYSTEM OF BOUNDARY PROCESS***

HE Ping*  YING Jiangang**

**Abstract**

In this paper, the authors investigate the joint distribution of end points of excursion away from a closed set straddling on a fixed time and use this result to compute the Lévy system and the Dirichlet form of the boundary process.

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## §1. Introduction

Excursion theory is one of the most important problems in the theory of Markov processes. Excursions away from a closed set are the collection of pieces of the path of a Markov process being away from a closed subset of the state space. K. Itô[9] first showed that the excursions away a single point maybe characterized by a Poisson point process with a characteristic measure which is called the excursion law. Since then many authors left their works on this problem. Among them, B. Maisonneuve[10] established an exit system, which consists of an excursion law and an additive functional, to characterize excursion in general framework.

In the present article, we are mainly concerned with excursions straddling on a fixed time $t$. Firstly in §2 we will compute the joint distribution of end points of such an excursion. In this direction, some work has been done by Getoor and Sharpe[6] where the Maisonneuve's exit system is assumed in advance. However we use the property of duality and our approach is similar to that employed in [8], where Hsu treated the reflecting Brownian motion. Thus our result is more concrete and useful. Then in §3 we shall apply this result and the technique of time change to constructing the Lévy system and find the Dirichlet form of the boundary process, which is obtained from the original process by being taken away those excursions.

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*Institute of Mathematics, Fudan University, Shanghai 200433, China. **E-mail:** mathpinghe@sina.com
**Institute of Mathematics, Fudan University, Shanghai 200433, China. **E-mail:** jgying@fudan.edu.cn
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## §2. Joint Distribution of Endpoints of Excursions

Let $E$ be a locally compact Hausdorff space with countable base. Suppose that we are given two strong Markov processes $X = \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x\}$ and $\widehat{X} = \{\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathcal{F}}_t, \widehat{X}_t, \hat{\theta}_t, \widehat{P}^x\}$ with $E$ the state space satisfying the following assumptions:

(A.1) Both $X$ and $\widehat{X}$ are continuous and conservative. For simplicity, assume that $\Omega$ is the set of continuous functions from $[0,\infty)$ to $E$ and $X$ and $\widehat{X}$ are both realized on $\Omega$. $\widehat{\Omega}$ is nothing but a copy of $\Omega$. Thus $X$ and $\widehat{X}$ are distinguished by their laws.

(A.2) Transition semigroups have densities in duality with a reference measure. That is, set

$$P_t(x,B) := P^x(X_t \in B),$$
$$\widehat{P}_t(x,B) = \widehat{P}^x(X_t \in B), \qquad x \in E,\ B \subset E.$$

Assume that there exists a non-negative Radon measure $m$ on $E$ and a non-negative continuous function $p(t,x,y)$ on $(0,+\infty) \times E \times E$ such that

$$P_t f(x) = \int_E p(t,x,y) f(y) m(dy),$$
$$\widehat{P}_t f(x) = \int_E p(t,y,x) f(y) m(dy) \qquad \text{for} \quad f \in C(E).$$

(A.3) Let $V$ be a fixed non-empty proper closed subset which is finely perfect and co-finely perfect, i.e., each point of $V$ is regular for itself, i.e., $P^x(T=0)=1$ and $\widehat{P}^x(\widehat{T}=0)=1$ for $x \in V$, where $T = T_V$ and $\widehat{T} = \widehat{T}_V$ are the first hitting times of $V$ for $X$ and $\widehat{X}$, respectively.

(A.4) Set $D = V^c$. There exist non-negative measure $\mu$ on $V$ and non-negative functions $h(x,b)$ and $\hat{h}(x,b)$ on $D \times V$ such that

$$E^x[f(X_T)] = \int_V h(x,b) f(b) \mu(db),$$
$$\widehat{E}^x[f(\widehat{X}_T)] = \int_V f(b) \hat{h}(x,b) \mu(db) \qquad \text{for} \quad f \in C(V).$$

Assume that the $\mu$-measure on every relatively open set of $V$ is strictly positive.

(A.5) There exist continuous functions $H_u(x,b)$ and $\widehat{H}_u(x,b)$ on $(0,+\infty) \times D \times V$ such that

$$h(x,b) = \int_0^{+\infty} H_u(x,b) du,$$
$$\hat{h}(x,b) = \int_0^{+\infty} \widehat{H}_u(x,b) du.$$

In addition,

$$\int_D P_v^0(x,dy) H_u(y,b) = H_{u+v}(x,b),$$
$$\int_D \widehat{H}_u(y,b) \widehat{P}_v^0(x,dy) = \widehat{H}_{u+v}(x,b),$$

where $P_t^0(x,B) := P^x(X_t \in B, t < T)$ and $\widehat{P}_t^0(x,B) := \widehat{P}^x(X_t \in B, t < T)$ for $x \in D, B \subset D$, which are the transition functions of the $X$ killed at leaving $D$, both are absolutely continuous with respect to $m$ and the density functions $p^0(t,\cdot,\cdot)$ and $\hat{p}^0(t,\cdot,\cdot)$ are dual with respect to $m$.

**Remark 2.1.** (A.1)–(A.3) are common for any conservative diffusion. In that situation, (A.4) and (A.5) will be satisfied for a set $V$ with smooth boundary. As being seen later, those conditions may be weaken for some of our consequences. A reflecting Brownian motion on a $C^3$ domain satisfies all above assumptions.

Before stating some lemmas, we would like to introduce some notation. Let $q > 0$. For $f \in C(E)$, set $u^q(x,y) = \int_0^\infty e^{-qt} p(t,x,y)dt$ and

$$U^q f(x) = E^x \int_0^\infty e^{-qt} f(X_t)dt.$$

Clearly, $U^q f(x) = \int_E u^q q(x,y)f(y)m(dy)$. Denote

$$U_D^q f(x) := E^x \int_0^T e^{-qt} f(X_t)dt \tag{2.1}$$

for a bounded measurable function $f$ on $D$, and

$$u_D^q(x,y) := u^q(x,y) - E^x[e^{-qT} u^q(X_T,y)] \qquad \text{for} \quad x,y \in E. \tag{2.2}$$

Then

$$U_D^q f(x) = \int_D u_D^q(x,y)f(y)m(dy) \tag{2.3}$$

for $x \in D$ and bounded function $f$ on $D$. Indeed, by Dynkin's formula,

$$\int_E u^q(x,y)f(y)m(dy) = E^x \int_0^{+\infty} e^{-qt} f(X_t)dt$$
$$= E^x \int_0^T e^{-qt} f(X_t)dt + E^x[e^{-qT} U^q f(X_T)].$$

Integrating both sides of (2.2) on $y$ with $f(y)m(dy)$ and combining this with the above expression, we can see that (2.3) is valid. Furthermore, for $x \in D$ and $y \in V$, set

$$h_q(x,b) := h(x,b) - q\int_D u_D^q(x,y)h(y,b)m(dy). \tag{2.4}$$

The joint distribution of $(T, X_T)$ is expressed as follows.

**Lemma 2.1.** *The assumption* (A.5) *holds if and only if for* $f \in C(V)$, $x \in D$ *and* $t > 0$,

$$E^x[e^{-qT} f(X_T)] = \int_V \int_0^\infty e^{-qu} H_u(x,b)f(b)\mu(db)du.$$

**Proof.** Assume (A.5) holds. At first, we would like to show that for $f \in C(V)$ and $x \in D$,

$$E^x[e^{-qT} f(X_T)] = \int_V h_q(x,b)f(b)\mu(db). \tag{2.5}$$

Set $hf(x) := \int_V h(x,b)f(b)\mu(db)$ for $f \in C(V)$. By (A.4) and (2.3),

$$q\int_D u_D^q(x,y)hf(y)m(dy) = q\int_D u_D^q(x,y)E^y[f(X_T)]m(dy).$$

By using the Markov property on $t$, the above expression is equal to

$$qE^x \int_0^T e^{-qt} E^{X_t}[f(X_T)]dt = q\int_0^{+\infty} e^{-qt} E^x[f(X_T); t < T]dt$$
$$= qE^x\Big[\int_0^T e^{-qt}dt f(X_T)\Big] = E^x[f(X_T)] - E^x[e^{-qT} f(X_T)].$$

Integrating both sides of (2.4) on $b$ with $f(y)m(dy)$ and combining this with the above, one obtains (2.5). Then it is enough to prove that for $x \in D$ and $b \in V$,

$$h_q(x,b) = \int_0^\infty e^{-qu} H_u(x,b)du.$$

In fact, by the definition (2.4) of $h_q(x,b)$,

$$h_q(x,b) = \int_0^\infty H_u(x,b)du - q\int_D u_D^q(x,y)m(dy)\int_0^\infty H_u(y,b)du.$$

The second term in the right side can be rewritten as

$$\begin{aligned} & q\int_0^\infty e^{-qt}dt \int_D P_t^0(x,dy)\int_0^\infty H_u(y,b)du \\ &= q\int_0^\infty \int_0^\infty e^{-qt} H_{u+t}(y,b)du\,dt \\ &= \int_0^\infty H_u(x,b)\int_0^u qe^{-qt}dt\,du \end{aligned}$$

by the assumption (A.5). Thus we have

$$\begin{aligned} h_q(x,b) &= \int_0^\infty H_u(x,b)\Big(1 - \int_0^u qe^{-qt}dt\Big)du \\ &= \int_0^\infty e^{-qu}H_u(x,b)du. \end{aligned}$$

Conversely, computing the Laplace transform, we have

$$\begin{aligned} & \int_0^\infty \int_D P_v^0(x,dy)\int_V H_u(y,b)f(b)\mu(db)e^{-qu}du \\ &= \int_D P_v^0(x,dy)\int_V h_q(y,b)f(b)\mu(db) \\ &= E^x[E^{X_v}[e^{-qT}f(X_T)]; v < T] \\ &= E^x[e^{-q(T-v)}f(X_T); T > v] \\ &= e^{qv}\int_v^\infty \int_V H_u(x,b)f(b)\mu(db)e^{-qu}du \\ &= \int_0^\infty \int_V H_{u+v}(x,b)f(b)\mu(db)e^{-qu}du. \end{aligned}$$

Thus (A.5) holds. That completes the proof.

When $x \in D$, by Markov property, we have

$$E^x[f(X_T), T > t] = \int_D P_t^0(x,dy)E^y[f(X_T)].$$

Thus (A.5) is also equivalent to that the right side is absolute continuous as a function of $t$. Denote $A_u(b,b') := \int_D \widehat{H}_v(x,b)H_{u-v}(x,b')m(dx)$ for $0 < v < u$, $b, b' \in V$.

**Lemma 2.2.** *The kernel $A_u$ defined above is independent of the choice of $v \in (0,u)$.*

**Proof.** From the duality, it follows that

$$m(dx)P_t^0(x,dy) = \widehat{P}_t^0(y,dx)m(dy).$$

Thus for $0 < v < w < u$, by (A.5),

$$\begin{aligned}
&\int_D \widehat{H}_w(x,b)H_{u-w}(x,b')m(dx)\\
&= \int_D\int_D \widehat{H}_v(y,b)\widehat{P}^0_{w-v}(x,dy)H_{u-w}(x,b')m(dx)\\
&= \int_D\int_D \widehat{H}_v(y,b)P^0_{w-v}(y,dx)H_{u-w}(x,b')m(dy)\\
&= \int_D \widehat{H}_v(y,b)H_{u-v}(y,b')m(dy).
\end{aligned}$$

This implies the lemma.

We define

$$N(b,b') := \int_0^\infty A_u(b,b')du, \qquad b,b' \in V,$$

which is called the Feller's kernel, as named in the case of Brownian motion. It also holds that

$$\int_u^\infty A_t(b,b')dt = \int_D \hat{h}(x,b)H_u(x,b')m(dx).$$

The next lemma gives the conditioned joint distribution of $(T, X_T)$.

**Lemma 2.3.** *Let $f \in C(V)$ and $t > 0$. Then for $s < t$ and $x, y \in E$,*

$$P^x(f(X_T); T < s \mid X_t = y) = \int_0^s \int_V H_u(x,z)f(z)\frac{p(t-u,z,y)}{p(t,x,y)}\mu(dz)du.$$

**Proof.** Take a Borel set $A \subset E$. By using the strong Markov property on $T$, we have

$$\begin{aligned}
&P^x(f(X_T), T < s, X_t \in A)\\
&= P^x(f(X_T), X_{t-T}(\theta_T(\cdot)) \in A, T < s)\\
&= E^x[f(X_T)E^x[X_{t-T}(\theta_T(\cdot)) \in A \mid \mathcal{F}_T], T < s]\\
&= E^x[f(X_T)P(t-T, X_T, A), T < s]\\
&= \int_A \int_0^s \int_V H_u(x,z)f(z)p(t-u,z,y)\mu(dz)du\,m(dy),
\end{aligned}$$

which shows our assertion.

We now give the main result of this section: the joint distribution of end points of excursions away from $V$ straddling on a fixed time. Our approach is similar to that employed in [8]. For a fixed $t > 0$, define

$$L(t) := \sup\{0 < s < t : X_s \in V\}, \quad R(t) := \inf\{s > t : X_s \in V\},$$

($\sup\emptyset = 0$ and $\inf\emptyset = \infty$ by convention) the last exit time of $V$ before $t$ and the first hitting time of $V$ after $t$ respectively. Since the path is continuous and $D$ is open, $X_t \in D$ implies that $L(t) < t < R(t)$. However fine and co-fine perfectness guarantees that the converse is true too. Clearly $X_{R(t)} \in V$ and $X_{L(t)} \in V$ since $V$ is closed. When $x \in V$, $P^x(L(t) > 0) = 1$ since $x$ is regular for $V$. The path $\{X_u : u \in (L(t), R(t))\}$ is called the excursion (away from $V$) straddling on $t$.

**Theorem 2.1.** *Let $t > 0$ and $a, b \in V$. For $s < t < u$, $s < t_1 < t_2 < \cdots < t_k < u$,*

$dy_1, dy_2, \cdots, dy_k \subset D$, *we have*

$$P^x(L(t)\in ds, X_{L(t)}\in da, X_{t_i}\in dy_i, i=1,2,\cdots,k, X_{R(t)}\in db, R(t)\in du)$$
$$= dsp(s,x,a)\mu(da)\widehat{H}_{t_1-s}(y_1,a)m(dy_1)\prod_{i=2}^{k}P^0_{t_i-t_{i-1}}(y_{i-1},dy_i)\mu(db)H_{u-t_k}(y_k,b)du.$$

**Proof.** For simplicity, we prove this formula for $k=2$. Take $\phi_j$, $\psi_j$, $\chi_j$ $(j=1,2)$ as non-negative bounded continuous functions on $[0,\infty)$, $V$ and $D$, respectively. For $t_1<t<t_2$, by Markov property at $t$,

$$\begin{aligned}&E^x[\phi_1(L(t))\psi_1(X_{L(t)})\chi_1(X_{t_1})\chi_2(X_{t_2})\phi_2(R(t))\psi_2(X_{R(t)})]\\&=E^x[E^x(\phi_1(L(t))\psi_1(X_{L(t)})\chi_1(X_{t_1})\chi_2(X_{t_2})\phi_2(R(t))\psi_2(X_{R(t)}))|X_t]\\&=\int_D E^x[\phi_1(L(t))\psi_1(X_{L(t)})\chi_1(X_{t_1})|X_t=y]\\&\quad\cdot E^x[\phi_2(R(t))\psi_2(X_{R(t)})\chi_2(X_{t_2})|X_t=y]p(t,x,y)m(dy),\end{aligned}$$

where

$$\begin{aligned}&E^x[\phi_2(R(t))\psi_2(X_{R(t)})\chi_2(X_{t_2})|X_t=y]\\&=E^y[\phi_2(T+t)\psi_2(X_T)\chi_2(X_{t_2-t});t_2-t<T]\\&=\int_D\chi_2(y_2)P^0_{t_2-t}(y,dy_2)\int_0^\infty\phi_2(u+t_2)du\int_V\psi_2(b)H_u(y_2,b)\mu(db).\end{aligned}$$

On the other hand, by the duality, under the probability $P^x(\,\cdot\,|X_t=y)$, the law of the reversed process $Y\equiv\{Y_s:=X_{t-s};\ 0\le s\le t\}$ is equal to the law of the dual process $\widehat{X}$ starting at $y$ and conditioned by $\widehat{X}_t=x$. Precisely if we define the reversed operator at $t$ as $r_t\omega(s):=\omega(t-s)$ for $\omega\in\Omega,\ s\in[0,t]$, then for any $A\in\mathcal{F}_t$, it holds that

$$P^x(A\circ r_t|X_t=y)=\widehat{P}^y(A|X_t=x).$$

Set $P^{x,X_t=y}(\cdot):=P^x(\cdot|X_t=y)$. It is easy to see that its transition function

$$P^{x,X_t=y}(X_s\in dz)=\frac{p(s,x,z)p(t-s,z,y)m(dz)}{p(t,x,y)},\quad s<t.$$

Obviously $L(t)\circ r_t=t-T$ and $X_{L(t)}\circ r_t=X_T$. Now using the Markov property of the conditioned law, we have by Lemmas 2.2 and 2.3,

$$\begin{aligned}&E^{x,X_t=y}[\phi_1(L(t))\psi_1(X_{L(t)})\chi_1(X_{t_1}),0\le L(t)<t_1]\\&=\widehat{E}^{y,X_t=x}[\phi_1(t-T)\psi_1(X_T)\chi_1(X_{t-t_1});t-t_1<T\le t]\\&=\widehat{E}^{y,X_t=x}[\chi_1(X_{t-t_1})\widehat{E}^{X_{t-t_1},X_{t_1}=x}[\phi_1(t_1-T)\psi_1(Y_T);T<t_1];t-t_1<T]\\&=\int_D\chi_1(y_1)\hat{p}^0(t-t_1,y,y_1)\frac{\hat{p}(t_1,y_1,x)}{\hat{p}(t,y,x)}m(dy_1)\\&\quad\cdot\int_0^{t_1}\phi_1(t_1-s)ds\int_V\psi_1(a)\widehat{H}_s(y_1,a)\frac{\hat{p}(t_1-s,a,x)}{\hat{p}(t_1,y_1,x)}\mu(da)\\&=\int_D\chi_1(y_1)p^0(t-t_1,y_1,y)m(dy_1)\\&\quad\cdot\int_0^{t_1}\phi_1(t_1-s)ds\int_V\psi_1(a)\widehat{H}_s(y_1,a)\frac{p(t_1-s,x,a)}{p(t,x,y)}\mu(da).\end{aligned}$$

Therefore

$$
\begin{aligned}
& E^x[\phi_1(L(t))\psi_1(X_{L(t)-})\chi_1(X_{t_1})\chi_2(X_{t_2})\phi_2(R(t))\psi_2(X_{R(t)})] \\
= & \int_0^{t_1} \phi_1(s)ds \int_V \psi_1(a)p(s,x,a)\mu(da) \int_D \chi_1(y_1)\widehat{H}_{t_1-s}(y_1,a)m(dy_1) \\
& \cdot \int_D \chi_2(y_2)m(dy_2) \int_D p^0(t_1-t,y_1,y)P^0_{t_2-t}(y,dy_2)m(dy) \\
& \cdot \int_V \psi_2(b)\mu(db) \int_{t_2}^{+\infty} \phi_2(u)H_{u-t_2}(y_2,b)du \\
= & \int_0^{t_1} \phi_1(s)ds \int_V \psi_1(a)p(s,x,a)\mu(da) \int_D \chi_1(y_1)\widehat{H}_{t_1-s}(y_1,a)m(dy_1) \\
& \cdot \int_D \chi_2(y_2)P^0_{t_2-t_1}(y_1,dy_2) \int_V \psi_2(b)\mu(db) \int_{t_2}^{+\infty} \phi_2(u)H_{u-t_2}(y_2,b)du.
\end{aligned}
$$

It gives the required result.

What we use frequently in the sequel is the following corollary.

**Corollary 2.1.** *For $x \in E$, $t > 0$, we have*

$$
\begin{aligned}
& P^x(L(t) \in ds, X_{L(t)} \in da, X_{R(t)} \in db, R(t) \in du) \\
= & ds\, p(s,x,a)\mu(da)A_{u-s}(a,b)\mu(db)\, du,
\end{aligned}
$$

*which holds on $(s,a,b,u) \in (0,t) \times V \times V \times (t,+\infty)$.*

**Proof.** Take $k = 1$ and $t = t_1$ in the theorem above. The result is immediate by the definition of $A_u$.

More precisely we may write the distribution above as

$$
\begin{aligned}
& P^x(L(t) \in ds, X_{L(t)} \in da, X_{R(t)} \in db, R(t) \in du) \\
= & 1_{(0,t)\times V\times V\times(t,+\infty)} ds\, p(s,x,a)\mu(da)A_{u-s}(a,b)\mu(db)\, du \\
& + \int_{y\in V} \epsilon_t(ds)\epsilon_t(du)\epsilon_y(da)\epsilon_y(db)p(t,x,y)m(dy).
\end{aligned}
$$

## §3. Lévy System of Boundary Processes

In this section, we shall use the results in §2 to compute some quantities related to excursion. At first define

$$J(\omega) := \{t : \ X_t(\omega) \in D\},$$

which is all of excursions away from $V$ of a path. Since $X$ is continuous, $J$ is open and let $I$ be the set of all left end points of excursion intervals in $J$. We first consider

$$A_t := \sum_{s\in I:\, 0<s\le t} f(X_{L(s)}, X_{R(s)}),$$

where $f$ is a non-negative continuous function on $V\times V$ vanishing on the diagonal: $f(a,a) = 0$ for any $a \in V$. Clearly $R(s)$ is a right continuous additive functional of $X$, and $s \in I$ if and only if $R(s-) < R(s)$ and $R(s-) = L(s)$. Thus

$$A_t = \sum_{0<s\le t:\, R(s-)<R(s)} f(X_{R(s-)}, X_{R(s)}).$$

Thus $A$ is an additive functional.

**Theorem 3.1.** *For $x \in E$, $t > 0$, we have*

$$E^x A_t = \int_0^t p(s,x,a) Nf(a)\mu(da)ds,$$

*where* $Nf(a) := \int_V N(a,b)f(a,b)\mu(db)$.

**Proof.** For $n \geq 1$, let $D_n := \{t_{n,k} = \frac{k}{2^n} : k \geq 0\}$ and $I_{n,k} = [t_{n,k-1}, t_{n,k})$. If $L(t) < t < R(t)$ for some $t \in D_n$, then we have $L(t) = L(t_{n,k}) \in I_{n,k}$ for one and only one $k$. On the other hand, for any $t > 0$, the excursion interval $(L(t), R(t))$ will have a binary point in $D_n$ for $n$ large enough. Thus any excursion interval will be counted finally and at most once in this way. It means that

$$A_t = \lim_{n\to\infty} \sum_{k\geq 1:\, t_{n,k}\leq t} f(X_{L(t_{n,k})}, X_{R(t_{n,k})}) 1_{\{L(t_{n,k})\in I_{n,k}\}},$$

which is an increasing limit. By Corollary 2.1, we may compute

$$\begin{aligned} &E^x \sum_{k\geq 1:\, t_{n,k}\leq t} f(X_{L(t_{n,k})}, X_{R(t_{n,k})}) 1_{\{L(t_{n,k})\in I_{n,k}\}} \\ &= \sum_{k\geq 1:\, t_{n,k}\leq t} \int_{I_{n,k}} dsp(s,x,a) \int_V \int_V f(a,b)\mu(da)\mu(db) \int_{t_{n,k}-s}^{\infty} A_u(a,b)du. \end{aligned}$$

By the monotone convergence theorem, we have

$$\begin{aligned} E^x A_t &= \int_V \int_V \int_0^t ds\, p(s,x,a) \int_0^\infty A_u(a,b)du\, f(a,b)\mu(da)\mu(db) \\ &= \int_V \int_0^t p(s,x,a) Nf(a)\mu(da)ds. \end{aligned}$$

That completes the proof.

One of the most successful applications of this result is that it provides a simple way to compute Lévy system of boundary process. Assume further that $E$ is compact, $m(V) = 0$ and write $\partial D := V$ in this case which is called boundary of $E$. It follows that

$$P^x(L(t) < t < R(t)) = 1 \quad \text{for any} \quad t > 0.$$

We also assume that $(P_t)$ and $(\widehat{P}_t)$ are Feller and for any $f \in C(\partial D)$,

$$K^q f(x) := \int_{\partial D} u^q(x,a) f(a)\mu(da) \in C(E).$$

It is easy to check that $K^q 1$ is $q$-uniformly excessive (refer to [1, Chapter IV]), there exists a continuous increasing additive functional (CAF) $\phi = (\phi(t))$, called a local time on $\partial D$, which satisfies that for any $q > 0, x \in E$,

$$K^q 1(x) = E^x \int_0^\infty e^{-qt} d\phi(t), \tag{3.1}$$

and if we define $R_\phi := \inf\{t > 0 : \phi(t) > 0\}$, then $R_\phi = T = T_{\partial D}$ (refer to [11]). Let $\tau = \{\tau_t\}$ be the right continuous inverse of $\phi$, i.e.,

$$\tau_t := \inf\{s : \phi(s) > t\},$$

and set $Y_t := X_{\tau_t}$, the time change of $X$ by $\tau$. The process $Y$ only lives on $\partial D$ and is called a boundary process of $X$.

**Theorem 3.2.** *The boundary process $Y$ defined above is purely discontinuous and its Lévy system is given by $(N(a,b)\mu(db), dt)$, i.e., for $f \in C(\partial D \times \partial D)$ vanishing on the* *diagonal, $x \in \partial D$, $q > 0$, we have*

$$E^x \sum_{0<t<+\infty} e^{-qt} f(Y_{t-}, Y_t) = E^x \int_0^\infty e^{-qt} N f(Y_t) dt.$$

**Proof.** Firstly it is known that

$$\tau_0 = R_\phi = T, \quad \phi(\tau_t) = t.$$

Then it follows that

$$\tau_{\phi(t)} = \inf\{s : \phi(s) > \phi(t)\} = \inf\{s : \phi(s-t) \circ \theta_t > 0\} = R_\phi \circ \theta_t + t = R(t).$$

Thus for any $t > 0$, $\tau_{\phi(t)} = R(t) > t$ a.s. It follows that $t \mapsto \tau_t$ takes only countable values, thus is purely discontinuous and $Y$ is also purely discontinuous. Secondly since it follows from (3.1) that

$$\int_V \int_0^t p(s, x, a) N f(a) \mu(da) ds = E^x \int_0^t N f(X_s) d\phi(s),$$

Theorem 3.1 shows that the dual predictable projection of $A$ is

$$\int_0^t N f(X_s) d\phi(s).$$

Thus, we have

$$\begin{aligned}
& E^x \sum_{0<t<+\infty} e^{-qt} f(Y_{t-}, Y_t) \\
&= E^x \sum_{0<t<+\infty} e^{-q\phi(t)} f(X(\tau_{\phi(t)-}), X(\tau_{\phi(t)})) \\
&= E^x \sum_{0<t<+\infty} e^{-q\phi(t)} f(X_{R(t-)}, X_{R(t)}) \\
&= E^x \int_0^\infty e^{-q\phi(t)} N f(X_t) d\phi(t) \\
&= E^x \int_0^\infty e^{-qt} N f(Y_t) dt.
\end{aligned}$$

That completes the proof.

Now we assume that $X$ is $m$-symmetric. Then it is characterized by a Dirichlet form $(\mathcal{E}, \mathcal{D})$ (refer to [5]):

$$\mathcal{D} = \Big\{ f \in L^2(m) : \lim_{t \downarrow 0} \frac{1}{2t} E^m [f(X_t) - f(X_0)]^2 < \infty \Big\},$$

$$\mathcal{E}(f, f) = \lim_{t \downarrow 0} \frac{1}{2t} E^m [f(X_t) - f(X_0)]^2.$$

We can write it this way since $X$ is conservative. Denote by $\mathcal{D}_e$ the extended Dirichlet space of $\mathcal{D}$. The following theorem generalizes the Douglas integral (refer to [5, (1.2.18)]).

**Theorem 3.3.** *Let $\mathcal{D}' := L^2(\partial D; \mu) \cap \mathcal{D}_e$, i.e., the set of $u \in L^2(\partial D; \mu)$ such that $u = v|_{\partial D}$ $\mu$-a.e. for some $v \in \mathcal{D}_e$. Then for $u \in \mathcal{D}'$,*

$$\mathcal{E}(hu, hu) = \frac{1}{2} \int_{\partial D} \int_{\partial D} [u(a) - u(b)]^2 N(a, b) \mu(da) \mu(db),$$

*where $hu(x) = \int_{\partial D} h(x, a) u(a) \mu(da)$, $x \in E$, as defined in §2.*

**Proof.** By a result of [4] (also see [5]), if we define for $u \in \mathcal{D}'$,

$$\mathcal{E}'(u,u) := \mathcal{E}(hu,hu),$$

then $(\mathcal{E}',\mathcal{D}')$ is the Dirichlet space on $L^2(\partial D,\mu)$, of the time changed process $Y$. To prove the theorem, it suffices to find the Dirichlet form of the boundary process $Y$. Since $Y$ is purely discontinuous, it is not hard to check that for any $u \in \mathcal{D}'$,

$$A_t^{[u]} := u(Y_t) - u(Y_0) = \lim_{n\to\infty} \sum_{0<s\le t} \Delta u(Y_s) 1_{\{|\Delta u(Y_s)|>\frac{1}{n}\}},$$

where $\Delta u(Y_s) := u(Y_s) - u(Y_{s-})$ and the right side converges uniformly in probability (see, e.g., [2]). Recall the Fukushima's decomposition,

$$A_t^{[u]} = M_t^{[u]} + N_t^{[u]},$$

as sum of a martingale AF and a CAF of zero energy. Thus the square bracket and angle bracket of $M^{[u]}$ are

$$[M^{[u]}]_t = \sum_{0<s\le t} (\Delta u(Y_s))^2,$$

$$\langle M^{[u]}\rangle_t = \int_0^t ds \int_E N(Y_s,b)(u(Y_s)-u(b))^2 \mu(db)$$

by Theorem 3.2. Thus

$$\mathcal{E}'(u,u) = e(M^{[u]}) = \sup_{t>0} \frac{1}{2t} E^\mu(\langle M^{[u]}\rangle_t)$$
$$= \frac{1}{2}\int_{\partial D}\int_{\partial D} [u(a)-u(b)]^2 N(a,b)\mu(da)\mu(db).$$

That completes the proof.

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