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ON THE HYPER ORDER OF SOLUTIONS OF HIGHER ORDER DIFFERENTIAL EQUATIONS**

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Abstract

The author investigates the hyper order of solutions of the higher order linear equation, and improves the results of M. Ozawa[15], G. Gundersen[6] and J. K. Langley[12].

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§1. Introduction and Results

For second order linear differential equation

$$f'' + e^{-z}f' + Q(z)f = 0, \tag{1.1}$$

where $Q(z)$ is an entire function of finite order, it is well known that each solution f of Equation (1.1) is an entire function, and that if f_1 and f_2 are any two linearly independent solutions of (1.1), then at least one of f_1, f_2 must have infinite order[8, pp.167–168]. Hence, "most" solutions of (1.1) will have infinite order. But Equation (1.1) with $Q(z) = -(1+e^{-z})$ possesses a solution $f = e^z$ of finite order.

Thus a natural question is: what condition on $Q(z)$ will guarantee that every solution $f \not\equiv 0$ of (1.1) has infinite order? Many authors, M. Frei, M. Ozawa, G. Gundersen and J. K. Langley have studied the problem.

M. Frei proved the following result in [4].

Theorem 1.1.[4] *If the equation*

$$f'' + e^{-z}f' + Cf = 0, \tag{1.2}$$

where $C(\neq 0)$ is a complex constant, possesses a solution $f \not\equiv 0$ of finite order, then $C = -k^2$, where k is a positive integer. Conversely for each positive integer k Equation (1.2) *with $C = -k^2$ possesses a solution f which is a polynomial in e^z of degree k.*

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M. Ozawa[15], I. Amemiya and M. Ozawa[1], and G. Gundersen[5] studied the case when $Q(z)$ is a particular polynomial. J. K. Langley proved the following result for the case when $Q(z)$ is a general polynomial in [12].

Theorem 1.2.[12] *Let $Q(z)$ be a non-constant polynomial. Then all nontrivial solutions of*

$$f'' + Ae^{-z}f' + Q(z)f = 0 \tag{1.3}$$

have infinite order, for any nonzero constant A.

For the case that $Q(z)$ is a transcendental entire function, G. Gundersen proved the following theorem.

Theorem 1.3.[5] *If $Q(z)$ is a transcendental entire function with order $\sigma(Q) \neq 1$, then every solution $f \not\equiv 0$ of (1.1) has infinite order.*

Here $\sigma(f)$ denotes the order of growth of meromorphic function $f(z)$. We assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory of meromorphic functions (e.g. see [9, 13]). In addition, we will use the notation $\sigma_2(f)$ to denote the hyper order of $f(z)$ (see [17]), which is defined to be

$$\sigma_2(f) = \varlimsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

The example in front shows that when $\sigma(Q) = 1$, Equation (1.1) can have a solution of finite order. Theorems 1.2, 1.3 show that if Q is a non-constant entire function and $\sigma(Q) \neq 1$, then every solution of (1.1) has infinite order. What condition on $Q(z)$ when $\sigma(Q) = 1$ will guarantee every solution $f \not\equiv 0$ of (1.1) has infinite order? The author investigates the problem and obtains the following results in [3].

Theorem 1.4.[3] *Let a, b be nonzero complex numbers and $a \neq b$, $Q(z)$ be a non-constant polynomial or $Q(z) = h(z)e^{bz}$ where $h(z)$ is nonzero polynomial. Then every solution $f(\not\equiv 0)$ of the equation*

$$f'' + e^{az}f' + Q(z)f = 0 \tag{1.4}$$

has infinite order and $\sigma_2(f) = 1$.

In this paper, we concern with the higher order differential equations and obtain the following results that greatly extend and perfect results of M. Frei, M. Ozawa, G. Gundersen, J. K. Langley and the author.

Theorem 1.5. *Suppose that $a_j\ (j = 0, \cdots, k-1)$ are complex numbers. There exist a_s and a_l such that $s < l$, $a_s = d_s e^{i\varphi}$, $a_l = -d_l e^{i\varphi}$, $d_s > 0$, $d_l > 0$, and for $j \neq s, l, a_j = d_j e^{i\varphi}\ (d_j \geq 0)$ or $a_j = -d_j e^{i\varphi}$, $\max\{d_j | j \neq s, l\} = d < \min\{d_s,\ d_l\}$. If $H_j = h_j e^{a_j z}$, where h_j are polynomials, $h_s h_l \not\equiv 0$, then every transcendental solution f of the differential equation*

$$f^{(k)} + H_{k-1}f^{(k-1)} + \cdots H_l f^{(l)} + \cdots + H_s f^{(s)} + \cdots + H_0 f = 0 \tag{1.5}$$

satisfies $\sigma(f) = \infty$ and $\sigma_2(f) = 1$.

Theorem 1.6. *Suppose that $H_j\ (j = 0, \cdots, k-1)$ satisfy additional hypotheses of Theorem 1.5, $g_j\ (j = 0, \cdots, k-1)$ are polynomials. Then every transcendental solution f of the differential equation*

$$f^{(k)} + (H_{k-1}+g_{k-1})f^{(k-1)} + \cdots (H_l+g_l)f^{(l)} + \cdots + (H_s+g_s)f^{(s)} + \cdots + (H_0+g_0)f = 0 \tag{1.6}$$

satisfies $\sigma(f)=\infty$ *and* $\sigma_2(f)=1$.

Corollary 1.1. *Suppose that* $h_j(z)\,(j=0,1,2)$ *are nonzero polynomials,* $g_j\,(j=0,1,2)$ *are polynomials. Then all nontrivial solutions of the following differential equations*

$$f'''+h_2e^zf''+h_1e^{-z}f'+h_0f=0, \tag{1.7}$$

$$f'''+(h_2e^z+g_2)f''+(h_1e^{-z}+g_1)f'+h_0f=0, \tag{1.8}$$

$$f'''+h_2f''+h_1e^zf'+h_0e^{-z}f=0, \tag{1.9}$$

$$f'''+h_2f''+(h_1e^z+g_1)f'+(h_0e^{-z}+g_0)f=0, \tag{1.10}$$

$$f'''+h_2e^zf''+h_1f'+h_0e^{-z}f=0, \tag{1.11}$$

$$f'''+(h_2e^z+g_2)f''+h_1f'+(h_0e^{-z}+g_0)f=0 \tag{1.12}$$

have infinite order and $\sigma_2(f)=1$.

§2. Lemmas for the Proofs of Theorems

Lemma 2.1.[2] *Let* $g(z)$ *be an entire function of infinite order with the hyper order* $\sigma_2(g)=\sigma$, *and let* $\nu(r)$ *be the central index of* g. *Then*

$$\overline{\lim_{r\to\infty}}\frac{\log\log\nu(r)}{\log r}=\sigma.$$

Lemma 2.2. *Let* $f(z)$ *be an entire function with* $\sigma(f)=\infty$ *and* $\sigma_2(f)=\alpha<+\infty$, *let set* $E\subset[1,\infty)$ *have finite logarithmic measure. Then there exists* $\{z_k=r_ke^{i\theta_k}\}$ *such that* $|f(z_k)|=M(r_k,f)$, $\theta_k\in[0,2\pi)$, $\lim\limits_{k\to\infty}\theta_k=\theta_0\in[0,2\pi)$, $r_k\notin E$, $r_k\to\infty$ *and for any given* $\varepsilon>0$, *for sufficiently large* r_k, *we have*

$$\overline{\lim_{k\to\infty}}\frac{\log\nu(r_k)}{\log r_k}=\infty, \tag{2.1}$$

$$\exp\{r_k^{\alpha-\varepsilon}\}<\nu(r_k)<\exp\{r_k^{\alpha+\varepsilon}\}, \tag{2.2}$$

where $\nu(r)$ *is the central index of* $f(z)$.

Proof. By Lemma 2.1 and $\sigma_2(f)=\alpha$, we have

$$\overline{\lim_{r\to\infty}}\frac{\log\log\nu(r)}{\log r}=\sigma_2(f)=\alpha<\infty.$$

There is a sequence $\{r'_k\}\,(r'_k\to\infty)$ satisfying

$$\lim_{r'_k\to\infty}\frac{\log\log\nu(r'_k)}{\log r'_k}=\alpha.$$

Set the logarithmic measure of E, $\operatorname{lm}E=\delta<\infty$. Then there is a point $r_k\in[r'_k,(\delta+1)r'_k]\backslash E$, so that since

$$\frac{\log\log\nu(r_k)}{\log r_k}\geq\frac{\log\log\nu(r'_k)}{\log[(\delta+1)r'_k]}=\frac{\log\log\nu(r'_k)}{\log r'_k\left[1+\frac{\log(\delta+1)}{\log r'_k}\right]},$$

we have

$$\lim_{r_k\to\infty}\frac{\log\log\nu(r_k)}{\log r_k}=\alpha.$$

Therefore, (2.2) holds. And (2.1) obviously holds. Now we take $z_k=r_ke^{i\theta_k}$, $\theta_k\in[0,2\pi)$, such that $|f(z_k)|=M(r_k,f)$. There is a subset $\{\theta_{k_j}\}$ of $\{\theta_k\}$, such that $\lim\limits_{j\to\infty}\theta_{k_j}=\theta_0\in[0,2\pi)$. Thus $\{z_{k_j}=r_{k_j}e^{i\theta_{k_j}}\}$ satisfies our assertion.

Lemma 2.3. *Suppose that $H = he^{az}$, where h is a nonzero polynomial, $a = de^{i\varphi}, \varphi \in [0, 2\pi)$, $d > 0$ is a constant, $z = re^{i\theta}$, $\theta \in [0, 2\pi)$. Then for any given $\varepsilon\,(> 0)$, we have as r sufficiently large,*

(i) *if* $\cos(\varphi + \theta) > 0$, *then*

$$\exp\{(1-\varepsilon)dr\cos(\varphi+\theta)\} \le |H(re^{i\theta})| \le \exp\{(1+\varepsilon)dr\cos(\varphi+\theta)\}; \tag{2.3}$$

(ii) *if* $\cos(\varphi + \theta) < 0$, *then*

$$\exp\{(1+\varepsilon)dr\cos(\varphi+\theta)\} \le |H(re^{i\theta})| \le \exp\{(1-\varepsilon)dr\cos(\varphi+\theta)\}; \tag{2.4}$$

(iii) *if* $\cos(\varphi + \theta) = 0$, *then*

$$|H(re^{i\theta})| \le r^M, \tag{2.5}$$

where $M\,(> 0)$ is a constant.

Proof. Since $|e^{az}| = \exp\{dr\cos(\varphi + \theta)\}$, we can easily prove Lemma 2.3.

By using the similar proof as that of Lemma 4 of [6], we can obtain the following Lemma 2.4.

Lemma 2.4. *Let $f(z)$ be an entire function and suppose that $|f^{(k)}(z)|$ is unbounded on some ray $\arg z = \theta$. Then there exists an infinite sequence of points $z_n = r_n e^{i\theta}$ $(n = 1, 2, \cdots)$, where $r_n \to \infty$, such that $f^{(k)}(z_n) \to \infty$ and*

$$\Big|\frac{f^{(j)}(z_n)}{f^{(k)}(z_n)}\Big| \le |z_n|^{k-j}(1 + o(1)), \quad j = 0, \cdots, k-1. \tag{2.6}$$

Lemma 2.5.[7] *Let f be a transcendental meromorphic function with $\sigma(f) = \sigma < \infty$, $H = \{(k_1, j_1), (k_2, j_2), \cdots, (k_q, j_q)\}$ be a finite set of distinct pairs of integers which satisfy $k_i > j_i \ge 0$ for $i = 1, \cdots, q$. And let $\varepsilon > 0$ be a given constant. Then there exists a set $E \subset [0, 2\pi)$ which has linear measure zero, such that if $\psi \in [0, 2\pi)\backslash E$, then there is a constant $R_0 = R_0(\psi) > 1$ such that for all z satisfying $\arg z = \psi$ and $|z| \ge R_0$ and for all $(k, j) \in H$, we have*

$$\Big|\frac{f^{(k)}(z)}{f^{(j)}(z)}\Big| \le |z|^{(k-j)(\sigma-1+\varepsilon)}. \tag{2.7}$$

Lemma 2.6.[14, p.214] *Let $f(z)$ be analytic in the region $D = \{z|\alpha < \arg z < \beta, r_0 < |z| < \infty\}$, and continuous on $\overline{D} = D \cup C$ (C is the boundary of D). If for any given small $\varepsilon > 0$, there exists $R(\varepsilon) > 0$ such that for $|z| \ge R(\varepsilon), z \in D$, we have $|f(z)| < \exp\{\varepsilon|z|^{\frac{\pi}{\beta-\alpha}}\}$, and for $z \in C$, we have $|f(z)| \le M(M > 0$ is a constant), then $|f(z)| \le M$ holds for all $z \in D$.*

Lemma 2.7. *Let $f(z)$ be an entire function with $\sigma(f) = \sigma < \infty$. Suppose that there exists a set $E \subset [0, 2\pi)$ which has linear measure zero, such that for any ray $\arg z = \theta_0 \in [0, 2\pi)\backslash E$, $|f(re^{i\theta_0})| \le Mr^k$ $(M = M(\theta_0) > 0$ is a constant, $k\,(> 0)$ is a constant independent of $\theta_0)$. Then $f(z)$ is a polynomial with $\deg f \le k$.*

Proof. Since E has linear measure zero, we can choose points $\theta_j \in [0, 2\pi)\backslash E\,(j = 1, \cdots, n, n+1)$ such that

$$0 \le \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi, \quad \theta_{n+1} = \theta_1 + 2\pi,$$
$$\max\{\theta_{j+1} - \theta_j | 1 \le j \le n\} < \frac{\pi}{\sigma + 1}.$$

For any given $\varepsilon > 0$, by $\sigma(f) = \sigma$, there is an $R(\varepsilon) > 0$, such that

$$\Big|\frac{f(z)}{z^k}\Big| \le \exp\{\varepsilon|z|^{\sigma+1}\}.$$

In the sectors

$$H_j = \{z|\theta_j \le \arg z \le \theta_{j+1}, |z| \ge R\}, \quad j = 1, \cdots, n,$$

$$\Big|\frac{f(z)}{z^k}\Big| \le \exp\{\varepsilon|z|^{\frac{\pi}{\theta_{j+1}-\theta_j}}\}$$

hold, and on the rays $\arg z = \theta_j, \theta_{j+1}$, $|\frac{f(z)}{z^k}| \le M$ holds. By Lemma 2.6, $|\frac{f(z)}{z^k}| \le M$ holds in each H_j. Hence $|f(z)| \le M|z|^k$ holds in whole plane. Therefore $f(z)$ is a polynomial with $\deg f \le k$.

Lemma 2.8.[7] *Let f be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exist a set $E \subset (1, +\infty)$ which has a finite logarithmic measure and a constant $B > 0$ depending only on α and (m, n) $(m, n \in \{0, \cdots, k\}$ and $m < n)$ such that for all z satisfying $|z| = r \notin [0, 1] \bigcup E$, we have*

$$\Big|\frac{f^{(n)}(z)}{f^{(m)}(z)}\Big| \le B\Big(\frac{T(\alpha r, f)}{r}(\log^\alpha r)\log T(\alpha r, f)\Big)^{n-m}. \tag{2.8}$$

Lemma 2.9. *Let $f(z)$ be a transcendental entire function. Then there is a set $E \subset (1, +\infty)$ having finite logarithmic measure such that when we take a point z satisfying $|z| = r \notin [0, 1] \bigcup E$ and $|f(z)| = M(r, f)$, we have*

$$\Big|\frac{f(z)}{f^{(s)}(z)}\Big| \le 2r^s, \quad s \in N. \tag{2.9}$$

Proof. From the Wiman-Valiron theory (see [10, 11, 13, 16]), we have

$$\frac{f^{(s)}(z)}{f(z)} = \Big(\frac{\nu(r)}{z}\Big)^s(1 + o(1)), \tag{2.10}$$

where $|z| = r \notin [0, 1] \bigcup E$, $E \subset (1, +\infty)$ is of finite logarithmic measure such that $|f(z)| = M(r, f)$ and $\nu(r)$ denotes the central index of $f(z)$. Since f is transcendental, $\nu(r) \to \infty$ $(r \to \infty)$. Hence when z satisfies $|z| = r \notin [0, 1] \bigcup E$ and $|f(z)| = M(r, f)$, by (2.10) we get (2.9).

Lemma 2.10. *Let H_j $(j = 0, \cdots, k-1)$ be entire functions with $\sigma(H_j) \le \sigma < \infty$. If $f(z)$ is a solution of the differential equation*

$$f^{(k)} + H_{k-1}f^{(k-1)} + \cdots + H_0 f = 0,$$

then $\sigma_2(f) \le \sigma$.

Proof. Using the Wiman-Valiron theory, we can easily prove Lemma 2.10.

§3. Proofs of Theorems 1.5 and 1.6, Corollary 1.1

Proof of Theorem 1.5. Assume that $f(z)$ is a transcendental solution of (1.5). First we show that $\sigma(f) = \infty$. Suppose to the contrary that $\sigma(f) = \sigma < \infty$. By Lemma 2.5, there exists a set $E_1 \subset [0, 2\pi)$ which has linear measure zero, such that if $\theta \in [0, 2\pi)\backslash E_1$, then there is a constant $R = R(\theta) > 1$ such that for all z satisfying $\arg z = \theta$ and $|z| \ge R$ and for $j = s+1, \cdots, k$, we have

$$\Big|\frac{f^{(j)}(z)}{f^{(s)}(z)}\Big| \le |z|^M, \tag{3.1}$$

where $M\,(>0)$ is a constant. Set

$$E_2=\{\theta|\cos(\varphi+\theta)=0\}.$$

Then E_2 is a finite set. For any $\theta\in[0,2\pi)\backslash(E_1\bigcup E_2)$, we have $\cos(\varphi+\theta)>0$ or $\cos(\varphi+\theta)<0$. We divide it into two cases: (a) $\cos(\varphi+\theta)>0$, and (b) $\cos(\varphi+\theta)<0$.

Case (a) $\cos(\varphi+\theta)>0$. By $h_sh_l\not\equiv 0$ we have $h_s\not\equiv 0$. By Lemma 2.3, for any given $\varepsilon\,(0<3\varepsilon<\frac{d_s-d}{d_s})$, we obtain for sufficiently large r,

$$\exp\{(1-\varepsilon)d_sr\cos(\varphi+\theta)\}\le|H_s(re^{i\theta})|\le\exp\{(1+\varepsilon)d_sr\cos(\varphi+\theta)\},\tag{3.2}$$

$$|H_j(re^{i\theta})|\le\exp\{(1+\varepsilon)dr\cos(\varphi+\theta)\},\quad j\ne s.\tag{3.3}$$

Now we prove that $|f^{(s)}(re^{i\theta})|$ is bounded on the ray $\arg z=\theta$. If $|f^{(s)}(re^{i\theta})|$ is unbounded on the ray $\arg z=\theta$, then by Lemma 2.4, there exists an infinite sequence of points $z_q=r_qe^{i\theta}\quad(q=1,2,\cdots)$ such that as $r_q\to\infty,\quad f^{(s)}(z_q)\to\infty$ and

$$\Big|\frac{f^{(j)}(z_q)}{f^{(s)}(z_q)}\Big|\le(1+o(1))|z_q|^{s-j},\quad j=0,\cdots,s-1.\tag{3.4}$$

Substituting (3.1)–(3.4) into (1.5), we obtain

$$\begin{aligned}&\exp\{(1-\varepsilon)d_sr_q\cos(\varphi+\theta)\}\le|H_s(z_q)|\\ \le&\Big|\frac{f^{(k)}(z_q)}{f^{(s)}(z_q)}\Big|+\cdots+\Big|H_{s+1}(z_q)\frac{f^{(s+1)}(z_q)}{f^{(s)}(z_q)}\Big|\\ &+\Big|H_{s-1}(z_q)\frac{f^{(s-1)}(z_q)}{f^{(s)}(z_q)}\Big|+\cdots+\Big|H_0(z_q)\frac{f(z_q)}{f^{(s)}(z_q)}\Big|\\ \le&\,k\exp\{(1+\varepsilon)dr_q\cos(\varphi+\theta)\}|z_q|^M.\end{aligned}\tag{3.5}$$

By (3.5), we obtain

$$\exp\Big\{\frac{1}{3}(d_s-d)r_q\cos(\varphi+\theta)\Big\}\le r_q^M.\tag{3.6}$$

This is a contradiction. Hence $|f^{(s)}(re^{i\theta})|\le M$ on $\arg z=\theta$. We can easily obtain

$$|f(re^{i\theta})|\le Mr^k\tag{3.7}$$

on $\arg z=\theta$.

Case (b) $\cos(\varphi+\theta)<0$. We can use the same reasoning as in Case (a) by replacing H_s with H_l to prove that

$$|f(re^{i\theta})|\le Mr^k\tag{3.8}$$

on the ray $\arg z=\theta$. By Lemma 2.7, combining (3.7), (3.8) and the fact that $E_1\bigcup E_2$ has linear measure zero, we know that $f(z)$ is a polynomial which contradicts our assumption. Therefore $\sigma(f)=\infty$.

Secondly, we show that $\sigma_2(f)=1$. Assume $\sigma_2(f)=\alpha<1$. By Lemma 2.8, we know that there is a set $E_3\subset(1,+\infty)$ which has finite logarithmic measure, and there is a constant $A>0$, such that

$$\Big|\frac{f^{(j)}(z)}{f^{(d)}(z)}\Big|\le A(T(2r,f))^{2k},\quad k\ge j>d\ge 0\tag{3.9}$$

hold for $|z|=r\notin E_3$ and for sufficiently large r.

By the Wiman-Valiron theory, we have basic formulas

$$\frac{f^{(j)}(z)}{f(z)}=\Big(\frac{\nu(r)}{z}\Big)^j(1+o(1)),\quad j=1,\cdots,k, \tag{3.10}$$

where z satisfies $|f(z)|=M(r,f),|z|=r\notin[0,1]\bigcup E_4, E_4\subset(1,\infty)$ has finite logarithmic measure, $\nu(r)$ is the central index of $f(z)$.

By Lemma 2.2, we can choose a point range $\{z_n=r_ne^{i\theta_n}\}$ such that $f(z_n)=M(r_n,f)$, $\theta_n\in[0,2\pi),\ \lim\limits_{n\to\infty}\theta_n=\theta_0,\ r_n\notin[0,1]\bigcup E_3\bigcup E_4,\ r_n\to\infty$ and for any given $\varepsilon_1\ (0<3\varepsilon_1<\min\{1-\alpha,\frac{d_s-d}{d_s}\})$, for sufficiently large r_n, we have

$$\overline{\lim_{n\to\infty}}\frac{\log\nu(r_n)}{\log r_n}=\infty, \tag{3.11}$$

$$\exp\{r_n^{\alpha-\varepsilon_1}\}\le\nu(r_n)\le\exp\{r_n^{\alpha+\varepsilon_1}\}. \tag{3.12}$$

For θ_0, we have $\mathrm{Re}\{a_sz\}=d_sr\cos(\varphi+\theta_0),\ \mathrm{Re}\{a_lz\}=-d_lr\cos(\varphi+\theta_0)$. For $\cos(\varphi+\theta_0)$, there are three cases: (i) $\cos(\varphi+\theta_0)>0$; (ii) $\cos(\varphi+\theta_0)<0$; (iii) $\cos(\varphi+\theta_0)=0$. We consider the three cases respectively.

Case(i) $\cos(\varphi+\theta_0)>0$. By (1.5) we have

$$\begin{aligned}h_se^{a_sz}=&\Big(\frac{f^{(k)}}{f^{(s)}}+h_{k-1}e^{a_{k-1}z}\frac{f^{(k-1)}}{f^{(s)}}+\cdots+h_{s+1}e^{a_{s+1}z}\frac{f^{(s+1)}}{f^{(s)}}\Big)\\&+\Big(h_{s-1}e^{a_{s-1}z}\frac{f^{(s-1)}}{f}+\cdots+h_1e^{a_1z}\frac{f'}{f}+h_0e^{a_0z}\Big)\frac{f}{f^{(s)}}.\end{aligned} \tag{3.13}$$

For sufficiently large $n,\cos(\varphi+\theta_n)>0$ hold since $\theta_n\to\theta_0$ and $\cos(\varphi+\theta_0)>0$. Hence for sufficiently large n, by Lemma 2.3, we know that θ_n satisfies

$$|h_se^{a_sz_n}|\ge\exp\{(1-\varepsilon_1)d_sr_n\cos(\varphi+\theta_n)\}, \tag{3.14}$$

$$|h_je^{a_jz_n}|\le r_n^M\exp\{(1+\varepsilon_1)dr_n\cos(\varphi+\theta_n)\}\qquad\text{for}\quad j\ne s. \tag{3.15}$$

By Lemma 2.9, we know that when $z_n=r_ne^{i\theta_n}$ satisfies $|f(z_n)|=M(r_n,f)$, we have

$$\Big|\frac{f(z_n)}{f^{(s)}(z_n)}\Big|\le r_n^k. \tag{3.16}$$

For the point range $\{z_n=r_ne^{i\theta_n}\}$, substituting (3.9), (3.14)–(3.16) into (3.13), as n sufficiently large, we obtain

$$\begin{aligned}&\exp\{(1-\varepsilon_1)d_sr_n\cos(\varphi+\theta_n)\}\\&\le Akr_n^M\exp\{(1+\varepsilon_1)dr_n\cos(\varphi+\theta_n)\}(T(2r_n,f))^{2k}.\end{aligned} \tag{3.17}$$

Hence

$$\exp\Big\{\frac{1}{3}(d_s-d)\cos(\varphi+\theta_n)r_n\Big\}\le Akr_n^M(T(2r_n,f))^{2k}. \tag{3.18}$$

By (3.18), we get $\sigma_2(f)\ge1$.

Case (ii) $\cos(\varphi+\theta_0)<0$. We can use the same reasoning as in the case (i) by replacing $h_se^{a_sz}$ with $h_le^{a_lz}$ to prove that if θ_0 satisfies the case (ii), then $\sigma_2(f)\ge1$.

Case (iii) $\cos(\varphi+\theta_0)=0$. Since $z_n=r_ne^{i\theta_n}$ satisfies $r_n\to\infty,\theta_n\to\theta_0$ as $n\to\infty$, the ray $\arg w=\varphi+\theta_0$ is an asymptotic line of $\{a_jz_n\}\,(j=0,\cdots,k-1)$. Hence there is an $N>0$, such that when $n>N$, by $\mathrm{Re}\{a_jr_ne^{i\theta_0}\}=0$, we have for $j=0,\cdots,k-1$,

$$-1<\mathrm{Re}\{a_jr_ne^{i\theta_n}\}<1,\quad\frac{1}{e}<|e^{a_jz_n}|<e,\quad r^{M_1}\le|h_je^{a_jz_n}|\le r^{M_2}, \tag{3.19}$$

where M_1, M_2 are two positive integers. By (1.5), (3.10) and (3.19), we have

$$\Big(-\frac{\nu(r_n)}{z_n}\Big)^k(1+o(1)) = H_{k-1}(z_n)\Big(\frac{\nu(r_n)}{z_n}\Big)^{k-1}(1+o(1)) + \cdots + H_0(z_n), \tag{3.20}$$

$$\nu(r_n)(1+o(1)) \le r^M. \tag{3.21}$$

(3.21) contradicts (3.11). This shows that the case (iii) can not occur. In the cases (i) and (ii), we have proved that $\sigma_2(f) \ge 1$. Combining this and Lemma 2.10, we have $\sigma_2(f) = 1$.

Proof of Theorem 1.6. Using the similar proof as that of Theorem 1.5, we can prove Theorem 1.6.

Proof of Corollary 1.1. Since any nonzero polynomial does not satisfy each of Equations (1.7)–(1.12), by Theorems 1.5 and 1.6, we know that Corollary 1.1 holds.

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