Chin. Ann. Math.
24B:4(2003),509–520.

METRIC GENERALIZED INVERSE OF LINEAR OPERATOR IN BANACH SPACE***

WANG Hui* WANG Yuwen**

Abstract

The Moore–Penrose metric generalized inverse T^+ of linear operator T in Banach space is systematically investigated in this paper. Unlike the case in Hilbert space, even T is a linear operator in Banach Space, the Moore-Penrose metric generalized inverse T^+ is usually homogeneous and nonlinear in general. By means of the methods of geometry of Banach Space, the necessary and sufficient conditions for existence, continuity, linearity and minimum property of the Moore-Penrose metric generalized inverse T^+ will be given, and some properties of T^+ will be investigated in this paper.

Keywords Banach space, Metric generalized inverse, Generalized orthogonal decomposition, Homogeneous operator
2000 MR Subject Classification 46B20
Chinese Library Classification O177.2 **Document Code** A
Article ID 0252-9599(2003)04-0509-12

§1. Introduction

Let X, Y be normed spaces, $T : X \to Y$ a linear operator. In order to get the best approximate solution (i.e. least extremal solution) for the ill–posed linear operator equation $Tx = y$, Nashed and Votruba[1−3] introduced the metric generalized inverse T^∂ of operator T, and the orthogonal partial inverse T^+_ρ of operator by means of the concept of orthogonally complemented subspace in normal linear space, while T^+_ρ is a single value linear selection of T^∂. By the orthogonal partial inverse T^+_ρ, the best approximate solution to the equation $Tx = y$ can be obtained easily. However, in general, in normal linear spaces, the orthogonally complemented subspaces are rare[1, p.39], so that the existence problem of the orthogonal partial inverse T^+_ρ of linear operator T is difficult to answer. Therefore, the problem of obtaining selections with nice property for the metric generalized inverse merits study, as Nashed and Votruba indicated in [1, 2].

Manuscript received December 29, 2001. Revised January 7, 2003.

*Department of Mathematics, Harbin Institute of Technology, Harbin 150080, China.
E-mail: sxx_wh@sina.com.cn
**Department of Mathematics, Harbin Normal University, Harbin 150080, China.
E-mail: sxx_zr@hrbnu.edu.cn

***Project supported by the National Natural Science Foundation of China (No.19971023) and the Heilongjiang Provincial Natural Science Foundation of China.

The purpose of this paper is to answer partly the above problem. Since the metric projectors on closed subspaces in Hilbert spaces are not only linear projectors, but also orthogonal projectors, and the orthogonal generalized inverse of linear operator T in Hilbert spaces[4] is just the linear metric generalized inverse. Such problems have been discussed by many authors[4–12]. While the metric projectors on closed subspaces in Banach space are no longer linear, and then the linear generalized inverse and the metric generalized inverse of linear operator in Banach spaces are quite different. The problem on linear generalized inverse in Banach space have been discussed by several authors[1–3, 13, 14]. Several special single valued metric generalized inverses of bounded linear operators or densely defined closed linear operators with closed range in Banach spaces and their applications have been investigated by Yu-Wen Wang et al[15–19] and Holmes[20].

In this paper, the single valued Moore-Penrose metric generalized inverses of linear operator in Banach space are investigated by means of the methods of geometry of Banach spaces.

§2. Existence of Moore-Penrose Metric Generalized Inverse

Throughout this paper, let X and Y be Banach spaces, and T be a linear operator from X to Y. On the definition of geometric properties of Banach spaces X and Y, we can find them in [20–22]. Let $\langle x^*,\ x\rangle$ denote the value of functional $x^* \in X^*$ at element $x \in X$, where X^* is the dual space of X.

Let $D(T),\ R(T),\ N(T)$ denote the domain, range and null space of T, respectively. $x_0 \in D(T)$ is called the best approximation solution (b.a.s.) to the operator equation $Tx = y$, if

$$\|Tx_0 - y\| = \inf\{\|Tx - y\ \| x \in D(T)\},$$
$$\|x_0\| = \inf\Big\{\|v\| | v \in D(T),\ \|Tv - y\| = \inf_{x\in D(T)} \|Tx - y\|\Big\},$$

where $y \in Y$.

Definition 2.1. *Let T be a linear operator from X to Y, $\overline{N(T)}$ and $\overline{R(T)}$ be the Chebyshev sets in X and Y, respectively. If there exists a homogeneous operator $T^+ : Y \to X$ such that*

(i) $TT^+T = T$;

(ii) $T^+TT^+ = T^+$;

(iii) $T^+T = I_{D(T)} - P_{\overline{N(T)}}$;

(iv) $TT^+ = P_{\overline{R(T)}}$,

then T^+ is called the Moore-Penrose metric generalized inverse of T, where $I_{D(T)}$ is the identity operator on $D(T)$, and $P_{\overline{N(T)}},\ P_{\overline{R(T)}}$ are the metric projectors onto $\overline{N(T)},\ \overline{R(T)}$ respectively.

If $N(T)$ and $R(T)$ are closed, T^+ is the same as Definition 2.9 in [17].

Theorem 2.1. *Let X and Y be strictly convex Banach spaces, T be a linear operator from X to Y. Then there exists a Moore-Penrose metric generalized inverse T^+ if and only if*

$$D(T) = N(T) \dot{+} C(T), \tag{2.1}$$

where $C(T) = \{x \in D(T) \mid F_X(x) \cap N(T)^\perp \neq \phi\}$.

Proof. Necessity. Suppose T has a Moore-Penrose metric generalized inverse T^+ given by Definition 2.1. Then $\overline{N(T)}$ and $\overline{R(T)}$ are Chebyshev subspaces in X and Y, respectively. For any $x \in D(T)$, by (i) in Definition 2.1, we have $TT^+Tx = Tx$, and hence $T^+Tx \in D(T)$. From (iii) in Definition 2.1, we obtain $P_{\overline{N(T)}}x = x - T^+Tx \in D(T)$. Thus, we get that

$$T(P_{\overline{N(T)}}x) = T(x - T^+Tx) = Tx - TT^+Tx = 0,$$

i.e.

$$P_{\overline{N(T)}}x \in N(T) \qquad \text{for any} \quad x \in D(T). \tag{2.2}$$

Because $\overline{N(T)}$ is a Chebyshev subspace in X, and $F_X^{-1}(\overline{N(T)}^{\perp}) = F_X^{-1}(N(T)^{\perp})$, from Lemma 3.2 in [28], for any $x \in D(T) \subset X$, there exists a unique decomposition

$$x = P_{\overline{N(T)}}x + x_2, \qquad x_2 \in F_X^{-1}(N(T)^{\perp}). \tag{2.3}$$

Hence $x_2 = x - P_{\overline{N(T)}}x \in D(T)$, and therefore $x_2 \in D(T) \cap F_X^{-1}(N(T)^{\perp}) = C(T)$. Thus, we obtain

$$D(T) = N(T) + C(T). \tag{2.4}$$

For any $x \in N(T) \cap C(T)$, since $F_X(x) \cap N(T)^{\perp} \neq \phi$, we may choose an $x^* \in F_X(x) \cap N(T)^{\perp}$. Hence $0 = \langle x^*,\ x\rangle = \|x\|^2 = \|x^*\|^2$. This implies that $x = 0$, i.e. (2.1) is true.

Sufficiency. We first show that T is one to one mapping from $C(T)$ to $R(T)$. In fact, for any $y \in R(T)$, there exists an $x \in D(T)$ such that $y = Tx$, (2.1) implies that $x = x_1 + x_2,\ x_1 \in N(T),\ x_2 \in C(T)$. Hence $y = Tx = Tx_2$, i.e. T is surjective mapping from $C(T)$ to $R(T)$. For any $x_1,\ x_2 \in C(T)$, if $Tx_1 = Tx_2$, then $x_1 - x_2 \in N(T)$. By the definition of $C(T)$, we have

$$F_X(x_i) \cap N(T)^{\perp} \neq \phi, \qquad i = 1, 2.$$

Choose $x_i^* \in F_X(x_i) \cap N(T)^{\perp}$ $(i = 1, 2)$, then $x_1^* - x_2^* \in N(T)^{\perp}$, and

$$\langle x_1^* - x_2^*,\ x_1 - x_2\rangle = 0. \tag{2.5}$$

Since X is strictly convex, by Proposition 2.14 in [22], F_X is strictly monotone. Hence (2.5) implies that $x_1 = x_2$, i.e. T is injective.

Let $T|_{C(T)}$ denote the restriction of the operator T to the set $C(T)$. Since the dual mapping F_X is homogeneous, so the set $C(T)$ is also homogeneous and $((T|_{C(T)}))^{-1}$ is a homogeneous operator from $R(T)$ to $C(T)$. Let $D^+ = R(T) \dot{+} F_Y^{-1}(R(T)^{\perp})$, where $F_Y^{-1}(R(T)^{\perp}) = \{y \in Y \mid F_Y(y) \cap R(T)^{\perp} \neq \phi\}$. It follows from the homogeneity of F_Y that the set D^+ is also homogenous set. Define an operator T^+ from D^+ to $C(T)$ as follows: for any $y \in D^+$, y has an unique decomposition $y = y_1 + y_2,\ y_1 \in R(T),\ y_2 \in F_Y^{-1}(R(T)^{\perp})$. Define

$$T^+y = (T|_{C(T)})^{-1}y_1. \tag{2.6}$$

On the other hand, since $y - y_1 = y_2 \in F_Y^{-1}(R(T)^{\perp})$, we have $F_Y(y - y_1) \cap R(T)^{\perp} \neq \phi$. For any $y \in D^+$ and $y \notin \overline{R(T)}$, we may choose $y_1^* \in F_Y(y - y_1) \cap R(T)^{\perp}$. Let $y^* = y_1^*/\|y - y_1\|$. We have $\|y^*\| = 1$, and

$$\langle y^*,\ y - y_1\rangle = \langle y_1^*, y - y_1\rangle/\|y - y_1\| = \|y - y_1\|^2/\|y - y_1\| = \|y - y_1\|.$$

Since $y_1^* \in R(T)^{\perp} = \overline{R(T)}^{\perp}$, we have $y^* \in \overline{R(T)}^{\perp}$. By Lemma 3.1 in [28] and strict convexity of the space Y, we have

$$y_1 = P_{\overline{R(T)}}y \in R(T). \tag{2.7}$$

It follows from (2.6) and (2.7) that

$$T^+y = (T|_{C(T)})^{-1}P_{\overline{R(T)}}y, \qquad y \in D^+. \tag{2.8}$$

By the homogeneity of $(T|_{C(T)})^{-1}$ and $P_{\overline{R(T)}}$ (see [20]), we know that T^+ is a homogeneous operator from D^+ to $D(T)$. For any $y \in D^+$, by (2.7), we have $P_{\overline{R(T)}}y \in R(T)$, and hence

$$TT^+y = T(T|_{C(T)})^{-1}P_{\overline{R(T)}}y = P_{\overline{R(T)}}y,$$

i.e. $TT^+ = P_{\overline{R(T)}}$ on D^+. Therefore, (iv) in Definition 2.1 follows.

For any $x \in D(T)$, by (2.1), x has an unique decomposition $x = x_1+x_2$, $x_1 \in N(T)$, $x_2 \in C(T)$, where $C(T) = D(T) \cap F_X^{-1}(N(T)^{\perp}) \subset F_X^{-1}(N(T)^{\perp})$. Since the space X is strictly convex, by the same argument as (2.7), we have $x_1 = P_{\overline{N(T)}}x \in N(T)$. Hence

$$x = P_{\overline{N(T)}}x + x_2, \quad x_2 \in C(T). \tag{2.9}$$

It follows that

$$\begin{aligned} T^+Tx &= T^+T(P_{\overline{N(T)}}x + x_2) = T^+Tx_2 \\ &= (T|_{C(T)})^{-1}P_{\overline{R(T)}}Tx_2 = x_2 = (I_{D(T)} - P_{\overline{N(T)}})x, \end{aligned} \tag{2.10}$$

i.e. $T^+T = I_{D(T)} - P_{\overline{N(T)}}$ on $D(T)$, thus (iii) in Definition 2.1 follows. $TT^+T = T$ on $D(T)$ is obvious.

For any $y \in D^+$, by (2.8), $T^+y \in C(T) \subset D(T)$. It follows from (2.10) that

$$T^+TT^+y = (I_{D(T)} - P_{\overline{N(T)}})T^+y = T^+y - P_{\overline{N(T)}}T^+y. \tag{2.11}$$

Since $T^+y \in C(T) \subset F_X^{-1}(\overline{N(T)}^{\perp})$, we obtain

$$F_X(T^+y) \cap \overline{N(T)}^{\perp} \neq \phi.$$

Take $x_1^* \in F_X(T^+y) \cap \overline{N(T)}^{\perp}$, and write $x^* = x_1^*/\|T^+y\|$, then

$$\|x^*\| = 1, \quad \langle x^*,\ T^+y - 0\rangle = \langle x_1^*,\ T^+y\rangle/\|T^+y\| = \|T^+y - 0\|.$$

Since $x_1^* \in \overline{N(T)}^{\perp}$, we have $x^* \in \overline{N(T)}^{\perp}$. Since X is strictly convex, by Lemma 3.1 in [28], we have $0 = P_{\overline{N(T)}}T^+y$. Hence, (2.11) shows that $T^+TT^+ = T^+$ on D^+, i.e. T^+ is the Moore-Penrose metric generalized inverse.

The main result in [17] can follow easily from above Theorem 2.1. We have

Corollary 2.1.[17] *Let X and Y be reflexive strictly convex Banach Spaces, T be a bounded linear operator or densely defined closed linear operator from X to Y. Then there exists the Moore-Penrose metric generalized inverse T^+ of operator T. Furthermore, if the range $R(T)$ is closed, then $D^+ = Y$.*

Proof. If T is a bounded linear operator or densely defined closed linear operator, the null space $N(T)$ is a closed subspace of X. Since X is reflexive, strictly convex, closed subspace $N(T)$ is Chebyshev. By Lemma 3.2 in [28], we have $D(T) = N(T) \dot{+} C(T)$, where $C(T) = D(T) \cap F_X^{-1}(N(T)^{\perp})$. It follows from Theorem 2.1 that there exists the Moore-Penrose metric generalized inverse T^+.

Furthermore, by the proof of Theorem 2.1, we get

$$D^+ = R(T) \dot{+} F_Y^{-1}(R(T)^{\perp}).$$

If $R(T)$ is a closed subspace of Y, while Y is reflexive and strictly convex, then $R(T)$ is a Chebyshev subspace of Y. By Lemma 3.2 in [28], we obtain

$$Y = R(T) \dot{+} F_Y^{-1}(R(T)^{\perp}) = D^+.$$

Remark 2.1. If X, Y are Hilbert space, then the Moore-Penrose metric generalized inverse T^+ is the Moore-Penrose generalized inverse under usual sense since the metric projector is linear orthogonal projector.

Denote $D^+ = D(T^+)$.

§3. Properties of Moore-Penrose Metric Generalized Inverse

Theorem 3.1. *Let X and Y be strictly convex Banach spaces, T be a linear operator from X to Y. If T has the Moore-Penrose metric generalized inverse T^+, then*

(1) *T^+ is unique on D^+, and $T^+y = (T|_{C(T)})^{-1}P_{\overline{R(T)}}y,\ y \in D^+$, where $D^+ = R(T) \dot{+} F_Y^{-1}(R(T)^{\perp})$;*

(2) *there exists a linear inner inverse $T^{(1)}$ from $R(T)$ to $D(T)$ (i.e. $TT^{(1)}T = T$) such that*

$$T^+y = (I_{D(T)} - P_{\overline{N(T)}})T^{(1)}P_{\overline{R(T)}}y, \qquad y \in D^+.$$

Proof. (1) Since T^+ exists, by Definition 2.1, we have that $\overline{N(T)}$ and $\overline{R(T)}$ are Chebyshev sets. It follows from Theorem 2.1 that $D(T) = N(T) \dot{+} C(T)$, where $C(T) = D(T) \cap F_X^{-1}(N(T)^{\perp})$. Take $D^+ = R(T) \dot{+} F_Y^{-1}(R(T)^{\perp})$. By the proof of sufficiency in Theorem 2.1, we have $P_{\overline{R(T)}}y \in R(T)$ for any $y \in D^+$ and homogeneous operator $T^{\#} \triangleq (T|_{C(T)})^{-1}P_{\overline{R(T)}}$ from D^+ to $C(T)$ is a Moore-Penrose metric generalized inverse of T. Let T^+ be any Moore-Penrose metric generalized inverse of T. For any $y \in D^+$, we shall prove that (2.8) is true. First of all, by (iv) in Definition 2.1, for any $y \in D^+$, we have

$$TT^+y = P_{\overline{R(T)}}y. \tag{3.1}$$

Since $T^+y \in D(T)$, by (iii) in Definition 2.1, we also have $T^+TT^+y = T^+y - P_{\overline{N(T)}}T^+y$. By (ii) in Definition 2.1, we know that

$$P_{\overline{N(T)}}T^+y = 0. \tag{3.2}$$

On the other hand, since $\overline{N(T)}$ is a Chebyshev subspace of X, by Lemma 3.2 in [28], $T^+y \in D(T)$ has a unique decomposition $T^+y = P_{\overline{N(T)}}T^+y + x_2,\ x_2 \in F_X^{-1}(\overline{N(T)}^{\perp})$. By (3.2), we get

$$T^+y = x_2 \in F_X^{-1}(\overline{N(T)}^{\perp}) \cap D(T) = C(T).$$

From the proof of sufficiency in Theorem 2.1, T is a one to one operator from $C(T)$ to $R(T)$ and $P_{\overline{R(T)}}y \in R(T)$ for any $y \in D^+$. Hence, by (3.1), we obtain

$$T^+y = (T|_{C(T)})^{-1}P_{\overline{R(T)}}y \quad \text{for any} \quad y \in D^+.$$

(2) Since T is linear operator from X to Y, it follows from Proposition 1.3 in [1] that there exists a linear inner inverse $T^{(1)}$ from $R(T)$ to $D(T)$ such that

$$TT^{(1)}T = T. \tag{3.3}$$

We shall show, for any $y \in R(T)$,

$$(T|_{C(T)})^{-1}y = (I_{D(T)} - P_{\overline{N(T)}})T^{(1)}y. \tag{3.4}$$

Indeed, for any $y \in R(T)$, $T^{(1)}y \in D(T)$. Since $\overline{N(T)}$ is a Chebyshev subspace of X, by Lemma 3.2 in [28], we have

$$T^{(1)}y = P_{\overline{N(T)}}T^{(1)}y + x_2, \quad x_2 \in F_X^{-1}(\overline{N(T)}^{\perp}). \tag{3.5}$$

It follows from $D(T) = N(T) \dot{+} C(T)$ and (3.5) that

$$P_{\overline{N(T)}}T^{(1)}y \in N(T). \tag{3.6}$$

Hence $x_2 \in F_X^{-1}(\overline{N(T)}^{\perp}) \cap D(T) = C(T)$. From the proof of the sufficiency in Theorem 2.1, condition (2.1) implies that operator T is one to one from $C(T)$ to $R(T)$. Hence, there exists a unique $x_0 \in C(T)$ such that $y = Tx_0 = T|_{C(T)}x_0$. Combining (3.3), (3.5) and (3.6), we obtain

$$y = Tx_0 = TT^{(1)}Tx_0 = T^{(1)}y = T(P_{\overline{N(T)}}T^{(1)}y + x_2) = Tx_2.$$

Since T is one to one from $C(T)$ to $R(T)$, we have

$$x_2 = x_0 = (T|_{C(T)})^{-1}y. \tag{3.7}$$

It follows from (3.5) and (3.7) that $(T|_{C(T)})^{-1}y = x_2 = (I_{D(T)} - P_{\overline{N(T)}})T^{(1)}y$ for any $y \in R(T)$. For any $y \in D^+$, since $D^+ = R(T) \dot{+} F_Y^{-1}(R(T)^{\perp})$, we have $P_{\overline{R(T)}}y \in R(T)$. Hence, from (3.4) and (1) we get

$$T^+y = (T|_{C(T)})^{-1}P_{\overline{R(T)}}y = (I_{D(T)} - P_{\overline{N(T)}})T^{(1)}P_{\overline{R(T)}}y.$$

Theorem 3.2. *Let X, Y and T be the same as in Theorem* 3.1. *Suppose that*

$$D(T) = N(T) \dot{+} C(T) \quad \textit{and} \quad D^+ = R(T) \dot{+} F_Y^{-1}(R(T)^{\perp}),$$

where $C(T) = D(T) \cap F_X^{-1}(N(T)^{\perp})$. If T^+ is a homogeneous operator from D^+ to $D(T)$, then the following statements are equivalent:

(1) *T^+ is the Moore-Penrose metric generalized inverse of T;*

(2) *For any $y \in D^+$, $x_0 = T^+y$ is the best approximate solution to the operator equation $Tx = y$;*

(3) *For any $y \in D^+$, $x_0 = T^+y$ is the minimal norm solution to the metric projector equation $Tx = P_{\overline{R(T)}}y$, i.e. $T^+y = P_{T^{-1}P_{\overline{R(T)}}y}0$, where $P_{T^{-1}P_{\overline{R(T)}}y}0$ is the metric projection of* 0 *onto $T^{-1}P_{\overline{R(T)}}y$.*

Proof. (1)⇒(2) Let T^+ be the Moore-Penrose metric generalized inverse from D^+ to $D(T)$. For any $y \in D^+$, take $x_0 = T^+y$, then $x_0 \in D(T)$. Definition 2.1 shows that

$$Tx_0 = TT^+y = P_{\overline{R(T)}}y, \tag{3.8}$$

and $P_{\overline{R(T)}}y \in R(T)$. For any $x \in D(T)$, since $Tx \in R(T)$, we get that $\|y - Tx_0\| = \|y - P_{\overline{R(T)}}y\| \le \|y - Tx\|$, i.e. x_0 is an extremal solution to $Tx = y$. For any $x \in D(T)$ with $Tx = P_{\overline{R(T)}}y$, from (3.8) we have

$$x_0 - x \in N(T). \tag{3.9}$$

Hence, by (ii), (iii) in Definition 2.1, we obtain that $x_0 = T^+y = T^+TT^+y = T^+Tx_0 = x_0 - P_{\overline{N(T)}}x_0$. Thus $P_{\overline{N(T)}}x_0 = 0$. Since $\overline{N}(T)$ is a Chebyshev subspace of X, by Lemma 3.2 in [28],

x_0 has a unique decomposition $x_0 = P_{\overline{N(T)}}x_0 + x_2,\ x_2 \in F_X^{-1}(N(T)^{\perp})$. Since $P_{\overline{N(T)}}x_0 = 0$, $x_0 = x_2 \in F_X^{-1}(N(T)^{\perp})$, i.e. $F_X(x_0) \cap N(T)^{\perp} \neq \phi$. Taking $x_0^* \in F_X(x_0) \cap N(T)^{\perp}$, it follows from (3.9) that $\langle x_0^*, x_0 - x\rangle = 0$. Therefore, by the definition of F_X, we get

$$\|x_0\|^2 = \langle x_0^*, x_0\rangle = \langle x_0^*, x\rangle \leq \|x_0^*\| \cdot \|x\| = \|x_0\| \cdot \|x\|,$$

i.e. $\|x_0\| \leq \|x\|$ for any $x \in D(T)$ and $Tx = P_{\overline{R(T)}}y$. Thus $x^0 = T^+y$ is the best approximation solution to $Tx = y$.

(2)$\Rightarrow$(3) It is obvious.

(3)$\Rightarrow$(1) Let $x_0 = T^+y$ be the minimal norm solution to the metric projector equation $Tx = P_{\overline{R(T)}}y$ for any $y \in D^+$. By the definition of metric projector and strictly convexity of X, we have

$$T^+y = \pi(T^{-1}P_{\overline{R(T)}}y|0) \overset{\triangle}{=} P_{T^{-1}P_{\overline{R(T)}}y}0, \tag{3.10}$$

where $T^{-1}P_{\overline{R(T)}}y = \{x \in D(T)|Tx = P_{\overline{R(T)}}y\}$. It remains to verify (i)–(iv) in Definition 2.1.

For any $y \in D^+$, from (3.10), we have $T^+y \in T^{-1}P_{\overline{R(T)}}y$, and hence

$$TT^+y = P_{\overline{R(T)}}y \qquad \text{for all} \quad y \in D^+, \tag{3.11}$$

i.e. (iv) in Definition 2.1 is true. It follows from (3.11) that (i), (ii) in Definition 2.1 are obvious. Since $D(T) = N(T) \dot{+} C(T)$, where $C(T) = D(T) \cap F_X^{-1}(N(T)^{\perp})$, it follows from Theorem 2.1 and Theorem 3.1 that there exists a unique Moore-Penrose metric generalized inverse of T. By Definition 2.1, $\overline{N(T)}$ and $\overline{R(T)}$ are Chebyshev subspace. Hence, by the same argument as (2.9), we have that for any $x \in D(T)$, x has a unique decomposition $x = P_{\overline{N(T)}}x + x_2$, where $P_{\overline{N(T)}}x \in N(T),\ x_2 \in C(T)$. Hence $Tx = Tx_2$, i.e. $x_2 \in T^{-1}Tx$, where $T^{-1}Tx = \{\widetilde{x} \in D(T)|T\widetilde{x} = Tx\}$. For any $x_1 \in T^{-1}Tx$, we have $x_1 - x_2 \in N(T)$. Take $x_0 = x_1 - x_2$, then $x_1 = x_0 + x_2$ and $x_0 \in N(T)$. Since $x_2 \in C(T) \subset F_X^{-1}(N(T)^{\perp})$, i.e. $F_X(x_2) \cap N(T)^{\perp} \neq \phi$, we may choose an $x_2^* \in F_X(x_2) \cap N(T)^{\perp}$. Thus, by the definition of F_X, we get

$$\langle x_2^*,\ x_1\rangle = \langle x_2^*,\ x_0\rangle + \langle x_2^*,\ x_2\rangle = \langle x_2^*,\ x_2\rangle = \|x_2^*\|^2 = \|x_2\|^2,$$

moreover

$$\|x_2\|^2 = \langle x_2^*,\ x_1\rangle \leq \|x_2^*\| \cdot \|x_1\| = \|x_2\| \cdot \|x_1\|.$$

Thus $\|x_2\| \leq \|x_1\|$ for any $x_1 \in T^{-1}Tx$. In other words, $x_2 \in \pi(T^{-1}Tx|0)$. It follows from the strict convexity of X and (3.10) that

$$T^+Tx = \pi(T^{-1}P_{\overline{R(T)}}Tx|0) = \pi(T^{-1}Tx|0) = x_2 = (I_{D(T)} - P_{\overline{N(T)}})x$$

for any $x \in D(T)$, i.e. (iii) in Definition 2.1 is also true. Thus, T^+ is just the Moore-Penrose metric generalized inverse of T.

§4. Necessary and Sufficient Conditions for Continuity and Linearity of the Moore-Penrose Metric Generalized Inverse

The Moore-Penrose metric generalized inverses of linear operators in Banach spaces are generally homogeneous and nonlinear, so that it is important to discuss the necessary and sufficient condition for continuity and linearity.

Banach space Y is said to have property H, if for any sequence $\{y_n\} \subset Y$ and element $y_0 \in Y$, $y_n \to y_0$ weakly and $\|y_n\| \to \|y_0\|$ $(n \to \infty)$ implies that $y_n \to y_0$ $(n \to \infty)$ in Y (see [21]).

Theorem 4.1. *Suppose that X and Y are reflexive, strictly convex Banach spaces, and have property H, T is a densely defined operator from X and Y. Let*

$$D^+ = R(T) \dot{+} F_Y^{-1}(R(T)^\perp) \quad \text{and} \quad C(T) = D(T) \cap F_X^{-1}(N(T)^\perp),$$

where F_X, F_Y are the dual mappings of X and Y. Then there exists a continuous Moore-Penrose metric generalized inverse T^+ from D^+ to $C(T)$, such that $\overline{R(T)} \subset D^+$ and $\overline{N(T)} \subset D(T)$ if and only if T is a closed operator with closed range.

Proof. Necessity. Suppose that there exists a continuous Moore-Penrose metric generalized inverse T^+ such that $\overline{R(T)} \subset D^+$ and $\overline{N(T)} \subset D(T)$. For any $y \in \overline{R(T)} \subset D^+$, by Definition 2.1, we get

$$y = P_{\overline{R(T)}}y = TT^+y \in R(T),$$

i.e. $R(T)$ is closed. Let $\{x_n\} \subset D(T)$, $x_0 \in X$, $y_0 \in Y$ such that $x_n \to x_0$, $Tx_n \to y_0$ $(n \to \infty)$. Take $y_n = Tx_n$. Then $y_n \in R(T)$ $(n = 1, 2, \cdots)$ and $y_n \to y_0$ $(n \to \infty)$, and hence $y_0 \in R(T) \subset D^+$. It follows from the continuity of T^+ that

$$\overline{x}_n = T^+y_n \to \overline{x} = T^+y_0 \quad (n \to \infty).$$

Since T^+ exists, it follows from Theorem 2.1 that $D(T) = N(T) \dot{+} C(T)$, where $C(T) = D(T) \cap F_X^{-1}(N(T)^\perp)$, and $\overline{N(T)}$ is a Chebyshev subspace of X. From above decomposition, by the same argument as (2.9), we have that for any $x \in D(T)$, x has a unique decomposition

$$x = P_{\overline{N(T)}}x + x', \tag{4.1}$$

where $P_{\overline{N(T)}}x \in N(T)$, $x' \in C(T)$. Hence for $x_n \in D(T)$, we have

$$x_n = P_{\overline{N(T)}}x_n + x'_n, \tag{4.2}$$

where $P_{\overline{N(T)}}x_n \in N(T)$, $x'_n \in C(T)$ $(n = 1, 2, \cdots)$. Let $T|_{C(T)}$ be the restriction of T onto $C(T)$. From the proof of sufficiency in Theorem 2.1, $T|_{C(T)}$ is one to one operator from $C(T)$ to $R(T)$. By (4.2), we obtain

$$y_n = Tx_n = Tx'_n = T|_{C(T)}x'_n, \qquad n = 1, 2, \cdots.$$

On the other hand, we obtain also that

$$y_n = P_{\overline{R(T)}}y_n = TT^+y_n = T\overline{x}_n = T|_{C(T)}\overline{x}_n, \qquad n = 1, 2, \cdots.$$

Hence $x'_n = \overline{x}_n$ $(n = 1, 2, \cdots)$. Thus

$$x'_n = T^+y_n \to T^+y_0 = \overline{x} \quad (n \to \infty), \tag{4.3}$$

and $\overline{x} \in C(T)$. It follows from (4.2) and (4.3) that

$$P_{\overline{N(T)}}x_n = x_n - x'_n \to x_0 - T^+y_0 \quad (n \to \infty).$$

Since $P_{\overline{N(T)}}x_n \in N(T)$ $(n = 1, 2, \cdots)$, thus $x_0 - T^+y_0 \in \overline{N(T)}$. Taking $\widetilde{x} = x_0 - T^+y_0$, we have

$$x_0 = \widetilde{x} + T^+y_0, \tag{4.4}$$

where $\widetilde{x} \in \overline{N(T)}$ and $T^+y_0 \in C(T)$. By the condition $\overline{N(T)} \subset D(T)$, it follows that $x_0 \in D(T)$. From the uniqueness of decomposition in (4.1) (replace x by x_0) and (4.4), we get

$$x_0 = P_{\overline{N(T)}}x_0 + T^+y_0,$$

where $P_{\overline{N(T)}}x_0 \in N(T)$. Hence

$$Tx_0 = T(P_{\overline{N(T)}}x_0 + T^+y_0) = TT^+y_0 = P_{\overline{R(T)}}y_0 = y_0,$$

i.e. T is closed operator.

Sufficiency. Let T be a closed linear operator with closed range $R(T)$. It follows that

$$\overline{N(T)} = N(T) \subset D(T) \quad \text{and} \quad \overline{R(T)} = R(T) \subset D^+.$$

By Corollary 2.1 there exists a Moore-Penrose metric generalized inverse T^+. It remains to show that T^+ is continuous. It follows from Theorem 3.2 that

$$T^+y = P_{T^{-1}P_{\overline{R(T)}}y}0. \tag{4.5}$$

In order to prove that T^+ is continuous on D^+, it is sufficient to show that

(i) For any $y_n \in D^+$ $(n = 0,\ 1,\ 2,\ \cdots)$ with $y_n \to y_0$ $(n \to \infty)$, we have

$$P_{\overline{R(T)}}y_n \to P_{\overline{R(T)}}y_0 \quad (n \to \infty).$$

(ii) For any $y_n \in R(T)$ $(n = 0,\ 1,\ 2,\ \cdots)$ with $y_n \to y_0$ $(n \to \infty)$, we have

$$P_{T^{-1}y_n}0 \to P_{T^{-1}y_0}0 \quad (n \to \infty).$$

For (i), since Y is reflexive, strictly convex and Y have property H, it is just the result from Corollary 4 in [23].

Next, we want to prove (ii). Define a norm on $D(T)$ by $\|x\|_{D(T)} = \|x\| + \|Tx\|,\ x \in D(T)$. Since T is closed linear operator, the $(D(T),\ \|\cdot\|_{D(T)})$ is a Banach space, denoted by $D(T)^{**}$. $R(T)$ is a closed subspace of Banach space Y, so that $R(T)$ is also a Banach space. It is easy to see that T is surjective and continuous linear from $D(T)^{**}$ to $R(T)$. By the open mapping theorem[24], there exists $l \geq 1$, such that for any $y,\ z \in R(T),\ x \in T^{-1}y$, there exists $w \in T^{-1}z$ such that

$$\|x - w\|_{D(T)} \leq l\|y - z\|. \tag{4.6}$$

By Theorem 2.2.1 in [24], the proper convex functional $\rho(y) = \inf\{\|x\|_{D(T)} | x \in T^{-1}y\},\ y \in R(T)$ is lower semicontinuous. For any $y \in R(T),\ T^+y \in T^{-1}y$ and for any $x \in T^{-1}y$, from (4.5), we have $\|T^+y\| \leq \|x\|$ and $Tx = y$, hence

$$\|T^+y\|_{D(T)} = \|T^+y\| + \|TT^+y\| = \|T^+y\| + \|y\| \leq \|x\| + \|Tx\| = \|x\|_{D(T)}.$$

Thus

$$\|T^+y\|_{D(T)} = \inf\{\|x\|_{D(T)} : x \in T^{-1}y\} = \rho(y) \tag{4.7}$$

for any $y \in R(T)$. In (4.6), take $x = T^+y,\ z = 0$, then there exists a $w \in T^{-1}0$ such that

$$\rho(y) = \|T^+y\|_{D(T)} \leq \|w\|_{D(T)} + l\|y\| < \infty \qquad \text{for any} \quad y \in R(T).$$

Hence $R(T)$ is the effective domain of $\rho(y)$. It follows from the lower semicontinuity of $\rho(y)$ on $R(T)$ and Propositioon 1.6 in [22] that $\rho(y)$ is continuous on $R(T)$. Let $y_n \in R(T)$ $(n = 0,\ 1,\ 2,\ \cdots)$ with $y_n \to y_0$ $(n \to \infty)$. We get

$$\|T^+y_n\|_{D(T)} = \rho(y_n) \to \rho(y_0) = \|T^+y_0\|_{D(T)}.$$

It follows from the definition of $\|\cdot\|_{D(T)}$ that $\|T^+y_n\| \to \|T^+y_0\| \quad (n\to\infty)$. Noticing that $T^+y_n = P_{T^{-1}y_n}0$ $(n = 0,\ 1,\ 2,\ \cdots)$, we obtain

$$\|P_{T^{-1}y_n}0\| \to \|P_{T^{-1}y_0}0\| \quad (n\to\infty).$$

Take $x_n = P_{T^{-1}y_n}0$ $(n = 0,\ 1,\ 2,\ \cdots)$, then

$$\|x_n\| \to \|x_0\| \quad (n\to\infty). \tag{4.8}$$

Since X has property H, from (4.8), in order to prove that $x_n \to x_0$ $(n\to\infty)$, it remains to show that

$$x_n \xrightarrow{w} x_0 \quad (n\to\infty). \tag{4.9}$$

Suppose that (4.9) were not true. Since X is reflexive, and $\{x_n\}$ is bounded, without lose of any generality, we may suppose that

$$x_n \xrightarrow{w} \overline{x} \neq x_0 \quad (n\to\infty). \tag{4.10}$$

Since $\overline{D(T)} = X$, and $X,\ Y$ are reflexive, T^* is well defined and $\overline{D(T^*)} = Y^*$. Hence T^{**} is also well defined and $T = T^{**}$. For any $w^* \in D(T^*)$, we get

$$\langle w^*,\ y_n\rangle = \langle w^*,\ Tx_n\rangle = \langle T^*w^*,\ x_n\rangle, \qquad n = 1,\ 2,\ \cdots.$$

Letting $n\to\infty$, by (4.10), we obtain $\langle w^*,\ y_0\rangle = \langle T^*w^*,\ \overline{x}\rangle$, but $\langle w^*,\ y_0\rangle = \langle w^*,\ Tx_0\rangle = \langle T^*w^*,\ x_0\rangle$. Hence

$$\langle T^*w^*,\ x_0 - \overline{x}\rangle = 0 \qquad \text{for any} \quad w^* \in D(T^*).$$

It follows from Banach closed range theorem that $x_0 - \overline{x} \in R(T^*)^\perp = N(T)$, i.e. $Tx_0 = T\overline{x}$. In other words, $\overline{x} \in T^{-1}Tx_0 = T^{-1}y_0$. Since the norm is lower semicontinuous weakly, it follows from (4.8) and (4.10) that

$$\|\overline{x}\| \le \liminf_{n\to\infty} \|x_n\| = \lim_{n\to\infty} \|x_n\| = \|x_0\|.$$

Because X is strictly convex, $T^{-1}y_0$ is a convex closed set, the minimal norm element is unique, and hence $\overline{x} = x_0$, which contradicts (4.10). Thus (4.9) is true.

Theorem 4.2. *Under the conditions in Theorem* 4.1, T^+ *is a linear operator if and only if both* $C(T) = D(T)\cap F_X^{-1}(N(T)^\perp)$ *and* $F_Y^{-1}(R(T)^\perp)$ *are linear subspaces.*

Proof. Necessity. If T^+ is a linear operator, then its range $R(T^+) = C(T)$ must be linear. In the following, we shall show that $N(T^+) = F_Y^{-1}(R(T)^\perp)$. From the linearity of $T^+, F_Y^{-1}(R(T)^\perp)$ is linear. By Theorem 3.1, we get

$$T^+ = (T|_{C(T)})^{-1}P_{\overline{R(T)}},$$

where $T|_{C(T)}$ is one to one operator from $C(T)$ to $R(T)$. Hence

$$N(T^+) = \{y \in D^+ | P_{\overline{R(T)}}y = 0\} \stackrel{\triangle}{=} P^{-1}_{\overline{R(T)}}0. \tag{4.11}$$

For any $y \in F_Y^{-1}(R(T)^\perp) = F_Y^{-1}(\overline{R(T)}^\perp)$, we have

$$F_Y(y)\cap \overline{R(T)}^\perp \neq \phi.$$

Take $y_0^* \in F_Y(y)\cap\overline{R(T)}^\perp$ and write $y^* = y_0^*/\|y_0\|$. Then

$$\|y^*\| = 1, \quad y^* \in \overline{R(T)}^\perp \quad \text{and} \quad \langle y^*,\ y-0\rangle = \|y\|^2/\|y\| = \|y - 0\|.$$

By Lemma 3.1 in [28] and the strict convexity of Y, it follows that $0 = P_{\overline{R(T)}}y$, and hence, by (4.11), $y \in N(T^+)$, i.e.

$$F_Y^{-1}(R(T)^\perp) \subset N(T^+). \tag{4.12}$$

Suppose on the contrary, for any $y \in N(T^+)$, from (4.11), $P_{\overline{R(T)}}y = 0$. Since $\overline{R(T)}$ is a Chebyshev subspace, by Lemma 3.2 in [28], we have

$$y = P_{\overline{R(T)}}y + y_2, \quad y_2 \in F_Y^{-1}(R(T)^\perp),$$

Hence $y = y_2 \in F_Y^{-1}(R(T)^\perp)$, i.e.

$$N(T^+) \subset F_Y^{-1}(R(T)^\perp). \tag{4.13}$$

Thus, it follows from (4.12) and (4.13) that

$$N(T^+) = F_Y^{-1}(R(T)^\perp). \tag{4.14}$$

Sufficiency. Let $C(T)$ and $F_Y^{-1}(R(T)^\perp)$ be linear. Then $T|_{C(T)}$ is one to one linear operator from $C(T)$ to $R(T)$, so is $(T|_{C(T)})^{-1}$ from $R(T)$ to $C(T)$, and $D^+ = R(T) \dot{+} F_Y^{-1}(R(T)^\perp)$ is a linear subspace of Y. For any $y \in D^+$, by the same argument as (2.7), we obtain $P_{\overline{R(T)}}y \in R(T)$. It follows from (4.11) and (4.14) that

$$P_{\overline{R(T)}}^{-1}0 = N(T^+) = F_Y^{-1}(R(T)^\perp).$$

Hence, $P_{\overline{R(T)}}^{-1}0$ is a linear subspace of Y. Proposition 4.7 in [25] implies that $P_{\overline{R(T)}}$ is a linear operator from D^+ to $R(T)$. Hence, by Theorem 3.1, $T^+ = (T|_{C(T)})^{-1}P_{\overline{R(T)}}$ is a linear operator from D^+ to $C(T)$.

Corollary 4.1. *Let X and Y be Hilbert spaces, T be a bounded linear or densely defined closed linear operator. If $R(T)$ is closed, then there exists an unique bounded linear operator T^+ from Y into X such that*

(i) $TT^+T = T$;

(ii) $T^+TT^+ = T^+$;

(iii) $T^+T = I_{D(T)} - P_{N(T)}$;

(iv) $TT^+ = I - P_{N(T^*)}$,

where $P_{N(T)}$, $P_{N(T^*)}$ are the orthogonal projectors.

Proof. If $R(T)$ is a closed subspace of Y, by Riesz orthogonal decomposition theorem, we get $I = P_{R(T)} + P_{R(T)^\perp}$. By Banach closed range theorem, we know that $R(T)^\perp = N(T^*)$. Hence

$$P_{R(T)} = I - P_{R(T)^\perp} = I - P_{N(T^*)}.$$

The others follow from Corollary 2.1, Theorem 4.1 and Theorem 4.2.

References

[1] Nashed, M. Z., Generalized inverse and application, Academic Press, New York-San Francisco-London, 1976.

[2] Nashed, M. Z. & Votruba, G. F., A unified operator theory of generalized inverse, Nashed (ed.) Academic Press, 1976, 1–109.

[3] Nashed, M. Z. & Votruba, G. F., A unified approach to generalized inverse of linear operators extremal and proximal properties, *Bull. Amer. Math. Soc.,* **80**(1974), 831–834.

[4] Ben-Israel, A. & Greville, T. N. E., Generalized inverse: theory and applications, Wiley-Interscience, New York, 1974.

[5] Tseng, Y. Y., Sur les solutions des equations operatrices funtionnelles entre les espaces unitaries, solutions extremales, solutions virtuelles, *C. R. Acad. Sci.*, Paris, **228**(1979), 640–641.
[6] Tseng, Y. Y., Generalized inverse of unbounded operators between two unitary spaces, *Dokl. Akad. Nauk. SSSR* (*N.S.*), **67**(1949), 431–434.
[7] Tseng, Y. Y., Properties and classification of generalized inverse of closed operators, *Dokl. Akad. Nauk. SSSR* (*N.S.*), **67**(1949), 607–610.
[8] Tseng, Y. Y., Virtual solutions and general inversions, *Vspehi. Mat. Nauk. SSSR* (*N.S.*), **11**(1956), 607–610.
[9] Locker, J., On constructing least squares solution to two-point boundary value problems, *Trans. Amer. Math. Soc.*, **203**(1975), 175–183.
[10] Locker, J., The generalized Green's function for an nth order linear differential operator, *Trans. Amer. Math. Soc.*, **228**(1977), 243–268.
[11] Locker, J., Functional analysis and two-point differential operators, Longman Scientific Technical, Joho Wiley & Sons, Inc., New York., 1986.
[12] Ma, J. P., Continuously sufficient and necessary conditions for Moore-Penrose inverse A_X^+, *Science in China, Series A,* **33**(1990), 1294–1302.
[13] Nashed, M. Z. & Chen, X., Convergence of Newton-like method for singular equations using outer inverse, *Number. Math.*, **66**(1993), 235–257.
[14] Ma, J. P., Rank theorems of operators between Banach spaces, *Science in China, Series A,* **43**(2000), 1–5.
[15] Li, Z. W. & Wang, Y. W., Singular optimal control with nonsmooth cost function in Banach spaces (in Chinese), *J. Sys. Sci.* & *Math. Scis.*, **15:**4(1995), 305–311.
[16] Wang, Y. W., The generalized inverse operators in Banach spaces, *Bulletin of the Polish Academy of Sciences,* **37:**7–12(1989), 433–441.
[17] Wang, Y. W. & Li, Z. W., Moore-Penrose generalized inverse in Banach spaces and ill-posed boundary value problem (in Chinese), *J. Sys. Sci.* & *Math. Scis.*, **15:**2(1995), 175–185.
[18] Wang, Y. W. & Ji, D. Q., The Tseng-metric generalized inverse of linear operator in Banach spaces (in Chinese), *J. Sys. Sci.* & *Math. Scis.*, **20:**2(2000), 203–209.
[19] Wang, Y. W. & Wang, H., The minimum norm control problems in Banach spaces (in Chinese), *J. Sys. Sci.* & *Math. Scis.*, **11:**1(1991), 1–5.
[20] Holmes, R. B., A course on optimization and best approximation, Lecture Notes **257**, Springer Verlag, Berlin-Heideberg -New York, 1972.
[21] Diestel, J., Geomertry of Banach spaces -selected topics, Lect. Math., Springer-Verlag, 1975.
[22] Barbu, V. & Precpuanu, T., Convexity and optimization in Banach spaces, Ed Acad. Rep. Soc., Romania, Bucuresti, 1978.
[23] Singer, I., Some remarks on approximative compactness, *Rev. Roum. Math. Pures et Appl.*, **9**(1964), 167–177.
[24] Aubin, J. P. & Frakowska, H., Set -valued analysis, systems and control, foundations and applications, Birkhauser, Boston, Basel, Berlin, 1990.
[25] Singer, I., The theory of best approximation and functional analysis, Springer-Verlay, Berlin-Heidelberg-New York, 1970.
[26] Lee, S. J. & Nashed, M. Z. N, Least-squares solutions of multivalued linear operator equations in Hilbert spaces, *J. Approx. Theory*, **38**(1983), 380–391.
[27] Lee, S. J. & Nashed, M. Z. N., Constrained Least-squares solutions of linear inclusion and singular control problems in Hilbert spaces, *Appl. Math. Optim*, **19**(1989), 225–242.
[28] Wang, Y. W. & Wang, H., Generalized orthogonal decomposition theorem in Banach space and generalized orthogonal complemented subspace (in Chinese), *Acta Math. Sinica*, **44:**6(2001), 1045–1050.