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ON CONE D.C. OPTIMIZATION AND CONJUGATE DUALITY

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Abstract

This paper derives first order necessary and sufficient conditions for unconstrained cone d.c. programming problems where the underlined space is partially ordered with respect to a cone. These conditions are given in terms of directional derivatives and subdifferentials of the component functions. Moreover, conjugate duality for cone d.c. optimization is discussed and weak duality theorem is proved in a more general partially ordered linear topological vector space (generalizing the results in [11]).

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§1. Introduction

Convexity is one of the well developed mathematical structures since 1950s and it plays an important role in optimization. An increasing number of problems arise from application that cannot be solved successfully by standard methods of linear or nonlinear programming. But a great many of these can be described by using difference of two convex functions called d.c. functions for short. One can utilize the concepts and tools of convex analysis to solve such problems. Many practical methods have been developed by several researchers to solve a d.c. optimization problem in a scalar case (see for example [2, 4, 10] and the references therein).

A natural generalization of this concept to vector optimization is named simply as cone d.c. programming (where cone refers here the ordering cone of the image set). A function is said to be cone d.c. if it can be described as the difference of two cone convex functions. The class of cone d.c. functions is very rich. For example, cone convex, cone concave functions and the combination of them fall under this category. On finite dimensional spaces one can easily verify that the extension of Hartman's theorem holds true also for vector valued functions, i.e., every locally cone d.c. function is globally cone d.c. Since convexity is present twice (in the reverse sense) in the decomposition of d.c. functions, many concepts and results from convex analysis will be used also for cone d.c. programming.

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The generalization of the results for scalar d.c. programming to Cone d.c. optimization was studied by Yin Zhiwen and Li Yuanxi[11]. However, their result is valid only for the case where the underlined space is an order-complete, finite dimensional vector space. Moreover, they used $*$-difference to give the minimality conditions. But this difference produces a set which is empty too often and may not give any information when the point is not efficient.

In this paper, we extend the results in [11] to a more general case and present some minimality conditions also using the conjugate duality. The duality case, here, is discussed on a more general partially ordered linear topological vector space extending Toland's duality theorem for scalar d.c. problems.

The paper is organized as follows. In Section 2 some necessary definitions and preliminary concepts, which are useful in the next part are discussed. In the third section the necessary and sufficient local weak optimality conditions are given. The last section contains the conjugate duality for cone d.c. programming. It gives some generalizations of Toland's duality theorem, which was given for scalar d.c. programming case in [9].

§2. Some Definitions and Preliminary Concepts

Let Y be a real topological vector space which is partially ordered by a pointed, closed, convex cone D with nonempty interior in Y. The following notations will be used in the sequel:

$$y \leq_D y' \quad \text{iff} \quad y' - y \in D \quad \text{and} \quad y <_D y' \quad \text{iff} \quad y' - y \in \operatorname{int} D.$$

The next definition of weak Supremum and Infimum of a set in partially ordered linear topological vector spaces was given by Tanino in [8]. This definition, which is on the basis of weak efficiency, is used to obtain the strong conjugate duality result in vector optimization (see [8] and [6]).

Let $\overline{Y}$ denote the extended space of Y (i.e., $\overline{Y} := Y \cup \{\pm\infty\}$). Given a set $Z \subset \overline{Y}$, we define the set $A(Z)$ of $\overline{Y}$ by $A(Z) = \{y \in \overline{Y} | y' <_D y \text{ for some } y' \in Z\}$ which is the set of all points above Z and the set $B(Z)$ of $\overline{Y}$ by $B(Z) = \{y \in \overline{Y} | y' >_D y \text{ for some } y' \in Z\}$ which is the set of all points below Z. Clearly $A(Z) \subseteq Y \cup \{+\infty\}$ and $B(Z) \subseteq Y \cup \{-\infty\}$.

Definition 2.1. *A point $z \in Z$ is said to be a D-minimal point of Z if there is no $y \in Z$ with $y <_D z$. The set of all D-minimal points of Z is called the D-minimum of Z and is denoted by $D\text{-}\mathrm{Min}\, Z$. The D-maximum of Z, $D\text{-}\mathrm{Max}\, Z$, is defined analogously.*

Definition 2.2. *Let Z be a nonempty subset of $\overline{Y}$ such that $Z \neq \{+\infty\}$. A point $p \in \overline{Y}$ is said to be a D-infimal point of a set Z, if there is no $y \in Z$ such that $y <_D p$ and if the relation $y' >_D p$ implies the existence of some $z \in Z$ such that $y' >_D z$. The set of all D-infimal points of Z is called a D-infimum of Z and is denoted by $D\text{-}\mathrm{Inf}\, Z$. If $Z = \emptyset$ and $Z = \{+\infty\}$, we define $D\text{-}\mathrm{Inf}\, Z = \{+\infty\}$. The D-Supremum of Z, $D\text{-}\mathrm{Sup}\, Z$ is defined similarly.*

As an easy consequence from the definition we deduce the following

(i) $-D\text{-}\mathrm{Max}\,(-Z) = D\text{-}\mathrm{Min}\, Z$ and $-D\text{-}\mathrm{Inf}\,(-Z) = D\text{-}\mathrm{Sup}\, Z$,

(ii) $D\text{-}\mathrm{Max}\, \emptyset = \emptyset$ and $D\text{-}\mathrm{Sup}\, \emptyset = \{-\infty\}$.

One observes that in the above definitions the D-infimum and the D-minimum are defined for the weak Pareto minimality (by using the order $>_D$). The following proposition which is proved in [8, Proposition 2.5] shows how a set can be partitioned by using the above defined concepts.

Proposition 2.1. $\overline{Y} = (D\text{-Sup}\, Z) \cup A(D\text{-Sup}\, Z) \cup B(D\text{-Sup}\, Z)$ *and the above three sets in the right-hand side are disjoint.*

Let X and Y be real locally convex topological vector spaces and let $L(X, Y)$ denote the space of all linear continuous operators from X into Y. Then we define the conjugate mapping and the subdifferential of a mapping $F : X \to \overline{Y}$. Hereafter by domain of F, $\operatorname{dom} F$ we mean the effective domain of F which is given by

$$\operatorname{dom} F \;=\; \{x \in X | F(x) \;\neq\; \emptyset,\; F(x) \;\neq\; \{+\infty\}\}$$

for a set-valued mapping F and by

$$\operatorname{dom} f \;=\; \{x \in X | f(x) <_D +\infty\}$$

for a vector valued function f.

Definition 2.3. *For a vector valued function $f : X \to \overline{Y}$, its conjugate map is a set valued mapping $f^* : L(X, Y) \to \overline{Y}$ defined by*

$$f^*(T) = D\text{-Sup}\{Tx - f(x) | x \in X\} \quad \textit{for} \quad T \in L(X, Y).$$

Moreover, its biconjugate mapping is a set-valued mapping $f^{**} : X \to \overline{Y}$ defined by

$$f^{**}(x) = D\text{-Sup} \bigcup_{T \in L(X,Y)} [Tx - f^*(T)] \quad \textit{for} \quad x \in \operatorname{dom} f.$$

Definition 2.4. *For a set-valued mapping $F : X \to \overline{Y}$, its conjugate map is a set valued mapping $F^* : L(X, Y) \to \overline{Y}$ defined by*

$$F^*(T) = D\text{-Sup} \bigcup_{x \in X} [Tx - F(T)] \quad \textit{for} \quad T \in L(X, Y).$$

Moreover, its biconjugate mapping is a set-valued mapping $F^{**} : X \to \overline{Y}$ defined by

$$F^{**}(x) = D\text{-Sup} \bigcup_{T \in L(X,Y)} [Tx - F^*(T)] \quad \textit{for} \quad x \in \operatorname{dom} F.$$

Assume that $\operatorname{dom} f \;\neq\; \emptyset$.

Definition 2.5 (Subgradients and Subdifferentials)**.** *Let $x_0 \in X$. An operator $T \in L(X, Y)$ is said to be a subgradient of a vector valued function f at x_0 if*

$$f(x) \not<_D f(x_0) + T(x - x_0) \quad \textit{for all} \quad x \in X,$$

or equivalently

$$f(x_0) - Tx_0 \in D\text{-Min}\{f(x) - T(x) |\; x \in X\}.$$

The set of all subgradients of f at x_0 is called the subdifferential of f at x_0 and is denoted by $\partial f(x_0)$. More precisely

$$\partial f(x_0) = \{T \in L(X, Y) | f(x) - f(x_0) \not<_D T(x - x_0) \;\; \textit{for all} \;\; x \in X\}.$$

For a set-valued mapping $F : X \to \overline{Y}$, let $x_0 \in X$ and $y_0 \in F(x_0)$. An operator $T \in L(X, Y)$ is called a subgradient of F at (x_0, y_0) if

$$Tx_0 - y_0 \in D\text{-Max} \bigcup_{x \in X} [Tx - F(x)].$$

The subdifferential of F at (x_0, y_0) is denoted by $\partial F(x_0, y_0)$. Moreover, we let

$$\partial F(x_0) = \bigcup_{y \in F(x_0)} \partial F(x_0, y).$$

Unlike to the scalar case, the subdifferential of a vector-valued function may not be a closed convex set even when f is a finite D-convex function (see [5, p.172]).

If the subdifferential of f at x_0 is nonempty (or if $\partial F(x_0, y_0) \neq \emptyset$ for every $y_0 \in F(x_0)$ when F is a set-valued mapping), then f is said to be subdifferentiable at x_0.

It is obvious from the definitions of conjugate maps and subgradients that $T \in \partial f(x)$ iff $Tx - f(x) \in f^*(T)$ and immediately from the definition of the subgradients one can see that

$$f(x^*) \in D\text{-Min}\{f(x)|x \in X\} \quad \text{if and only if} \quad 0 \in \partial f(x^*).$$

Moreover it is well known that $f(x_0) \in f^{**}(x_0)$ for any $x_0 \in X$.

A vector valued function $f : X \to \overline{Y}$ is said to be D-convex iff its D-epigraph, D-epi $f = \{(x, y) \in X \times Y | y \in f(x) + D\}$ is a convex set in $X \times \overline{Y}$, or iff for every $x^1, x^2 \in X$ and for any $\alpha \in [0, 1]$,

$$\alpha f(x^1) + (1 - \alpha)f(x^2) - f(\alpha x^1 + (1 - \alpha)x^2) \in D.$$

Similarly a set-valued mapping $F : X \to \overline{Y}$ is said to be D-convex iff its D-epigraph, D-epi $F = \{(x, y) \in X \times Y | y \in F(x) + D\}$ is convex, or iff for all $t \in [0, 1]$ and $x^1, x^2 \in X$,

$$tF(x^1) + (1 - t)F(x^2) \ \subseteq \ F(tx^1 + (1 - t)x^2) + D.$$

§3. Optimality Conditions

Let X be a real linear topological vector space and we assume for this section that $Y = \mathrm{R}^p$. Let $D \subset Y$ be a pointed, closed, convex cone which has a nonempty interior in Y. Assume that Y is partially ordered by the cone D.

Now consider the following unconstrained vector minimization problem

(P1) $\qquad$ minimize $\quad f(x) \quad$ s.t. $\quad x \in X.$

Definition 3.1. *A function $f : X \to Y$ is said to be locally D-Lipschitz if for every $x \in X$, there exist a neighborhood $\mathcal{B}(x, \delta)$ of x with radius $\delta > 0$ and $L \in D$ such that*

$$-L\|x - y\| \leq_D f(x) - f(y) \leq_D L\|x - y\|, \quad \forall x, y \in \mathcal{B}(x, \delta).$$

Here, L is called Lipschitz constant.

For a vector valued function $f : X \to \overline{Y}$ the directional derivative of f at a point $x_0 \in X$ in the direction $u \in X$ is given by

$$f'(x_0; u) := \lim_{t \to 0^+} \frac{f(x_0 + tu) - f(x_0)}{t}$$

if the limit exists. It is clear that for D-convex functions this is always the case.

Lemma 3.1. *Suppose that f is directionally differentiable vector-valued function. Then $T \in \partial f(x_0)$ iff $f'(x_0; u) \not<_D Tu$ for any $u \in X$.*

Proof. Let $T \in \partial f(x_0)$. Then we have $tf'(x_0; u) + o(t) = f(x_0 + tu) - f(x_0) \not<_D T(tu)$ for all $t > 0$ and for all $u \in X$, where $o(t)/t \to 0$ *as* $t \to 0$, or $f'(x_0; u) + o(t)/t \not<_D Tu$ for all $t > 0$, which in turn implies that $f'(x_0; u) \not<_D Tu$ for all $u \in X$.

The proof of the other side is obvious.

Note that for a D-convex function f, we have a stronger condition

$$f'(x_0; u) = \lim_{t \to 0^+} \frac{f(x_0 + tu) - f(x_0)}{t} \leq_D f(x_0 + u) - f(x_0) \quad \text{for all} \quad u \in X, \tag{3.1}$$

which together with the above lemma gives us

$$f'(x_0; u) \in D\text{-Max}\{Tu | T \in \partial f(x_0)\}. \tag{3.2}$$

A point $x_0 \in \mathrm{dom} f$ is said to be a local efficient point for (P1) if there exists a neighborhood of x_0 such that $f(x) \not<_D f(x_0)$ for all x in the neighborhood.

Using the above two definitions we give the following optimality condition for (P1) which is a necessary condition for local weak efficiency.

Proposition 3.1. *Let f be directionally differentiable at $x_0 \in X$. If x_0 is a local efficient point for* (P1)*, then*

$$f'(x_0; u) \not<_D 0 \quad \text{for all} \quad u \in X.$$

Proof. Since f is directionally differentiable at the point x_0, we have $f(x_0 + tu) = f(x_0) + tf'(x_0; u) + o(t)$, $\forall t \geq 0, \forall u \in X$, where $o(t) := o(t, x_0, u)$ and $\lim\limits_{t\to 0^+} t^{-1}[o(t, x_0, u)] = 0$, $\forall u \in X$. Then $f'(x_0; u) = t^{-1}[f(x_0 + tu) - f(x_0)] - o(t)/t$. Since x_0 is a local minimum point of f, the assertion of the proposition follows.

A slight modification (i.e., adding a local D-Lipschitz criterion) to the above necessary condition will make it also sufficient as the following proposition shows.

Proposition 3.2. *Let f be a function which is directionally differentiable at a point $x_0 \in X$. If f is locally D-Lipschitz in a neighborhood of x_0 and if $f'(x_0; u) \not\leq_D 0$ for all $u \in X$ and $u \neq 0$, then x_0 is a local efficient point of f.*

Proof. Suppose the contrary. Then there exists a net of vectors $\{u_i\}$ with $\|u_i\| = 1$, $\forall i$ in a neighborhood of x_0 and a sequence $\{t_i\}$ with $t_i \geq 0$, $t_i \to 0$ and $u_i \to u$ as $i \to \infty$ such that $f(x_0 + t_i u_i) <_D f(x_0)$. But $f(x_0 + t_i u_i) - f(x_0) = f(x_0 + t_i u) - f(x_0) + f(x_0 + t_i u_i) - f(x_0 + t_i u) <_D 0$. Since f is locally D-Lipschitz, there exists $L \in D$ such that for sufficiently large i,

$$-Lt_i\|u_i - u\| \leq_D f(x_0 + t_i u_i) - f(x_0 + t_i u) \leq_D Lt_i\|u_i - u\|.$$

Then $[f(x_0 + t_i u) - f(x_0)]/t_i \leq_D L||u_i - u||$ for sufficiently large i. Hence

$$f'(x_0; u) = \lim_{i\to\infty} \frac{f(x_0 + t_i u) - f(x_0)}{t_i} \leq_D \lim_{i\to\infty} L\|u_i - u\| = 0,$$

i.e., $f'(x_0; u) \leq_D 0$, contradicting the assumption. Therefore, the conclusion of the proposition is true.

A D-convex function f is said to be proper if $f(x) >_D -\infty$ for all $x \in X$.

Definition 3.2. *A vector valued function $f : X \to \overline{Y}$ is said to be a D-d.c. function if and only if it can be written as a difference of two proper cone-convex functions, i.e., $f(x) = g(x) - h(x)$, where g and h are D-convex and proper vector valued functions.*

Let $g, h : X \to \overline{Y}$ be D-convex proper vector valued functions. Then the function $f : X \to \overline{Y}$ given by $f(x) := g(x) - h(x)$ is a D-d.c. function on X and it is easy to verify that f is locally D-Lipschitz at each points of X and is directionally differentiable on X with $f'(x_0; u) = g'(x_0; u) - h'(x_0; u)$, $\forall u, x_0 \in X$.

Consider the following D-d.c. optimization problem

(P) $\quad$ minimize $f(x)$ $\quad$ s.t. $x \in X$,

where $f = g - h$ and g and h are as above. We adopt in the sequel the convention $+\infty - (+\infty) = +\infty$. To state the necessary condition for minimality we first define the strong subdifferential of a vector valued function f at a point x_0, denoted by $\partial_s f(x_0)$, by

$$\partial_s f(x_0) := \{T \in L(X, Y) | T(x - x_0) \leq_D f(x) - f(x_0) \quad \text{for all} \quad x \in X\}.$$

It is easy to show that

$$T \in \partial_s f(x_0) \quad \text{iff} \quad Tu \leq_D f'(x_0; u) \quad \text{for all} \quad u \in X.$$

Applying the condition (3.1) one can see easily that for D-convex f, $\partial_s f(x_0)$ is nonempty as it at least contains the directional derivative of f at the point x_0. Note also that $\partial_s f(x_0) \subseteq \partial f(x_0)$ for any x_0 at which f is subdifferentiable.

Theorem 3.1 (Necessary Condition). *For $f = g - h$ to attain its local D-minimal value at a point $x_0 \in X$, it is necessary that $\partial_s h(x_0) \subseteq \partial g(x_0)$.*

Proof. If x_0 is a local minimum point for f, then there exists a neighborhood $\mathcal{U}$ of x_0 such that $f(x) \not<_D f(x_0)$ for all $x \in \mathcal{U}$, or $g(x) - g(x_0) \not<_D h(x) - h(x_0)$, $\forall x \in \mathcal{U}$.

But for $T \in \partial_s h(x_0)$ we have $T(x - x_0) \leq_D h(x) - h(x_0)$ for all $x \in X$. Then one can conclude that $g(x) - g(x_0) \not<_D T(x - x_0)$, or $T \in \partial g(x_0)$.

For otherwise, if $g(x) - g(x_0) <_D T(x - x_0)$, together with the relation $T(x - x_0) \leq_D h(x) - h(x_0)$ we will have

$$g(x) - g(x_0) <_D h(x) - h(x_0),$$

which is a contradiction. Hence the theorem is proved.

Theorem 3.2 (Sufficient Condition). *If $\partial_s h(x_0) \subseteq \operatorname{int} \partial g(x_0)$, then the criterion vector x_0 is a local efficient point for* (P).

Proof. Let $f(x) := g(x) - h(x)$ for all $x \in X$. Then clearly f is directionally differentiable on X and it is locally D-Lipschitz. From the assumption of the theorem, we have $\partial_s h(x_0) \subseteq \operatorname{int} \partial g(x_0)$, and from the relation (3.2) it follows immediately that

$$g'(x_0; u) \not\leq_D h'(x_0; u), \qquad \forall u \in X,\ u \neq 0.$$

Hence invoking Proposition 3.2 we have the conclusion of the theorem.

§4. Conjugate Duality in Cone D.C. Optimization

Let X be a real linear topological vector space and Y be a locally convex linear topological vector space. Assume that Y is partially ordered by a pointed, closed, convex cone D which has a nonempty interior. Let g and h be vector valued D-convex functions from X to $\overline{Y}$. In the next part we assume that the functions g and h are proper (note that, a D-convex function p is said to be proper in X iff $p(x) >_D -\infty$ for all $x \in X$).

Now consider the D-d.c. optimization problem

(P) minimize $f(x)$ s.t. $x \in X$.

Solving this problem means to find the set

$$D\text{-Inf(P)} = D\text{-Inf}\{g(x) - h(x) \mid x \in X\}.$$

Let $U \subseteq X$ be another locally convex linear topological vector space and U^* be its dual space. We introduce a special perturbation function $\varphi : X \times U \to \overline{Y}$ such that

$$\varphi(x, u) = h(x + u) - g(x) \quad \text{for all} \quad (x, u) \in X \times U.$$

Then clearly $\varphi(x, 0) = -f(x)$ for all $x \in X$. For $\Lambda \in \mathcal{M} := L(U, Y)$, the space of all linear continuous operators from U to Y, let the Lagrangian of problem (P) be given by

$$\begin{aligned} -\mathcal{L}(x, \Lambda) &= D\text{-Sup}\{\Lambda u - \varphi(x, u) \mid u \in U\} \\ &= D\text{-Sup}\{\Lambda(x + u) + g(x) - h(x + u) - \Lambda x \mid u \in U\} \\ &= h^*(\Lambda) + g(x) - \Lambda x, \end{aligned}$$

where $h^*(\Lambda)$ denotes the conjugate map of h. Now we put $-J(\Lambda) := D\text{-Sup} \bigcup\limits_{x \in X} \mathcal{L}(x, \Lambda)$, which is equal to $g^*(\Lambda) - h^*(\Lambda)$. Then the dual optimization problem for (P) is written as

(Dual) minimize $h^*(\Lambda) - g^*(\Lambda), \quad \Lambda \in \mathcal{M},$

which is equivalent to the formulation

(Dual) $$D\text{-Inf} \bigcup_{\Lambda \in \mathcal{M}} [h^*(\Lambda) - g^*(\Lambda)].$$

We can observe the symmetry between the primal problem and the dual one. But since both $h^*(\cdot)$ and $g^*(\cdot)$ are set valued maps the dual problem (Dual) is not a usual vector optimization problem. However, it can be understood as determining the set $D\text{-Inf} \bigcup_{\Lambda \in \mathcal{M}} [h^*(\Lambda) - g^*(\Lambda)]$.

On the other hand,

$$\begin{aligned} D\text{-Sup} \bigcup_{x \in X} [-\mathcal{L}(x, \Lambda)] &= D\text{-Sup} \bigcup_{x \in X} D\text{-Sup}\{\Lambda u - \varphi(x, u) \mid u \in U\} \\ &= D\text{-Sup} \bigcup_{(x,u) \in X \times U} \{\Lambda u + 0x - \varphi(x, u)\} \\ &= \varphi^*(0, \Lambda). \end{aligned}$$

Therefore, $\varphi^*(0, \Lambda) = h^*(\Lambda) - g^*(\Lambda)$. This will help us in proving the following weak duality results. In the next theorem it is proved that any feasible value of the primal problem is not above any feasible value of the dual problem.

Theorem 4.1. *For any $x \in X$ and $\Lambda \in \mathcal{M}$, $f(x) \notin A(h^*(\Lambda) - g^*(\Lambda))$ and thus*

$$D\text{-Inf}(\mathrm{P}) \cap A(D\text{-Inf}(\mathrm{Dual})) = \emptyset.$$

Proof. Suppose the contrary. Then there exists $y \in h^*(\Lambda) - g^*(\Lambda)$ such that $y <_D f(x)$. But since

$$h^*(\Lambda) - g^*(\Lambda) = D\text{-Sup} \bigcup_{(x,u) \in X \times U} \{\Lambda u + 0x - \varphi(x, u)\},$$

$y \not<_D \Lambda u - \varphi(x, u)$ for all $u \in U$. In particular, if we put $u = 0$ and noting that $f(x) = -\varphi(x, 0)$, it follows that $y \not<_D -\varphi(x, 0) = f(x)$, $\forall y \in h^*(\Lambda) - g^*(\Lambda)$, which contradicts our assumption. Hence the theorem is proved.

The above theorem assures us that for any x, $f(x) \in D\text{-Inf} \bigcup_{\Lambda \in \mathcal{M}} [h^*(\Lambda) - g^*(\Lambda)]$. If we can find some $\Lambda_0 \in \mathcal{M}$ such that $f(x_0) \in h^*(\Lambda_0) - g^*(\Lambda_0)$ for some x_0, then it means that Λ_0 solves (Dual). The next theorem reflects this fact.

Theorem 4.2. *If x_0 solves* (P), *then there exists some $\Lambda_0 \in \mathcal{M}$ which solves* (Dual).

Proof. If x_0 solves (P), then since $f(x) = -\varphi(x, 0)$, the same x_0 solves the problem

(P′) $$\min_{x \in X} -\varphi(x_0, 0).$$

Then there exists some $\Lambda_0 \in \mathcal{M}$ such that $(0, \Lambda_0) \in \partial\varphi(x_0, 0)$. But this in turn implies that $(0, \Lambda_0)(x_0, 0)^T - \varphi(x_0, 0) \in \varphi^*(0, \Lambda_0)$, which means that $f(x_0) = -\varphi(x_0, 0) \in \varphi^*(0, \Lambda_0) = h^*(\Lambda_0) - g^*(\Lambda_0)$.

Now assume that Λ_0 does not solve (Dual). Then there exists $\Lambda \in \mathcal{M}$ such that

$$(h^*(\Lambda) - g^*(\Lambda)) \cap B\,(h^*(\Lambda_0) - g^*(\Lambda_0)) \neq \emptyset.$$

Since $f(x_0) \in h^*(\Lambda_0) - g^*(\Lambda_0)$, there exists $y \in h^*(\Lambda) - g^*(\Lambda)$ such that $y <_D f(x_0)$. But this contradicts the statement in Theorem 4.1. Hence Λ_0 solves (Dual).

Corollary 4.1. *If x_0 solves* (P) *and $\Lambda_0 \in \partial_s h(x_0)$, then Λ_0 solves* (Dual).

Proof. From the assumption we have $\partial_s h(x_0) \subseteq \partial g(x_0)$, and the relation $T \in \partial f(x_0)$ iff $Tx_0 - f(x_0) \in f^*(T)$ gives

$$g(x_0) - h(x_0) = (\Lambda_0 x_0 - h(x_0)) - (\Lambda_0 x_0 - g(x_0)) \in h^*(\Lambda_0) - g^*(\Lambda_0).$$

That is, $f(x_0) \in h^*(\Lambda_0) - g^*(\Lambda_0)$. Hence Λ_0 solves (Dual).

Proposition 4.1. *Let $x_0 \in \operatorname{dom} f$. If $f(x_0) \in h^*(\Lambda_0) - g^*(\Lambda_0)$ for some $\Lambda_0 \in \mathcal{M}$, then x_0 solves* (P) *at least locally.*

Proof. By the assumption we have

$$-\varphi(x_0, 0) = f(x_0) \in h^*(\Lambda_0) - g^*(\Lambda_0) = \varphi^*(0, \Lambda_0),$$

which is equivalent to

$$(0, \Lambda_0) \in \partial\varphi(x_0, 0). \tag{4.1}$$

Then from the relation $-\varphi(x_0, 0) = f(x_0)$ and (4.1), we can see that $0 \in \partial f(x_0)$, or x_0 is a local minimum of $f(x)$.

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References

[1] Clarke, F. H., Optimization and nonsmooth analysis, Wiley Interscience, New York, 1983.
[2] Hiriart Urruty, J. B., From convex optimization to nonconvex optimization, Part I: Necessary and sufficient conditions for global optimality, in Nonsmooth Optimization and Related Topics, F. H. Clarke et al (eds.), Plenum, New York, 1989, 219–239.
[3] Luc, D. T., Theory of vector optimization, Lecture Notes in Economics and Mathematical Systems, Springer-Verlag, 1989.
[4] Pham Dinh, T. & Hoai An, L. T., A d.c. optimization algorithm for solving the trust region subprogram, *SIAM Journal of Optimization,* **8**(1998), 476–505.
[5] Sawaragi, Y., Nakayama, H. & Tanino, T., Thoery of multiobjective optimization, Academic Press, 1985.
[6] Song, W., Conjugate duality in set-valued vector optimization, *Journal of Mathemathical Analysis and Applications,* **216**(1997), 265–283.
[7] Staib, T., On necessary and sufficient optimality conditions for multicriterial optimization problems, *ZOR-Zeitschrift fuer Operations Research,* **35**(1991), 231–248.
[8] Tanino, T., Conjugate duality in vector optimization, *Journal of Mathemathical Analysis and Applications,* **167**(1992), 84–97.
[9] Toland, J. F., Duality in nonconvex optimization, *Journal of Mathemathical Analysis and Applications,* **66**(1978), 399–415.
[10] Tuy, H., D.C. optimization: theory, methods and algorithms, in Hand Book of Global Optimization, R. Horst & P. M. Paradalos (eds.), Kluwer Academic Publishers, 1995, 149–216.
[11] Yin Zhiwen & Li Yuanxi, Optimization and duality of cone d.c. programming, *Chin. Ann. Math.,* **16B:**4(1995), 499–508.