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REGULARITY OF SOLUTIONS TO THE DIRICHLET PROBLEM FOR DEGENERATE ELLIPTIC EQUATION

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Abstract

In this paper, the author studies the regularity of solutions to the Dirichlet problem for equation $Lu = f$, where L is a second order degenerate elliptic operator in divergence form in Ω, a bounded open subset of R^n $(n \geq 3)$.

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§1. Introduction

The aim of this paper is to study the Dirichlet problem

$$\begin{cases} Lu = -(a_{ij}u_{x_i})_{x_j} = f, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of R^n $(n \geq 3)$, $a_{ij}(x)$ are symmetric, measurable and there exists $\nu > 0$, such that for all $\xi \in \mathrm{R}^n$ and a.a. $x \in \mathrm{R}^n$,

$$\nu^{-1}\omega(x)|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \nu\omega(x)|\xi|^2, \tag{1.2}$$

where $\omega(x)$ is a suitable weighted function.

Compared with that of uniformly elliptic operator, $\omega(x)$ in this situation may either vanish, or be infinite, or both, and such operator will be called degenerate operator. There have been a lot of classical results about uniformly elliptic equations. So it is natural to ask whether there exist corresponding results for degenerate ones.

In 1960–70's, De Giorgi-Nash's theorem and Harnack's inequality were extended to the degenerate case by some people[9, 11, 14, 15]. They imposed some restrictions on the weighted function $\omega(x)$. In three important papers [6, 4, 5], E. B. Fabes, C. E. Kenig, D. Jerrison and R. P. Serapioni studied the degenerate operator L with two kinds of weighted functions and got many results. In their papers, $\omega(x)$ belongs to A_2 or QC. Here A_2 is Muckenhoupt class, and $\omega(x) \in QC$ means $\omega(x) = |f'(x)|^{1-\frac{2}{n}}$, where $f : \mathrm{R}^n \to \mathrm{R}^n$ is a quasiconformal mapping and $|f'(x)|$ denotes the absolute of the Jacobian determinant of f. Since then, many people have made further researches in degenerate elliptic equations[8, 1, 16]. Among them, two classes of weighted Morrey spaces were introduced in [16], and they make it possible

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extending the regularity in Morrey spaces in uniformly elliptic case to that in degenerate case.

In this paper, we consider the following two problems:

(I) What regularity property does the solution to (1.1) have for a given f?

(II) What are the minimal conditions to be imposed on f to ensure given regularity of the solution to (1.1)?

The solution we consider is a very weak one introduced in [10] because in general the Dirichlet problem (1.1) does not have a weak solution under our assumption on f. Of course, we will point out that the very weak solution to (1.1) is actually a weak one in some conditions. We wish to say that the weighted function $\omega(x)$ belongs to A_2 throughout the paper. Finally, we say that our results extend the corresponding ones in [2, 3] to the degenerate case.

§2. Preliminaries

In this section, we will give some definitions, spaces and known results. Because of the local character of our results it is sufficient to assume $\Omega \equiv B_R(0)$.

We give the definition of A_p weight (or Muckenhoupt class) first.

Definition 2.1. *Let $\omega(x) > 0$, $\omega(x) \in L^1_{\rm loc}(\mathrm{R}^n)$, $1 < p < +\infty$. We say that $\omega(x)$ is an A_p weight, which denoted by $\omega(x) \in A_p$, if*

$$\sup_Q \Big(\frac{1}{|Q|}\int_Q \omega(y)dy\Big)\Big(\frac{1}{|Q|}\int_Q \omega(y)^{-\frac{1}{p-1}}dy\Big)^{p-1} \le C < +\infty,$$

where Q is a cube in $\mathrm{R}^{\rm n}$.

Let ω be an A_2 weight, $1 \le p < +\infty$. We give the definitions of weighted Lebesgue spaces and weighted Sobolev spaces.

$L^p(\Omega,\omega)$ is the space of measurable f in Ω, such that

$$\|f\|_{L^p(\Omega,\omega)} = \Big(\int_\Omega |f(x)|^p\omega(x)dx\Big)^{\frac{1}{p}} < +\infty.$$

$L^p_w(\Omega,\omega)$ is the space of measurable f in Ω, such that

$$\|f\|_{L^p_w(\Omega,\omega)} = \Big[\sup_{t>0} t^p\omega(\{x\in\Omega : |f(x)| > t\})\Big]^{\frac{1}{p}} < +\infty.$$

Obviously, $L^p(\Omega,\omega) \subset L^p_w(\Omega,\omega) \subset L^q(\Omega,\omega) \quad (1 \le q < p < +\infty)$.

$L^\infty(\Omega,\omega)$ is the space of measurable f in Ω, such that

$$\|f\|_{L^\infty(\Omega,\omega)} = \inf\{a \ge 0 : \omega(\{x\in\Omega : |f(x)| > a\}) = 0\} < +\infty.$$

$\mathrm{Lip}(\overline{\Omega})$ denotes the class of Lipschitz functions in $\overline{\Omega}$. $\mathrm{Lip}_0(\Omega)$ denotes the class of functions $f \in \mathrm{Lip}(\overline{\Omega})$ with compact support contained in Ω. If $f \in \mathrm{Lip}(\overline{\Omega})$, we can define the norm

$$\|f\|_{H^{1,p}(\Omega,\omega)} = \|f\|_{L^p(\Omega,\omega)} + \|\nabla f\|_{L^p(\Omega,\omega)}. \tag{2.1}$$

$H^{1,p}(\Omega,\omega)$ denotes the closure of $\mathrm{Lip}(\overline{\Omega})$ under the norm (2.1). We say that $f \in H^{1,p}_{\rm loc}(\Omega,\omega)$ if $f \in H^{1,p}(\Omega',\omega)$ for every $\Omega' \subset\subset \Omega$. $H^{1,p}_0(\Omega,\omega)$ denotes the closure of $\mathrm{Lip}_0(\Omega)$ under the norm (2.1). And $H^{-1,p'}(\Omega,\omega)$ is the dual space of $H^{1,p}_0(\Omega,\omega)$, where $\frac{1}{p}+\frac{1}{p'}=1$.

Next, we define some weighted spaces.

Definition 2.2.[16] *Let $\sigma > 0$, $C > 0$ and $0 < r < 2R$, we define*

$$M_\sigma(\Omega,\omega) = \Big\{f \in L^1(\Omega,\omega) : \sup_{x\in\Omega}\int_{\Omega_r(x)} |f(y)|\Big(\int_{|x-y|}^{4R}\frac{sds}{\omega(B_s(x))}\Big)\omega(y)dy \le Cr^\sigma\Big\},$$

where $\Omega_r(x) = \Omega \cap B_r(x)$.

Definition 2.3.[16] *Let* $\sigma \in \mathrm{R}$, *we define*

$$L^{1,\sigma}(\Omega,\omega) = \Big\{ f \in L^1(\Omega,\omega) : \|f\|_{1,\sigma} = \sup_{\substack{x\in\Omega \\ 0<r<2R}} \frac{r^{2-\sigma}}{\omega(B_r(x))} \int_{\Omega_r(x)} |f(y)|\omega(y)dy < +\infty \Big\}.$$

Above two spaces, called weighted Morrey spaces, were introduced by C. Vitanza and P. Zamboni when they studied Hölder continuity of solutions to degenerate Schrödinger equations.

Definition 2.4.[8,16] *Let* $\eta(r)$ *be a nondecreasing function defined in* $(0, +\infty)$ *such that* $\lim_{r\to 0} \eta(r) = 0$, *we set*

$$S(\Omega,\omega) = \Big\{ f \in L^1(\Omega,\omega) : \sup_{x\in\Omega} \int_{\Omega_r(x)} |f(y)| \Big(\int_{|x-y|}^{4R} \frac{sds}{\omega(B_s(x))} \Big) \omega(y)dy \le \eta(r) \Big\}.$$

This space was introduced by C. E. Gutierrez and it is the classical Stummel-kato class when $\omega \equiv 1$.

Definition 2.5. *We set*

$$\tilde{S}(\Omega,\omega) = \Big\{ f \in L^1(\Omega,\omega) : \sup_{\substack{x\in\Omega \\ r>0}} \int_{\Omega_r(x)} |f(y)| \Big(\int_{|x-y|}^{4R} \frac{sds}{\omega(B_s(x))} \Big) \omega(y)dy < +\infty \Big\}.$$

Definition 2.6. *Let* $1 \le p < +\infty$. *We set*

$$S^p(\Omega,\omega) = \Big\{ f \in L^1(\Omega,\omega) : \int_\Omega \Big[\int_{\Omega_r(x)} |f(y)| \Big(\int_{|x-y|}^{4R} \frac{sds}{\omega(B_s(x))} \Big) \omega(y)dy \Big]^p \cdot \omega(x)dx < +\infty \ \ \text{for some } r > 0 \Big\}.$$

When $p = \infty$, *we define* $S^\infty(\Omega,\omega) = \tilde{S}(\Omega,\omega)$.

The classical Schechter classes were introduced by Schechter in [12] and [13] for different reasons. $S^p(\Omega,\omega)$ are the corresponding ones in the weighted case.

We now give the definitions of weak solution and very weak solution.

Definition 2.7. *Let* μ *be a bounded variation measure on* Ω. *We say that* $u \in H_0^{1,2}(\Omega,\omega)$ *is a weak solution to the Dirichlet problem*

$$\begin{cases} Lu = \mu, & in\ \Omega, \\ u = 0, & on\ \partial\Omega, \end{cases} \tag{2.2}$$

if we have

$$\int_\Omega a_{ij} u_{x_i} \phi_{x_j} = \int_\Omega \phi d\mu, \qquad \forall \phi \in H_0^{1,2}(\Omega,\omega).$$

Definition 2.8. *We say that* $u \in L^1(\Omega,\omega)$ *is a very weak solution to* (2.2), *if we have*

$$\int_\Omega uL\phi = \int_\Omega \phi d\mu, \quad \forall \phi \in H_0^{1,2}(\Omega,\omega) \cap C^0(\overline{\Omega}) \quad s.t. \quad L\phi \in C^0(\overline{\Omega}).$$

Theorem 2.1.[4] *Let* μ *be a bounded variation measure on* Ω. *Then there exists a unique very weak solution to* (2.2).

Similarly to the uniformly elliptic case, we introduce Green's function in the degenerate case and give some basic propositions.

Definition 2.9. *Let* δ_y *be the Dirac delta at* $y \in \Omega$. *Then the very weak solution to the Dirichlet problem* $\begin{cases} Lu = \delta_y, & in\ \Omega, \\ u = 0, & on\ \partial\Omega \end{cases}$ *will be called the Green's function for* L *and* Ω *with pole at* y. *We denote it by* $g(x, y)$.

Theorem 2.2. *Let μ be a bounded variation measure on Ω. Then $u(x)=\int_\Omega g(x,y)d\mu(y)$ exists almost everywhere, moreover u is a very weak solution to* (2.2).

Theorem 2.3. *There exist constants C_1 and C_2 such that for any $x,y\in\Omega$, we have*

$$C_1\int_{|x-y|}^{4R}\frac{sds}{\omega(B_s(x))}\le g(x,y)\le C_2\int_{|x-y|}^{4R}\frac{sds}{\omega(B_s(x))}.$$

Proofs of the above theorems and other properties of $g(x,y)$ can be seen in [4].

Finally, we give three important propositions of A_2 weight.

Proposition 2.1. *There exists $C>1$ such that*

$$\omega(B_{2r}(x))\le C\omega(B_r(x)). \tag{2.3}$$

Proposition 2.2. *There exists $M<1$ such that*

$$\omega(B_r(x))\le M\omega(B_{2r}(x)). \tag{2.4}$$

Proposition 2.3. *Let $0<2\rho_1\le\rho_2$. Then there is a constant $C_1>0$ such that*

$$\int_{\rho_1}^{\rho_2}\frac{sds}{\omega(B_s(x))}\ge C_1\frac{\rho_1^2}{\omega(B_{\rho_1}(x))}. \tag{2.5}$$

Let $0<\rho_1<\rho_2$, $M<1/4$ (M is the constant in (2.4)*). Then there is a constant $C_2>0$ such that*

$$\int_{\rho_1}^{\rho_2}\frac{sds}{\omega(B_s(x))}\le C_2\frac{\rho_1^2}{\omega(B_{\rho_1}(x))}. \tag{2.6}$$

Proofs of Propositions 2.1 and 2.2 can be seen in [7] and we can find the proof of Proposition 2.3 in [16].

§3. Regularities I

In this section, we solve the problem (I) mentioned in Introduction. Now we give our results.

Theorem 3.1. *Let $f/\omega\in L^{1,\sigma}(\Omega,\omega)$, $\sigma<0$, $M<1/4$ (M is the constant in* (2.4)*). Then the very weak solution u to* (1.1) *belongs to $L_\omega^{\frac{\sigma-2}{\sigma}}(\Omega,\omega)$. In particular, $u\in L^p(\Omega,\omega)\,(p<\frac{\sigma-2}{\sigma})$.*

Proof. By Theorems 2.1 and 2.2, we have

$$u(x)=\int_\Omega g(x,y)f(y)dy.$$

Then

$$\begin{aligned}|u(x)|&\le\int_\Omega g(x,y)|f(y)|dy\\&=\int_{\{y\in\Omega:|x-y|<\epsilon\}}g(x,y)|f(y)|dy+\int_{\{y\in\Omega:|x-y|\ge\epsilon\}}g(x,y)|f(y)|dy\\&=\mathrm{I}_1+\mathrm{I}_2.\end{aligned}$$

Set

$$\begin{aligned}R_k&=\{y\in\Omega:2^{-k-1}\epsilon\le|x-y|<2^{-k}\epsilon,k\in\overline{Z^-}\},\\R_k'&=\{y\in\Omega:2^{k}\epsilon\le|x-y|<2^{k+1}\epsilon,k\in\overline{Z^-}\}.\end{aligned}$$

Then by Theorem 2.3,

$$|\mathrm{I}_1|\le\sum_{k=0}^\infty\int_{R_k}g(x,y)|f(y)|dy\le C\sum_{k=0}^\infty\int_{R_k}\Big(\int_{2^{-k-1}\epsilon}^{4R}\frac{sds}{\omega(B_s(x))}\Big)|f(y)|dy.$$

And by (2.6), we get

$$|\mathrm{I}_1| \le C\sum_{k=0}^{\infty}\frac{(2^{-k-1}\epsilon)^2}{\omega(B_{2^{-k-1}\epsilon}(x))}\int_{R_k}|f(y)|dy \le C\epsilon^2 M_\omega(f/\omega),$$

where $M_\omega(f)=\sup\limits_{r>0}\frac{1}{\omega(B_r)}\int_{B_r}|f|\omega$. The last inequality uses the double condition (2.3) of ω.

Similarly, we have

$$\begin{aligned}|\mathrm{I}_2| &\le \sum_{k=0}^{\infty}\int_{R'_k} g(x,y)|f(y)|dy\\ &\le C\sum_{k=0}^{\infty}\int_{R'_k}\Big(\int_{2^k\epsilon}^{4R}\frac{sds}{\omega(B_s(x))}\Big)|f(y)|dy\\ &\le C\sum_{k=0}^{\infty}\frac{(2^k\epsilon)^2}{\omega(B_{2^k\epsilon}(x))}\int_{R'_k}|f(y)|dy\\ &\le C\sum_{k=0}^{\infty}(2^k\epsilon)^\sigma\|f/\omega\|_{1,\sigma}\le C\epsilon^\sigma\|f/\omega\|_{1,\sigma}.\end{aligned}$$

Choosing $\epsilon=\left[\frac{\|f/\omega\|_{1,\sigma}}{M_\omega(f/\omega)}\right]^{\frac{1}{2-\sigma}}$, we obtain

$$|u(x)|\le C\left[M_\omega(f/\omega)\right]^{\frac{\sigma}{\sigma-2}}\|f/\omega\|_{1,\sigma}^{\frac{2}{2-\sigma}}.$$

Thus

$$\begin{aligned}&t^{\frac{\sigma-2}{\sigma}}\omega\left(\{x\in\Omega:|u(x)|>t\}\right)\\ \le\ & t^{\frac{\sigma-2}{\sigma}}\omega\big(\{x\in\Omega: M_\omega(f/\omega)>t^{\frac{\sigma-2}{\sigma}}C^{\frac{2-\sigma}{\sigma}}\|f/\omega\|_{1,\sigma}^{\frac{2}{\sigma}}\}\big)\\ \le\ & C_\sigma\|f/\omega\|_{L^1(\Omega,\omega)}^{\frac{\sigma-2}{\sigma}}\|f/\omega\|_{1,\sigma}^{\frac{-2}{\sigma}}.\end{aligned}$$

The last inequality holds true because M_ω is weak type (1,1).

And this completes the proof.

Remark 3.1. The assumption $M<1/4$ in Theorem 3.1 contains the case $\omega\equiv 1$. That is to say, Theorem 3.1 is the corresponding extension of Theorem 2.1 in [2] which is in the uniformly elliptic case. So are the following theorems.

Theorem 3.2. *Let $f/\omega\in L^{1,0}(\Omega,\omega)$, $M<1/4$. Then the very weak solution u to* (1.1) *is in* $\mathrm{BMO}_{\mathrm{loc}}(\Omega,\omega)$, *i.e., $\forall\Omega'\subset\subset\Omega$, $d=\mathrm{dist}(\Omega',\partial\Omega)$, $\exists C=C(d)>0$ s.t. $\forall B_r(x_0)$, $x_0\in\Omega'$, $0<r<d/2$, we have*

$$\frac{1}{\omega(B_r(x_0))}\int_{B_r(x_0)}|u(x)-u_{B_r(x_0)}|\omega(x)dx\le C\|f/\omega\|_{1,0},$$

where $u_{B_r(x_0)}=\frac{1}{\omega(B_r(x_0))}\int_{B_r(x_0)}u(x)\omega(x)dx$.

Proof. Set $B\equiv B_r(x_0)$, $B_*\equiv B_{2r}(x_0)$, $f_1=f\chi_{B_*}$, $f_2=f(1-\chi_{B_*})$. Then $u=u_1+u_2$, where $u_1(x)=\int_\Omega g(x,y)f_1(y)dy$ and $u_2(x)=\int_\Omega g(x,y)f_2(y)dy$ are the very weak solutions to $\begin{cases}Lu_1=f_1, & \text{in }\Omega,\\ u_1=0, & \text{on }\partial\Omega\end{cases}$ and $\begin{cases}Lu_2=f_2, & \text{in }\Omega,\\ u_2=0, & \text{on }\partial\Omega\end{cases}$ respectively.

We now estimate the BMO norm of u_1 and u_2.

As for u_1, we have

$$\frac{1}{\omega(B)}\int_B |u_1(x)-u_{1_B}|\omega(x)dx \le 2|u_1|_B \le \frac{2}{\omega(B)}\int_B\Big(\int_{B_*} g(x,y)|f(y)|dy\Big)\omega(x)dx = \frac{2}{\omega(B)}\int_{B_*}|f(y)|\Big(\int_B g(x,y)\omega(x)dx\Big)dy.$$

Set

$$R_k=\{y\in B_*: 2^{-k-1}3r\le |x-y|<2^{-k}3r, k\in \overline{Z^-}\}.$$

We get

$$\int_B g(x,y)\omega(x)dx \le \sum_{k=0}^{\infty}\int_{R_k} g(x,y)\omega(x)dx \le C\sum_{k=0}^{\infty}\int_{R_k}\Big(\int_{2^{-k-1}3r}^{4R}\frac{sds}{\omega(B_s(x))}\Big)\omega(x)dx \le C\sum_{k=0}^{\infty}4^{-k}r^2\le Cr^2.$$

Thus

$$\frac{1}{\omega(B)}\int_B |u_1(x)-u_{1_B}|\omega(x)dx \le Cr^2\frac{1}{\omega(B_*)}\int_{B_*}|f(y)|dy\le C\|f/\omega\|_{1,0}.$$

Next, we estimate for u_2.

$$\begin{aligned}
&\frac{1}{\omega(B)}\int_B |u_2(x)-u_{2_B}|\omega(x)dx\\
&=\frac{1}{\omega(B)}\int_B\Big|\int_{\Omega\setminus B_*} g(x,y)f(y)dy-\frac{1}{\omega(B)}\int_B\Big(\int_{\Omega\setminus B_*} g(z,y)f(y)dy\Big)\omega(z)dz\Big|\omega(x)dx\\
&=\frac{1}{\omega(B)}\int_B\Big|\int_{\Omega\setminus B_*}\Big[g(x,y)-\frac{1}{\omega(B)}\int_B g(z,y)\omega(z)dz\Big]f(y)dy\Big|\omega(x)dx\\
&\le\frac{1}{\omega(B)}\int_B\int_{\Omega\setminus B_*}\Big|g(x,y)-\frac{1}{\omega(B)}\int_B g(z,y)\omega(z)dz\Big||f(y)|dy\omega(x)dx\\
&=\int_{\Omega\setminus B_*}|f(y)|\Big[\frac{1}{\omega(B)}\int_B\Big|g(x,y)-\frac{1}{\omega(B)}\int_B g(z,y)\omega(z)dz\Big|\omega(x)dx\Big]dy\\
&=\mathrm{I}+\mathrm{II},
\end{aligned}$$

where I and II represent the integrals on Ω_d and Ω^d respectively. Here, we set $\Omega_d=\{y:|y-x_0|\le d\}\cap(\Omega\setminus B_*)$ and $\Omega^d=\{y:|y-x_0|>d\}\cap(\Omega\setminus B_*)$.

We estimate the integral on Ω_d first.

When restricted to B, $g(\cdot,y)$ is a weak solution of $Lu=0$. So using Theorems 2.3.12 and

2.3.8 in [6], we have

$$
\begin{aligned}
&\frac{1}{\omega(B)}\int_B \Big|g(x,y)-\frac{1}{\omega(B)}\int_B g(z,y)\omega(z)dz\Big|\omega(x)dx\\
\le{}& \frac{1}{\omega(B)}\int_B \Big(\frac{1}{\omega(B)}\int_B |g(x,y)-g(z,y)|\omega(z)dz\Big)\omega(x)dx\\
\le{}& C\frac{1}{\omega(B)}\int_B \Big[\frac{1}{\omega(B)}\int_B \Big(\frac{1}{\omega(B_{|y-x_0|})(x_0)}\int_{B_{|y-x_0|}(x_0)} g(x,y)^2\omega(x)dx\Big)^{1/2}\\
&\cdot\Big|\frac{x-z}{y-x_0}\Big|^{\alpha}\omega(z)dz\Big]\omega(x)dx\\
\le{}& C\frac{1}{\omega(B)}\int_B \Big[\frac{1}{\omega(B)}\int_B g(x_0,y)\Big(\frac{r}{|y-x_0|}\Big)^{\alpha}\omega(z)dz\Big]\omega(x)dx\\
={}& Cr^{\alpha}g(x_0,y)|y-x_0|^{-\alpha}.
\end{aligned}
$$

Set

$$\Omega_{d,k}=\left\{y\in\Omega_d:\ 2^k r\le|y-x_0|<2^{k+1}r,\ k\in \mathrm{N}\right\}.$$

Then we get

$$
\begin{aligned}
\mathrm{I}\le{}& Cr^{\alpha}\int_{\Omega_d}|f(y)||g(x_0,y)||y-x_0|^{-\alpha}dy\\
\le{}& Cr^{\alpha}\sum_{k=0}^{\infty}\int_{\Omega_{d,k}}|f(y)||g(x_0,y)||y-x_0|^{-\alpha}dy\\
\le{}& Cr^{\alpha}\sum_{k=0}^{\infty}\int_{\Omega_{d,k}}|f(y)|\Big(\int_{2^k r}^{4R}\frac{sds}{\omega(B_s(x_0))}\Big)(2^k r)^{-\alpha}dy\\
\le{}& C\sum_{k=0}^{\infty}\int_{\Omega_{d,k}}|f(y)|\frac{(2^k r)^2}{\omega(B_{2^k r}(x_0))}2^{-k\alpha}dy\\
\le{}& C\sum_{k=0}^{\infty}2^{-k\alpha}\frac{(2^k r)^2}{\omega(B_{2^{k+1}r}(x_0))}\int_{B_{2^{k+1}r}(x_0)}|f(y)|dy\\
\le{}& C\|f/\omega\|_{1,0}.
\end{aligned}
$$

Next, we estimate the integral on Ω^d.

For any $x\in B$, we have

$$|x-y|\ge|x_0-y|-|x-x_0|>|x_0-y|-r>d/2,$$

then

$$g(x,y)\le\int_{d/2}^{4R}\frac{sds}{\omega(B_s(x))}\le\frac{Cd^2}{\omega(B_{d/2}(x))}.$$

So we have

$$
\begin{aligned}
&\frac{1}{\omega(B)}\int_B \Big|g(x,y)-\frac{1}{\omega(B)}\int_B g(z,y)\omega(z)dz\Big|\omega(x)dx\\
\le{}& \frac{Cd^2}{\omega(B)}\int_B\frac{\omega(x)dx}{\omega(B_{d/2}(x))}\\
\le{}& \frac{Cd^2}{\omega(B_{d/2}(x_0))}.
\end{aligned}
$$

The truth of the last inequality is due to

$$\omega(B_{d/2}(x_0)) \le \omega(B_{d/2+|x-x_0|}(x)) \le \omega(B_d(x)) \le C\omega(B_{d/2}(x)).$$

Thus

$$\mathrm{II} \le \frac{Cd^2}{\omega(B_{d/2}(x_0))}\int_{\Omega^d}|f(y)|dy \le C\|f/\omega\|_{1,0}.$$

This completes the proof.

Theorem 3.3. *Let $f/\omega \in \tilde{S}(\Omega,\omega)$. Then the very weak solution u to* (1.1) *is in $L^\infty(\Omega)$.*

Proof. We have

$$\begin{aligned}|u(x)| &\le \int_\Omega g(x,y)|f(y)|dy\\ &\le C\int_\Omega |f(y)|\Big(\int_{|x-y|}^{4R}\frac{sds}{\omega(B_s(x))}\Big)dy\\ &\le C\sup_{\substack{x\in\Omega\\ r>0}}\int_\Omega |f(y)|\Big(\int_{|x-y|}^{4R}\frac{sds}{\omega(B_s(x))}\Big)dy.\end{aligned}$$

Remark 3.2. $L^\infty(\Omega,\omega) = L^\infty(\Omega)$ because ωdx and dx are absolutely continous mutually. So $L^\infty(\Omega)$ in Theorem 3.3 can be replaced by $L^\infty(\Omega,\omega)$.

Theorem 3.4. *Let $f/\omega \in S(\Omega,\omega)$. Then the very weak solution u to* (1.1) *is continous in Ω.*

Theorem 3.5. *Let $f/\omega \in M_\sigma(\Omega,\omega)$. Then the very weak solution u to* (1.1) *is locally Hölder-continous in Ω.*

Proofs of the above two theorems are very close to those of Theorems 3.1 and 3.2 in [16] if we substitute f for Vu. So we omit the proofs.

Remark 3.3. By Lemma 3.3 in [8], we know that $f/\omega \in S(\Omega,\omega) \Rightarrow f \in H^{-1,2}(\Omega,\omega)$. And from definition, $M_\sigma(\Omega,\omega) \subset S(\Omega,\omega)$. Then by Theorem 2.2 in [6], when $f/\omega \in S(\Omega,\omega)$ or $M_\sigma(\Omega,\omega)$, (1.1) has a unique weak solution and the very weak solution to (1.1) is the weak one.

Remark 3.4. In order to see how the regularity of the solution to (1.1) changes under the variation of f, we give the containing relations of the spaces appearing in the above theorems. We have

$$M_\sigma(\Omega,\omega) \subset S(\Omega,\omega) \subset \tilde{S}(\Omega,\omega) \subset L^{1,0}(\Omega,\omega) \subset L^{1,\lambda}(\Omega,\omega) \quad (\lambda < 0),$$

where $\tilde{S}(\Omega,\omega) \subset L^{1,0}(\Omega,\omega)$ demands $M < 1/4$.

§4. Regularities II

In this section, we solve the problem (II) mentioned in Introduction.

We give two propositions of weighted Schechter class $S^p(\Omega,\omega)$ first.

Proposition 4.1. *Let $M < 1/4$, $1 \le p < \frac{n}{n-1}$. Then $S^p(\Omega,\omega) = L^1(\Omega,\omega)$.*

Proof. By Remark 2.13 in [16], if $\sigma \le 2-2n$, we have $L^{1,\sigma}(\Omega,\omega) = L^1(\Omega,\omega)$. Set

$$I(f)(x) = \int_\Omega |f(y)|\Big(\int_{|x-y|}^{4R}\frac{sds}{\omega(B_s(x))}\Big)\omega(y)dy.$$

Then if $f \in L^1(\Omega,\omega)$, we can get

$$I(f) \in L^p(\Omega,\omega)\ \Big(1 \le p < \frac{\sigma-2}{\sigma}\Big)$$

by using Theorem 3.1. $\sigma \le 2-2n$ implies $\frac{\sigma-2}{\sigma} \le \frac{n}{n-1}$, so we get

$$f \in S^p(\Omega,\omega)\ \Big(1 \le p < \frac{n}{n-1}\Big).$$

Proposition 4.2. *Let $M<1/4$. Then $S^\infty(\Omega,\omega) \subset L^{1,0}(\Omega,\omega) \subset \bigcap_{1\le p<\infty} S^p(\Omega,\omega)$.*

Proof. Definition implies $S^\infty(\Omega,\omega)=\tilde{S}(\Omega,\omega)$. And we have $S^\infty(\Omega,\omega) \subset L^{1,0}(\Omega,\omega)$ by Remark 3.4.

Now we prove the second containing relation.

$f \in L^{1,0}(\Omega,\omega)$ implies $f \in L^{1,\sigma}(\Omega,\omega)$ $(\forall \sigma<0)$. By Theorem 3.1, $I(f) \in L_\omega^{\frac{\sigma-2}{\sigma}}(\Omega,\omega)$. By Proposition 4.1, when $1 \le p < \frac{n}{n-1}$, $S^p(\Omega,\omega)=L^1(\Omega,\omega)$. So we will only consider the case of $\frac{n}{n-1} \le p < \infty$. Because of $1 < \frac{\sigma-2}{\sigma} < \infty$, $I(f) \in L^p(\Omega,\omega)$ $(\frac{n}{n-1} \le p < \infty)$ and so we have $f \in S^p(\Omega,\omega)$ $(1 \le p < \infty)$.

We now give our results of regularity.

Theorem 4.1. *Let $f/\omega \in L^1(\Omega,\omega)$, $M<1/4$ and $u \in L^1(\Omega,\omega)$ be a very weak solution to* (1.1)*. Then*

$$u \in L^p_{\rm loc}(\Omega,\omega) \Longleftrightarrow f/\omega \in S^p_{\rm loc}(\Omega,\omega) \quad (1 \le p < \infty).$$

Proof. We first prove the theorem in the case $f \ge 0$. Let K be a compact subset of Ω. If $u \in L^p(K,\omega)$, then

$$\begin{aligned}\int_K |u(x)|^p \omega(x)dx &= \int_K \Big(\int_\Omega g(x,y)f(y)dy\Big)^p \omega(x)dx \\ &\ge \int_K \Big(\int_{\Omega_r(x)} g(x,y)f(y)dy\Big)^p \omega(x)dx.\end{aligned}$$

We have the following conclusion: let $M<1/4$, $x\in\Omega$ and $|x-y| < \frac{1}{2}{\rm dist}(x,\partial\Omega)=\frac{1}{2}d_x$, then

$$g(x,y) \ge C\int_{|x-y|}^{4R} \frac{sds}{\omega(B_s(x))}.$$

In fact, Theorem 4 in [5] implies

$$g(x,y) \ge C\int_{|x-y|}^{d_x} \frac{sds}{\omega(B_s(x))}.$$

And by (2.5) and (2.6),

$$\int_{|x-y|}^{d_x} \frac{sds}{\omega(B_s(x))} \ge \frac{|x-y|^2}{\omega(B_{|x-y|}(x))} \ge C\int_{|x-y|}^{4R} \frac{sds}{\omega(B_s(x))}.$$

So we get the conclusion.

If we choose $r>0$ such that $r < \frac{{\rm dist}(K,\partial\Omega)}{2}$, we have

$$\int_K |u(x)|^p\omega(x)dx \ge C\int_K \Big[\int_{\Omega_r(x)} f(y)\Big(\int_{|x-y|}^{4R}\frac{sds}{\omega(B_s(x))}\Big)dy\Big]^p\omega(x)dx.$$

So, $u \in L^p_{\rm loc}(\Omega,\omega) \Longrightarrow f/\omega \in S^p_{\rm loc}(\Omega,\omega)$.

Now the converse. For $r > 0$,

$$\begin{aligned} u(x) &= \int_\Omega g(x,y)f(y)dy \\ &\le C\int_{\Omega_r(x)} f(y)\Big(\int_{|x-y|}^{4R}\frac{sds}{\omega(B_s(x))}\Big)dy \\ &\quad + C\int_{\{y\in\Omega:|y-x|\ge r\}} f(y)\Big(\int_{|x-y|}^{4R}\frac{sds}{\omega(B_s(x))}\Big)dy \\ &= \mathrm{I}+\mathrm{II}. \end{aligned}$$

$f/\omega \in S^p(K,\omega)$ implies I$\in L^p(K,\omega)$.

For II, we have

$$\begin{aligned} \mathrm{II} &\le C\int_\Omega f(y)\Big(\int_r^{4R}\frac{sds}{\omega(B_s(x))}\Big)dy \\ &\le \frac{Cr^2}{\omega(B_r(x))}\int_\Omega f(y)dy \\ &\le C(\omega,\Omega)r^{2-2n}\|f/\omega\|_{L^1(\Omega,\omega)}, \end{aligned}$$

where the last inequality holds true because of Lemma 2.1(c) in [16] and we get II $\in L^p(K,\omega)$. So

$$f/\omega \in S^p_{\rm loc}(\Omega,\omega) \Longrightarrow u \in L^p_{\rm loc}(\Omega,\omega).$$

For general case, we write $u = u^+ - u^-$, where $u^+ = \max(u,0)$ and $u^- = -\min(u,0)$. Then we have $u^\pm \in L^1(\Omega,\omega)$. Set $Lu^\pm = f^\pm$. We obtain $f^\pm \ge 0$ and $f = f^+ - f^-$ from weak maximum principle[6] as well as Theorem 2.1. Thus, using the result in the case $f \ge 0$, we complete the proof.

Remark 4.1. Let $1 \le p < \infty$, we define weighted L-Schechter class as

$$S^p_L(\Omega,\omega) = \Big\{f\in L^1(\Omega,\omega) : \int_\Omega\Big[\int_{\Omega_r(x)}|f(y)|g(x,y)\omega(y)dy\Big]^p\omega(x)dx < \infty \text{ for some } r>0\Big\},$$

where $g(x,y)$ is the Green's function for L and Ω.

Using $S^p_L(\Omega,\omega)$ to take the place of $S^p(\Omega,\omega)$ in Theorem 4.1, we could get a similar result of global nature.

The following theorem shows that Theorem 4.1 still holds true for $p = +\infty$.

Theorem 4.2. *Let $f/\omega \in L^1(\Omega,\omega)$, $M < 1/4$ and $u \in L^1(\Omega,\omega)$ be a very weak solution to* (1.1). *Then*

$$u \in L^\infty_{\rm loc}(\Omega) \Longleftrightarrow f/\omega \in \tilde{S}_{\rm loc}(\Omega,\omega).$$

Proof. Just as in the proof of Theorem 4.1, we only need to prove the case $f \ge 0$. Let K be a compact subset of Ω, $x \in K$ and $|x-y| < r < \frac{1}{2}{\rm dist}(K,\partial\Omega)$. Applying the conclusion in the proof of Theorem 4.1, we have

$$\begin{aligned} \int_{\Omega_r(x)} f(y)\Big(\int_{|x-y|}^{4R}\frac{sds}{\omega(B_s(x))}\Big)dy &\le C\int_{\Omega_r(x)} f(y)g(x,y)dy \\ &\le Cu(x) \le C\|u\|_{L^\infty(K)}. \end{aligned}$$

So

$$u \in L^\infty_{\rm loc}(\Omega) \Longrightarrow f/\omega \in \tilde{S}_{\rm loc}(\Omega,\omega).$$

The sufficient part can be found in Theorem 3.3.

Theorem 4.3. *Let $f/\omega \in L^1(\Omega,\omega)$, $M < 1/4$ and $u \in L^1(\Omega,\omega)$ be a very weak solution to* (1.1). *Then*

$$u \in C^0(\Omega) \Longleftrightarrow f/\omega \in S_{\rm loc}(\Omega,\omega).$$

Proof. Just as in the proof of Theorem 4.1, we only need to prove the case $f \geq 0$. Let

$$u_r(x) = \int_{\{y\in\Omega:|x-y|\geq r\}} f(y)g(x,y)dy, \quad r > 0.$$

We have

$$\begin{aligned}
&f(y)\left|g(x,y)\chi_{\{|x-y|\geq r\}}(y) - g(x_0,y)\chi_{\{|x_0-y|\geq r\}}(y)\right| \\
&\leq Cf(y)\Big(\int_r^{4R}\frac{sds}{\omega(B_s(x))} + \int_r^{4R}\frac{sds}{\omega(B_s(x_0))}\Big) \\
&\leq C(\omega,\Omega)r^{2-2n}f(y) \in L^1(\Omega).
\end{aligned}$$

By dominated convergence theorem, we get $\lim\limits_{x\to x_0} u_r(x) = u_r(x_0)$, which means $u_r(x) \in C^0(\Omega)$. Moreover $0 \leq u_r(x) \leq u(x)$ and $u_r(x) \to u(x)$ monotonically for every $x \in \Omega$. If K is a compact subset of Ω, then by Dini's theorem we get

$$\sup_{x\in K}(u(x) - u_r(x)) = \sup_{x\in K}\int_{\Omega_r(x)} f(x)g(x,y)dy \longrightarrow 0.$$

Choosing $r < \frac{1}{2}\mathrm{dist}(K,\partial\Omega)$, we have

$$\sup_{x\in K}\int_{\Omega_r(x)} f(y)\Big(\int_{|x-y|}^{4R}\frac{sds}{\omega(B_s(x))}\Big)dy \longrightarrow 0.$$

So

$$u \in C^0(\Omega) \Longrightarrow f/\omega \in S_{\rm loc}(\Omega,\omega).$$

The sufficient part can be found in Theorem 3.4.

Lemma 4.1. *Let $f/\omega \in L^1(\Omega,\omega)$, $f \geq 0$ and u be a bounded very weak solution to* (1.1). *Then u is the weak solution to the same Dirichlet problem.*

The proof is very similar to Remark 4.6 in [3], so we omit it.

Lemma 4.2. *Let $f/\omega \in L^1(\Omega,\omega)$, $f \geq 0$, $0 < \alpha < 1$ and $u \in C^{0,\alpha}(\Omega)$ be a very weak solution to* (1.1). *Then we have*

$$\int_{B_r(x_0)}|\nabla u|^2\omega \leq C\Big[r^\alpha\int_{B_{2r}(x_0)} f + r^{2\alpha-2}\omega(B_r(x_0))\Big]$$

for all $B_{2r} \subset \Omega$.

Proof. We can prove this lemma by Lemma 4.1 in [16] and Lemma 4.1.

Theorem 4.4. *Let $f/\omega \in L^1(\Omega,\omega)$, $M < 1/4$ and $u \in L^1(\Omega,\omega)$ be a very weak solution to* (1.1). *Then we have*

$$u \in C^{0,\alpha}(\Omega) \Longleftrightarrow f/\omega \in L^{1,\alpha}(\Omega,\omega) \qquad (0 < \alpha < 1).$$

Proof. Just as in the proof of Theorem 4.1, we only need to prove the case $f \geq 0$. We can get the necessary part by Theorem 4.2 in [16] and Lemma 4.2. And we can get the sufficient one by Theorem 3.5 and the fact that $L^{1,\alpha}(\Omega,\omega) \subset M_\alpha(\Omega,\omega)$ when $M < 1/4$.

Remark 4.2. We would not need the assumption $M < 1/4$ when we get the part $u \in C^{0,\alpha}(\Omega) \Longrightarrow f/\omega \in L^{1,\alpha}(\Omega,\omega)$ in Theorem 4.4.

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