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GLOBAL UNIQUENESS IN THE INVERSE ACOUSTIC SCATTERING PROBLEM WITHIN POLYGONAL OBSTACLES***

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Abstract

The authors prove the uniqueness in the inverse acoustic scattering problem within convex polygonal domains by a single incident direction in the sound-soft case and the sound-hard case, and by two incident directions in the case of the impedance boundary condition. The proof is based on analytic continuation on a straight line.

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§ 1 . Introduction

Let $k \in \mathbb{R}$, $\lambda > 0$ and $i = \sqrt{-1}$. We consider an acoustic scattering problem by an impenetrable obstacle $D \subset \mathbb{R}^2$:

$$\Delta u + k^2 u = 0, \qquad \text{in } \mathbb{R}^2 \setminus \overline{D}, \tag{1.1}$$

$$u(x) = e^{ikx\cdot d} + u^S(x), \tag{1.2}$$

$$\lim_{|x|\to\infty} |x|^{\frac{1}{2}} \left(\frac{\partial}{\partial |x|} u^S(x) - iku^S(x) \right) = 0 \tag{1.3}$$

associated with

$$u = 0, \qquad \text{on } \partial D\text{: the sound-soft obstacle} \tag{1.4}$$

$$\frac{\partial u}{\partial \nu} + i\lambda u = 0, \qquad \text{on } \partial D\text{: the sound-hard obstacle if } \lambda = 0 \text{ and the impedance boundary condition if } \lambda > 0. \tag{1.5}$$

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Here and henceforth, $x = (x_1, x_2) \in \mathbb{R}^2$, $|x| = \sqrt{x_1^2 + x_2^2}$, $S^1 = \{x \in \mathbb{R}^2; |x| = 1\}$, $\overline{D}$ denotes the closure of a set $D \subset \mathbb{R}^2$, $d \in S^1$ and $\frac{\partial}{\partial \nu}$ denotes the normal derivative: $\frac{\partial u}{\partial \nu} = \nabla u \cdot \nu$, where $\nu = \nu(x)$ is the outward unit normal vector to ∂D at x. Moreover, $k \in \mathbb{R}$ is the wave number and we can interpret $e^{ikx\cdot d}$ as an incident wave with the direction d. Equation (1.3) at ∞ is called the Sommerfeld radiation condition and u^S is called the scattering field. As for the details, we refer to [4, 5, 9, 12, 13, 16].

It is known (e.g., [2, 15]) that there exists a unique solution $u^S(D; d) \in H^1_{\mathrm{loc}}(\mathbb{R}^2 \setminus \overline{D})$ to (1.1)–(1.4) or (1.1)–(1.3) and (1.5) for $d \in S^1$ if ∂D is Lipschitz continuous. In the case (1.5) with $\lambda > 0$, we denote the solution by $u^S(D, \lambda; d)$ for specifying the dependence on λ. Moreover, similarly to Theorem 2.5 in [5], we can prove that there exists $u_\infty(D; d)(x)$ or $u_\infty(D, \lambda; d)(x)$ in the case (1.5) with $\lambda > 0$, $x \in S^1$, such that

$$u^S(D; d)(x) = \frac{e^{ik|x|}}{|x|^{\frac{1}{2}}} \left(u_\infty(D; d) \left(\frac{x}{|x|} \right) + O\left(\frac{1}{|x|} \right) \right) \qquad \text{as } |x| \longrightarrow \infty. \tag{1.6}$$

We call $u_\infty(D; d)(x)$ or $u_\infty(D, \lambda; d)(x)$ the far field pattern. Throughout this paper, we fix $k \in \mathbb{R}$. In the direct problem, we are required to find $u_\infty(D; d)(\cdot)$ for a given domain D. Now our main interest is in the inverse problem of determining D from the far field pattern $u_\infty(D; d)(\cdot)$ which can be observed. The most important theoretical issue is

Uniqueness. Let $S \subset S^1$ be a prescribed subset. Then is the correspondence

$$\{u_\infty(D; d)\}_{d \in S} \longleftrightarrow D$$

one to one?

In the sound-soft case, there are several uniqueness results within bounded C^2-domains D's:

(1) Schiffer proved that if $u_\infty(D_1; d) = u_\infty(D_2; d)$ on S^1 for $d \in S$: an infinite set, then $D_1 = D_2$ (e.g., [5, 13]).

(2) Colton and Sleeman in [6] proved that: Let D_1, D_2 be included in $\{x; |x| < R\}$ with $R > 0$. Then there exists $N(R) \in \mathbb{N}$ such that if S has $N(R)$ elements and $u_\infty(D_1; d) = u_\infty(D_2; d)$ on S^1 for all $d \in S$, then $D_1 = D_2$.

One can estimate $N(R)$, and if $kR < \pi$, then we can take $N(R) = 1$. That is, a single incident wave yields the uniqueness; let $D_1, D_2 \subset \{x \in \mathbb{R}^2; |x| < R\}$ and let $kR < \pi$. Then $u_\infty(D_1; d) = u_\infty(D_2; d)$ on S^1 for a single d implies $D_1 = D_2$.

As for stability results, see [11, Chapter 6], and [16, Chapter 1] contains a survey of theoretical results.

In the case (1.5), the uniqueness with a finite number of incident waves is not known, and see Theorem 5.6 in [5] for the uniqueness with all the incident directions. Even in the sound-soft case (1.4), the general uniqueness with a single d seems still an open problem. On the other hand, especially in the sound-soft case, it seems known as folklore that we can determine a convex polygon with an arbitrarily given single incident wave.

The main purpose of this paper is to prove the uniqueness in all the cases of sound-soft, sound-hard obstacles and the impedance boundary condition within convex polygonal obstacles, by means of at most two incident waves (a single incident wave in the sound-soft case and the sound-hard case). More precisely, in the case of impedance boundary condition, we can prove the uniqueness of λ in (1.5) as well as of an obstacle. Our proof is based essentially on the analyticity of the solution $u^S(D; d)$ to the direct problem, and so our argument is applicable to D's of more general geometry in higher dimensions and to other scattering problems such as electromagnetic scattering. See e.g., [3] for the determination of non-convex polygonal obstacles in the sound-soft and sound-hard cases. However, for showing the essence, in this paper, we will exclusively discuss the two dimensional inverse acoustic scattering problem within convex polygonal D's.

The rest of this paper is organized as follows: Section 2. Main result in the sound-soft case and the sound-hard case. Section 3. Main result in the case of impedance boundary condition. Section 4. Concluding remarks.

§ 2. Main Result in the Sound-Soft Case and the Sound-Hard Case

Let $u(D;d)(x)$ be a weak solution in $H^1_{\rm loc}(\mathbb{R}^2\setminus\overline{D})$ to (1.1)–(1.4) or (1.1)–(1.3) and (1.5) with $\lambda=0$ for a given $d\in S^1$ and let $u_\infty(D;d)$ be the corresponding far field pattern. We note that $u(D;d)$ is smooth in any compact subset in $\mathbb{R}^2\setminus\overline{D}$ (e.g., [10]). By a polygon, we mean an interior bounded by a piecewise linear closed curve.

We are ready to state our first main result.

Theorem 2.1. *Let $k\in\mathbb{R}$ and $d\in S^1$ be arbitrarily chosen. We assume that $k\neq 0$ in the case* (1.5) *with $\lambda=0$. Let $D_1, D_2\subset\mathbb{R}^2$ be bounded convex polygons. If $u_\infty(D_1;d)(x)=u_\infty(D_2;d)(x)$ for any $x\in S^1$, then $D_1=D_2$.*

Remark 2.1. Let $k=0$ in the case (1.5) with $\lambda=0$. Then within $H^1(\mathbb{R}^2\setminus\overline{D})$, there exists a unique solution $u^S(D;d)(x)=0$ for any D and any $d\in S^1$. Therefore we cannot expect the uniqueness in our inverse problem.

By the theorem, in the sound-soft case and the sound-hard case, an arbitrarily chosen single incident wave guarantees the uniqueness within convex polygonal obstacles. As an admissible set of unknown obstacles D's, all the polygonal domains are restrictive, but in some practical cases, piecewise linear shapes in the scattering may be reasonable. See e.g., [17] in the case of the scattering by periodic structures, and [1] for the numerics in the case of the inverse acoustic scattering with polygons. Moreover, as is seen from the proof below, we can apply our argument to D's whose boundaries are piecewise analytic and every analytic piece can be extended analytically to ∞, and we can prove the uniqueness in determining such obstacles.

Our proof is based on a classical argument for the inverse scattering theory by Rellich's lemma (e.g., Lemma 2.11 in [5]) and the analytic extension of solutions to Helmholtz equation (1.1).

Proof of Theorem 2.1. We set $u_j=u(D_j;d)$, $j=1,2$. For sufficiently large $R>0$,

$$u_1(x)-u_2(x)=|x|^{-\frac{1}{2}}e^{ik|x|}O\left(\frac{1}{|x|}\right)\qquad \text{if } \ |x|>R$$

by (1.6) and $u_\infty(D_1;d)\equiv u_\infty(D_1;d)$. Therefore $\int_{|x|=r}|u_1(x)-u_2(x)|^2ds=O\left(\frac{1}{r}\right)$ if $r>R$, which implies that $u_1(x)=u_2(x)$ if $|x|>R$ by Rellich's lemma (e.g., Lemma 2.11 in [5, p.32]). Since u_1-u_2 satisfies the Helmholtz equation in $\mathbb{R}^2\setminus\overline{(D_1\cup\ D_2)}$, the unique continuation implies

$$u_1(x)=u_2(x),\qquad x\in\mathbb{R}^2\setminus\overline{(D_1\cup\ D_2)}. \tag{2.1}$$

Assume that $D_1\neq D_2$. Then there exists a vertex O of ∂D_1 such that $O\in\mathbb{R}^2\setminus\overline{D}_2$ or a vertex O' of ∂D_2 such that $O'\in\mathbb{R}^2\setminus\overline{D}_1$. Without loss of generality, we may assume the former case. Moreover we can choose points P and Q on edges of a convex polygon D_1 such that the segments OP and OQ are on the two edges and that $OP, OQ\subset\mathbb{R}^2\setminus\overline{D}_2$. Moreover, since D_2 is convex, we can extend the segments OP and OQ (at least over respective one end point) to ∞ in $\mathbb{R}^2 \setminus \overline{D}_2$.

We will discuss only the sound-hard case, because the sound-soft case is treated in the same way. Therefore, by (1.5) with $\lambda = 0$, we have

$$\frac{\partial u_1}{\partial \nu}(x) = 0, \qquad x \in OP \cup OQ. \tag{2.2}$$

Hence in the trace sense, from (2.1) and (2.2) it follows that $\frac{\partial u_2}{\partial \nu}(x) = 0$, $x \in OP \cup OQ$. By ν_1 and ν_2 we denote the unit outward normal vectors respectively to $OP \subset \partial D_1$ and $OQ \subset \partial D_1$, and by τ_1 and τ_2 we denote unit tangential vectors on OP and OQ respectively. Since OP and OQ are two edges of D_1, we see that ν_1 and ν_2 are linearly independent. Without loss of generality, we may take $O = (0,0)$ and

$$\{s\tau_1;\, s < 0\},\ \{s\tau_2;\, s < 0\} \subset \mathbb{R}^2 \setminus \overline{D}_2. \tag{2.3}$$

Consequently we have

$$\nabla u_2(s\tau_1) \cdot \nu_1 = \nabla u_2(s\tau_2) \cdot \nu_2 = 0 \tag{2.4}$$

for $0 < s \le 1$. Since u_2 is real-analytic in $x \in \mathbb{R}^2 \setminus \overline{D}_2$ (e.g., [5]), in terms of (2.3), we have (2.4) for all $s \in (-\infty, 1)$. By Green's formula (e.g., Theorem 2.4 in [5]), we can prove

$$\lim_{|x| \to \infty} |\nabla u^S(D; d)(x)| = 0 \tag{2.5}$$

in a way similar to the proof of (1.6) (e.g., Theorem 2.5 in [5]). Therefore (2.4) and (2.5) yield

$$\lim_{s \to -\infty} |ik(d \cdot \nu_1) e^{iks(\tau_1 \cdot d)}| = \lim_{s \to -\infty} |ik(d \cdot \nu_2) e^{iks(\tau_2 \cdot d)}| = 0.$$

Since $|e^{iks(\tau_1 \cdot d)}| = |e^{iks(\tau_2 \cdot d)}| = 1$ and $k \ne 0$, we have $(d \cdot \nu_1) = (d \cdot \nu_2) = 0$. Because ν_1 and ν_2 are linearly independent, we have $d = 0$, which contradicts $|d| = 1$. Thus the proof of Theorem 2.1 is complete.

§ 3. Main Result for the Impedance Boundary Condition

In this section, we consider the case (1.1)–(1.3) and (1.5) with $\lambda > 0$. We can state our uniqueness result.

Theorem 3.1. (the impedance boundary condition) *Let $k \in \mathbb{R}$ and $\lambda_1, \lambda_2 > 0$ in (1.5), and let $d^1, d^2 \in S^1$ be distinct. Assume that $D_1, D_2 \subset \mathbb{R}^2$ are bounded convex polygons. If $u_\infty(D_1, \lambda_1; d^j)(x) = u_\infty(D_2, \lambda_2; d^j)(x)$, $j = 1, 2$, for any $|x| = 1$, then $D_1 = D_2$ and $\lambda_1 = \lambda_2$.*

Proof. We set $u_j^\ell(x) = u(D_j, \lambda_j; d^\ell)(x)$ for $j = 1, 2$ and $\ell = 1, 2$. Assume that $D_1 \ne D_2$. Similarly to the proof of Theorem 2.1, we may choose a vertex O of ∂D_1 and points P, Q such that $OP, OQ \subset \mathbb{R}^2 \setminus \overline{D}_2$ and

$$\begin{aligned} &\nabla u_1^\ell(x) \cdot \nu_1 + i\lambda_1 u_1^\ell(x) = 0, \qquad \ell = 1, 2,\ x \in OP, \\ &\nabla u_1^\ell(x) \cdot \nu_2 + i\lambda_1 u_1^\ell(x) = 0, \qquad \ell = 1, 2,\ x \in OQ. \end{aligned} \tag{3.1}$$

Here ν_1 and ν_2 are the unit outward normal vectors to $OP \subset \partial D_1$ and $OQ \subset \partial D_1$ respectively, and τ_1 and τ_2 are unit tangential vectors on OP and OQ respectively. Similarly in obtaining (2.4) from (2.1), by (3.1) we can prove

$$\begin{aligned} &\nabla u_2^\ell(s\tau_1) \cdot \nu_1 + i\lambda_1 u_2^\ell(s\tau_1) = 0, \\ &\nabla u_2^\ell(s\tau_2) \cdot \nu_2 + i\lambda_1 u_2^\ell(s\tau_2) = 0, \qquad \ell = 1, 2,\ s \le 1. \end{aligned} \tag{3.2}$$

Here we have chosen $O = (0,0)$, and assume that (2.3) holds true.

Noting (1.6), (2.5) and

$$\nabla u_2^\ell(s\tau_j) = \nabla u^S(D_2, \lambda_2; d^\ell)(s\tau_j) + ikd^\ell e^{iks(\tau_j \cdot d^\ell)}, \qquad j = 1,2,\ \ell = 1,2,$$

we see that Equation (3.2) implies

$$\lim_{s\to-\infty}(k(d^\ell \cdot \nu_1) + \lambda_1)e^{iks(\tau_1\cdot d^\ell)} = 0,$$
$$\lim_{s\to-\infty}(k(d^\ell \cdot \nu_2) + \lambda_1)e^{iks(\tau_2\cdot d^\ell)} = 0, \qquad \ell = 1,2.$$

Therefore we have

$$k(d^\ell \cdot \nu_j) + \lambda_1 = 0, \qquad j = 1,2,\ \ell = 1,2. \tag{3.3}$$

By $\lambda_1 > 0$, we see from (3.3) that $k \neq 0$, so that $(d^1 \cdot \nu_j) = (d^2 \cdot \nu_j)$, $j = 1,2$. Since ν_1 and ν_2 are linearly independent, we have $d^1 = d^2$, which contradicts the assumption that $d^1 \neq d^2$. Therefore we see that $D_1 = D_2$.

Finally we have to prove that $\lambda_1 = \lambda_2$. By (2.1), we have $u_1 = u_2$ and $\frac{\partial u_1}{\partial\nu} = \frac{\partial u_2}{\partial\nu}$ on $\partial D_1 = \partial D_2$ in the sense of traces. Hence, by (1.5), we obtain $(\lambda_1 - \lambda_2)u_1 = 0$ almost everywhere on ∂D_1. Assume contrarily that $\lambda_1 \neq \lambda_2$. Then we have $u_1 = \frac{\partial u_1}{\partial\nu} = 0$ on ∂D_1. Let $P \in \partial D_1$ be a point such that ∂D_1 is flat in a neighbourhood $\mathcal{U}$ of P. Hence $u_1 \in H^1((\mathbb{R}^2 \setminus \overline{D}_1) \cap \mathcal{U})$ satisfies

$$\Delta u_1 + k^2 u_1 = 0, \qquad \text{in } (\mathbb{R}^2 \setminus \overline{D}_1) \cap \mathcal{U}$$
$$u_1 = \frac{\partial u_1}{\partial\nu} = 0, \qquad \text{on } \partial D_1 \cap \mathcal{U}.$$

Therefore the unique continuation yields $u_1 = 0$ in $(\mathbb{R}^2 \setminus \overline{D}_1) \cap \mathcal{U}$. For example, see the proof of Theorem 3.3.1 in [11, pp.53–54]. Note that for the unique continuation, it is sufficient to consider H^1-solutions. By taking into consideration that u_1 is smooth in any compact subset in $\mathbb{R}^2 \setminus \overline{D}_1$, the classical unique continuation implies that $u_1 = 0$ in $\mathbb{R}^2 \setminus \overline{D}_1$. This contradicts the behaviour of the total field u_1 at ∞. Consequently $\lambda_1 = \lambda_2$. Thus the proof of Theorem 3.1 is complete.

§ 4. Concluding Remarks

(1) In this paper, we exclusively consider convex polygons. In the cases (1.4) and (1.5), provided that non-convex polygons under consideration satisfy some geometric constraints, the uniqueness is proved (see [3]). In a forthcoming paper, for the case (1.5), we will prove the uniqueness in identifying non-convex polygonal obstacles under some geometric constraints by means of at most three incident waves.

(2) The numerical reconstruction for the inverse scattering problem is extremely important and we can refer to [4, 5, 9, 12, 16] for example. The uniqueness in our inverse problem is useful for that purpose. For example, if we reconstruct a convex polygonal domain by an optimization method, then our uniqueness results guarantee that we need not take a subsequence of the minimizing sequence for the cost functional and that any minimizing sequence itself converges to a uniquely determined obstacle (see e.g., [7, p.1316]).

(3) Our argument is essentially based on the analyticity of solutions, so that it is applicable to other inverse scattering problems such as electromagnetic scattering provided that the coefficients of the governing equations are real-analytic. The three dimensional case

is treated similarly. A similar argument is used for an inverse problem in periodic diffractive optics (see [8]).

(4) Some related uniqueness results within the class of balls can be found in [14], [18] etc.

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