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# ZERO SET OF SOBOLEV FUNCTIONS WITH NEGATIVE POWER OF INTEGRABILITY

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**Abstract**

Here the authors are interested in the zero set of Sobolev functions and functions of bounded variation with negative power of integrability. The main result is a general Hausdorff dimension estimate on the size of zero set. The research is motivated by the model on van der waal force driven thin film, which is a singular elliptic equation. After obtaining some basic regularity result, the authors get an estimate on the size of singular set; such set corresponds to the thin film rupture set in the thin film model.

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## § 1 . Introduction

Let $\Omega \subset \mathbf{R}^n$ $(n \geq 1)$ be an open set, $p \geq 1$. We consider functions $u \in W^{1,p}(\Omega)$, such that, for some $\alpha > 0$,

$$\int_\Omega |u|^{-\alpha} < \infty. \tag{1.1}$$

When $p = 1$, it is also natural to consider $u \in BV(\Omega)$ which satisfies (1.1).

For any $u \in W^{1,p}(\Omega)$, $p \geq 1$ or $u \in BV(\Omega)$, we define its zero set by

$$\Sigma = \Big\{ x \in \Omega : \lim_{r \to 0^+} \frac{1}{|B_r(x)|} \int_{B_r(x)} |u| \text{ exists, and is equal to } 0 \Big\}. \tag{1.2}$$

Our main theorem is the following:

**Theorem 1.1.** *$\mathcal{H}^s(\Sigma) = 0$, where* $s = \max\Big\{0, n - p + \dfrac{p^2}{p+\alpha}\Big\}$.

We note that, when $n - p + \frac{p^2}{p+\alpha} \leq 0$, one has in particular that $\Sigma$ is empty. This latter fact follows also from the Sobolev imbedding theorem: $u \in W^{1,p}(\Omega)$, $p > n$, implies $u \in C^\beta(\Omega)$, $\beta = 1 - \frac{n}{p}$. Indeed, if $u(x_0) = 0$ for some $x_0 \in \Omega$, then

$$|u(x)| \equiv |u(x) - u(x_0)| \leq c(u)\,|x - x_0|^\beta .$$

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Thus

$$\int_\Omega |u|^{-\alpha}(x)\,dx \geq c_2(u,\alpha)\int_\Omega \frac{1}{|x-x_0|^{\alpha\beta}} = +\infty$$

whenever $\alpha\beta \geq n$. That is equivalent to $n-p+\frac{p^2}{p+\alpha} \leq 0$. In other words, under the additional assumption $\int_\Omega |u|^{-\alpha} < \infty$, one would have $u$ never vanish if $n-p+\frac{p^2}{p+\alpha} \leq 0$.

We also note that, for a $W^{1,p}(\Omega)$ function $u$, the set

$$\Sigma^* = \left\{x \in \Omega : \lim_{r\to 0^+} \frac{1}{|B_r(x)|}\int_{B_r(x)} |u| \text{ does not exist, or is equal to } \infty\right\} \tag{1.3}$$

is of Hausdorff dimension at most $n-p$, by a theorem of Federer-Ziemer [3]. And for functions of bounded variation, we have $H^{n-1}(\Sigma^*) = 0$, see Section 5.9 of [1] for more information on fine properties of BV functions. Hence our result concerning $\Sigma$, the Lebesgue set of $u$ of the value zero makes sense because $s > n-p$.

In order to show Theorem 1.1, we will need a Poincaré type inequality which will be proven in the next section.

We will concentrate our proof on the cases that $n \geq 2$ and $p > 1$. When $n = 1$, any function in $W^{1,p}(\Omega)$ is Hölder continuous for $p > 1$ and absolutely continuous when $p = 1$. So it is easy to check that our theorem is valid in this case. In the case $p = 1$, since $W^{1,1}(\Omega) \subset BV(\Omega)$, the theorem can be proved in the same manner as in the case $p > 1$, with the help of Theorem 2.2.

Our motivation for studying the zero set of general Sobolev function comes from considerations on the so-called rupture set of thin films, see [5]. Indeed, we consider a nonnegative solution $u$ of

$$\triangle u - \frac{1}{u^\alpha} + h(x) = 0 \tag{1.4}$$

in $\Omega \subset R^n$, where $\alpha > 1$, and $h$ is a smooth function in $\Omega$. The value $u$ represents the thickness of the thin films, and the zero set of $u$ represents the ruptures. Naturally one is interested in how big can such rupture sets be.

We say that $u \geq 0$ is a finite energy solution of (1.4), if $u$ is nonnegative and continuous, satisfying (1.4) in $\{x \in \Omega : u(x) > 0\}$, and such that

$$\int_\Omega \left[|\nabla u|^2 - \frac{1}{\alpha-1}|u|^{-\alpha+1}\right]dx$$

is of finite value.

Applying Theorem 1.1, we have

**Corollary 1.1.** *$\mathcal{H}^{\mu_1}(\Sigma_1) = 0$, where $\Sigma_1 = \{x \in \Omega : u(x) = 0\}$, and $\mu_1 = n-2+\dfrac{4}{\alpha+1}$.*

Alternatively, we say $u \geq 0$ is a weak solution of (1.4) in $\Omega$, if $u \in L^1(\Omega)$, $u^{-\alpha} \in L^1(\Omega)$, and (1.4) holds in the sense of distribution. Then we have the following:

**Theorem 1.2.** *$u \in H^1_{\rm loc}(\Omega)$. Moreover, for any $K \subset\subset \Omega$, there is a constant $C > 0$, such that for any $x \in K$, $0 < r < \dfrac{1}{2}\mathrm{dist}(K, \partial\Omega)$, we have*

$$\int_{B_r(x)} |\nabla u|^2(y)\,dy \leq Cr^{n-2}.$$

*Furthermore, $\mathcal{H}^\mu(\Sigma) = 0$ for $\mu = n-2+\dfrac{4}{\alpha+2}$, where $\mu$ is defined in (1.2).*

The above estimates on the zero set of weak solutions of (1.4 ) is probably the first of its kind. However, we expect better estimates may be valid. The reason is that very little information of $u$ being a weak solution of (1.4) is used in the proof of Theorem 1.2.

The paper is organized as follows: We will prove several Poincaré type inequalities in Section 2. Then we prove Theorem 1.1 in Section 3, and then discuss its application to (1.4) in the last section.

## § 2 . Poincaré Type Inequality

If $p > n$ and $u \in W^{1,p}(B_R)$, then $u$ is Hölder continuous, and hence we have the following classical Poincaré lemma:

**Proposition 2.1.** *Let $p > n$, and $B_R$ be any ball in $\mathbf{R}^n$ with radius $R$. Then for any $u \in W^{1,p}(B_R)$ such that $u(x) = 0$ for some point $x \in B_R$, we have*

$$\int_{B_R} u^p \leq c\,(n,p)\,R^p \int_{B_R} |\nabla u|^p\,.$$

If $1 < p \leq n$, then $u(x) = 0$ for some point $x \in B_R$ is not well defined. However, we still have Poincaré inequality if the zero set is large enough.

**Theorem 2.1.** *Let $n \geq 2$, $1 < p \leq n$, and $n - p < s \leq n$, let $B_R$ be any ball in $\mathbf{R}^n$ with radius $R$, and $T \subset B_R$ be a $\mathcal{H}^s$-measurable set, such that*

$$\mathcal{H}^s\,(T) \geq \theta_1 R^s,$$

*and that for any $x \in \mathbf{R}^n$, and $r > 0$,*

$$\mathcal{H}^s\,(T \cap B_r\,(x)) \leq \theta_2 r^s$$

*holds. Then for any $u \in W^{1,p}(B_R)$ such that $T \subset \Sigma$, where $\Sigma$ is defined in* (1.2)*, we have*

$$\int_{B_R} u^p \leq c\,(n,p,s)\,\frac{\theta_2^p}{\theta_1^p} R^p \int_{B_R} |\nabla u|^p\,.$$

**Proof.** After a scaling, we can always assume $R = 1$.

**Step I.** Let $\mu = \mathcal{H}^s\lfloor_T$. Then $\mu$ is a Radon measure supported on $B_1$, such that

$$\mu(B_1) \geq \theta_1,$$

and for any $B_r(x) \subset \mathbf{R}^n$,

$$\mu(B_r\,(x)) \leq \theta_2 r^s.$$

By applying Theorem 4.7.5 in [6], we have $\mu \in (W^{1,p}(\mathbf{R}^n))^*$. Furthermore,

$$\begin{aligned}\|\mu\|_{(W^{1,p}(\mathbf{R}^n))^*} &\leq c(n,p)\Big(\int_{\mathbf{R}^n}\int_0^\infty \Big(\frac{\mu(B_r(y))}{r^{n-p}}\Big)^{\frac{1}{p-1}}\frac{dr}{r}d\mu(y)\Big)^{\frac{p-1}{p}}\\ &\leq c(n,p,s)\theta_2,\end{aligned}$$

where in the last inequality, we used the fact that $\mu$ is supported on $B_1$ and $\mu(B_r(y)) \leq \theta_2\min\{1, r^s\}$. Since every function $u \in W^{1,p}(B_1)$ can be extended to $\tilde{u}$ defined on whole $R^n$, so that

$$||\tilde{u}||_{W^{1,p}(R^n)} \leq c(n,p)\|u\|_{W^{1,p}(B_1)},$$

we can define for any $u \in W^{1,p}(B_1)$, $\mu(u) = \mu(\tilde{u})$, where $\tilde{u} \in W^{1,p}(R^n)$ is an extension of $u$. Thus $\mu$ can be viewed as a member of $(W^{1,p}(B_1))^*$, with

$$\|\mu\|_{(W^{1,p}(B_1))^*} \leq c(n,p,s)\theta_2.$$

**Step II.** Applying Lemma 4.1.4 in [6], we have

$$\begin{aligned}\int_{B_1} |u - \mu(u)|^p &\leq c\,(n,p,s)\,\frac{\|\mu\|^p_{(W^{1,p}(B_1)^*}}{(\mu(1))^p}\int_{B_1}|\nabla u|^p \\ &\leq c\,(n,p,s)\,\frac{\theta_2^p}{\theta_1^p}\int_{B_1}|\nabla u|^p\,.\end{aligned}$$

**Step III.** Finally, we need to show $\mu(u) = 0$ under our assumption. To see this, let $\tilde{u}$ be the extension of $u$, and $\tilde{u}^M$ be the cutoff function of $\tilde{u}$ so that $|\tilde{u}^M| \leq M$, then $\tilde{u}^M \to \tilde{u}$ in $W^{1,p}(\mathbf{R}^n)$ as $M \to \infty$. Let $\tilde{u}^M_\varepsilon$ be the standard mollification of $\tilde{u}^M$, then $\tilde{u}^M_\varepsilon \to 0$, $\mu$-a.e. as $\varepsilon \to 0^+$. Hence the Lebesgue dominated convergence theorem implies

$$\int_{R^n} \tilde{u}^M_\varepsilon d\mu \to 0$$

as $\varepsilon \to 0$. On the other hand, since $\tilde{u}^M_\varepsilon \to \tilde{u}^M$ in $W^{1,p}(\mathbf{R}^n)$, and $\mu$ is a bounded operator on $W^{1,p}(\mathbf{R}^n)$, we have as $\varepsilon \to 0$,

$$\int_{R^n} \tilde{u}^M_\varepsilon d\mu \to \mu(\tilde{u}^M).$$

Therefore, one has $\mu(\tilde{u}^M) = 0$. Letting $M \to \infty$, we deduce $\mu(u) = \mu(\tilde{u}) = 0$.

When $u$ is a function of bounded variation, we have a similar Poincaré type inequality:

**Theorem 2.2.** *Let $n \geq 2$, $n-1 \leq s \leq n$, and $B_R$ be any ball in $\mathbf{R}^n$ with radius $R$, and $T \subset B_R$ be a $\mathcal{H}^s$-measurable set, such that*

$$\mathcal{H}^s\,(T) \geq \theta_1 R^s,$$

*and that for any $x \in \mathbf{R}^n$, and $r > 0$,*

$$\mathcal{H}^s\,(T \cap B_r\,(x)) \leq \theta_2 r^s$$

*holds. Then for any $u \in BV\,(B_R)$ such that $T \subset \Sigma$, where $\Sigma$ is defined in (1.2), we have*

$$\int_{B_R} |u| \leq c(n) R\Big(\int_{B_R} u^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \leq c\,(n)\,\frac{\theta_2}{\theta_1} R \int_{B_R} |\nabla u|\,.$$

**Proof.** We assume $R = 1$. Let $\mu = \mathcal{H}^s\lfloor_T$. Then $\mu$ is a Radon measure supported on $B_1$, such that

$$\mu(B_1) \geq \theta_1,$$

and for any $B_r(x) \subset \mathbf{R}^n$,

$$\mu(B_r\,(x)) \leq \theta_2 r^s.$$

Since $s \geq n-1$, we have for any $B_r(x) \subset \mathbf{R}^n$,

$$\mu(B_r\,(x)) \leq \theta_2 r^{n-1}$$

when $r \leq 1$, and when $r > 1$,

$$\mu(B_r(x)) \leq \mu(B_1(0)) \leq \theta_2 \leq \theta_2 r^{n-1}.$$

So in either case, we always have

$$\mu(B_r(x)) \leq \theta_2 r^{n-1}.$$

The theorem then follows from Theorem 5.12.7 in [6].

**Corollary 2.1.** *Let $p \geq 1$, $s = \max\left\{n - p + \dfrac{p^2}{p+\alpha}, 0\right\}$ for some $\alpha > 0$ and consider $u \in W^{1,p}(B_R)$ or $u \in BV(B_R)$ with $R \leq 1$. Suppose either $p > n$ and $u(x) = 0$ for some $x \in B_R$, or $1 \leq p \leq n$ and there exists $T$ such that $u$ satisfies conditions in Theorem 2.1 or 2.2. Then under the assumption $\int_{B_R} |u|^{-\alpha} < \infty$, we have*

$$\int_{B_R} |\nabla u|^p + \int_{B_R} |u|^{-\alpha} \geq cR^s,$$

*where $c = c(n,p)$ if $p > n$ and $c = c(n,p,\alpha,\theta_1,\theta_2)$ if $1 \leq p \leq n$.*

**Proof.** Applying the Poincaré inequalities we just proved, we have

$$\begin{aligned}\int_{B_R} |\nabla u|^p + \int_{B_R} |u|^{-\alpha} &\geq cR^{-p} \int_{B_R} |u|^p + \int_{B_R} |u|^{-\alpha} \\ &\geq cR^{n-p+\frac{p^2}{p+\alpha}} \\ &\geq cR^s.\end{aligned}$$

Here we have used Young's inequality.

## § 3. Hausdorff Dimension Estimate for Zero Set

**Proof of Theorem 1.1.** We prove it by contradiction. Suppose $H^s(\Sigma) > 0$ (possibly with infinite measure). Then since $\Sigma$ is a Souslin set, Theorem 5.6 and its proof in [2] say that, there is a closed subset $T \subset \Sigma$, with $0 < H^s(T) < \infty$, and for some constant $\theta > 0$,

$$H^s(T \cap B_r(x)) \leq \theta r^s$$

holds for any $x \in R^n$, $r > 0$.

For such $T$, the basic density lemma says that for $H^s$-a.e. $x \in T$,

$$\frac{1}{2^s} \leq \limsup_{r\to 0} \frac{H^s(B_r(x) \cap T)}{\alpha(s)\, r^s} \leq 1.$$

Let

$$T^* = \left\{x \in T : \limsup_{r\to 0} \frac{H^s(B_r(x) \cap T)}{\alpha(s)\, r^s} \geq \frac{1}{2^s}\right\}.$$

Then for any $\delta > 0$ and for any $U$ open, such that $T^* \subset U$,

$$\left\{B_r(x) : x \in T^*,\ 0 < r < \frac{1}{2}\delta,\ B_r(x) \subset U \text{ and } \frac{H^s(B_r(x) \cap T)}{\alpha(s)\, r^s} \geq \frac{1}{2^{s+1}}\right\}$$

is a fine covering of $T^*$. Hence, by Vitali covering lemma, there is a pairwise disjoint subcollection $\{B_{r_k}(x_k)\}_{k=1}^{\infty}$, such that $T^* \subset \bigcup_{k=1}^{\infty} B_{5r_k}(x_k)$. Hence, applying Corollary 2.1, we have

$$\begin{aligned} H_{5\delta}^s(T^*) &\le \sum_{k=1}^{\infty} \alpha(s)(5r_k)^s \\ &\le c(n,p,s,\theta) \sum_{k=1}^{\infty} \int_{B_{r_k}(x_k)} [|\nabla u|^p + |u|^{-\alpha}] \\ &\le c(n,p,s,\theta) \int_U [|\nabla u|^p + |u|^{-\alpha}]. \end{aligned}$$

Since $H^s(T^*) < \infty$, we can choose $U$ with arbitrary small $H^n$-measure so that the right hand side of the inequality can be arbitrary small. Thus we would have $H_{5\delta}^s(T^*) = 0$. Letting $\delta \to 0$, we conclude $H^s(T^*) = 0$, hence $H^s(T) = 0$, which gives the contradiction.

## § 4. Rupture Set of Thin Film Model

**Proof of Corollary 1.1.** Since the energy is finite, we have $u \in W^{1,2}(\Omega)$ and $u^{-\alpha+1} \in L^1(\Omega)$, hence the result follows from Theorem 1.1.

Now we turn to the proof of Theorem 1.2. Actually, we would like to prove the theorem in a more general setting. Let $\Omega \subset R^n$, $f \in L^1(\Omega)$, $f \ge 0$ in $\Omega$ and $g \in L^q(\Omega)$ for some $q \ge \frac{n}{2}$. We consider nonnegative solutions of

$$\triangle u = f + g \qquad \text{in} \ \ \Omega. \tag{4.1}$$

Since $f + g \in L^1(\Omega)$, classical elliptic theory implies $u \in W_{\text{loc}}^{1,p}(\Omega)$ for any $1 \le p < \frac{n}{n-1}$. In our setting, we expect better results. First, we have

**Lemma 4.1.** *Let $f$, $g$, $u$ be as above with $q > \frac{n}{2}$. For any $B_{2R} \subset \Omega$, and for any $p > 1$ such that $\|u\|_{L^p(B_{2R})} < \infty$, we have*

$$\sup_{B_R} u \le c(n,p,q)\,(R^{-\frac{n}{p}} \|u\|_{L^p(B_{2R})} + R^{2-\frac{n}{q}} \|g\|_{L^q(B_{2R})}).$$

**Proof.** This follows from the fact that $u$ is a subsolution of $\triangle u = g$. We could apply Theorem 8.17 in [4] directly if we have $u \in H_{\text{loc}}^1(\Omega)$. So naturally, we consider $u_\varepsilon$, the standard mollification of $u$, then we have

$$\triangle u_\varepsilon = f_\varepsilon + g_\varepsilon.$$

Now $u_\varepsilon$ is smooth, so we can apply Theorem 8.17 in [4] to the mollified equation, and get

$$\sup_{B_R} u_\varepsilon \le c(n,p,q)\,(R^{-\frac{n}{p}} \|u_\varepsilon\|_{L^p(B_{2R})} + R^{2-\frac{n}{q}} \|g_\varepsilon\|_{L^q(B_{2R})}).$$

The lemma follows by letting $\varepsilon \to 0^+$.

Next, we need the following technical lemma.

**Lemma 4.2.** *Let $f$, $g$ and $u$ be as above and $q = \frac{n}{2}$. Then for any $x \in \Omega$, $0 < r \le \min\{1, \operatorname{dist}(x, \partial\Omega)\}$, we have*

$$r^{2-n} \int_{B_{\frac{r}{2}}(x)} f(y)\, dy \le c\Big(\sup_{B_r} u + \|g\|_{L^{\frac{n}{2}}(B_r)}\Big),$$

*where $c = c(n)$.*

**Proof.** For any $x \in \Omega$, $0 < r \le \min\{1, \operatorname{dist}(x, \partial\Omega)\}$, we have $B_r(x) \subset \Omega$. Now let $\varphi$ be the first eigenfunction of the Laplacian on the unit ball of $\mathbf{R}^n$:

$$\begin{cases} -\triangle\varphi = \lambda_1\varphi(x), & \text{in} \quad B_1(0), \\ \varphi = 0, & \text{on} \quad \partial B_1(0), \\ \varphi > 0, & \text{in} \quad B_1(0), \end{cases}$$

and define $\varphi_r(x) = \varphi(\frac{x}{r})$. Now let $\eta_m \in C_0^\infty(B_r(x))$ be a smooth cutoff function such that $\eta_m = 1$ in $B_{\frac{(m-1)r}{m}}(x)$ and $|\nabla\eta_m| \le \frac{2m}{r}$. Using $\eta_m\varphi_r$ as a test function, we have

$$\begin{aligned} &- \int_{B_r(x)} \varphi_r \nabla u \cdot \nabla\eta_m + \int_{B_r(x)} u\nabla\varphi_r \cdot \nabla\eta_m \\ &= \int_{B_r(x)} u\triangle\varphi_r\eta_m + \int_{B_r(x)} f\varphi_r\eta_m + \int_{B_r(x)} g\varphi_r\eta_m. \end{aligned}$$

Now let $m \to \infty$, then $\eta_m \to \chi_{B_r(x)}$, and $\nabla\varphi_r \cdot \nabla\eta_m \to \frac{c_0}{r} H^{n-1}\lfloor_{\partial B_r(x)}$, where $c_0 = \frac{\partial\varphi}{\partial r}|_{r=1}$. Hence we deduce

$$\lim_{m\to\infty} \int_{B_r(x)} u\nabla\varphi_r \cdot \nabla\eta_m = \frac{c_0}{r} \int_{\partial B_r(x)} u.$$

On the other hand, since $\varphi_r\nabla\eta_m$ is uniformly bounded in $B_r(x)$ and tends to 0 a.e., we have

$$\lim_{m\to\infty} \int_{B_r(x)} \varphi_r \nabla u \cdot \nabla\eta_m = 0.$$

Combining these limits, we have

$$\begin{aligned} r^{2-n} \int_{B_r(x)} f\varphi_r &= c_0 r^{1-n} \int_{\partial B_r(x)} u - \lambda_1 r^{-n} \int_{B_r(x)} u\varphi_r + r^{2-n} \int_{B_r(x)} g\varphi_r \\ &\le c(n)(\|u\|_{L^\infty(B_r(x))} + r^{2-n}\|g\|_{L^1(B_r(x))}) \\ &\le c\,(n)\,(\|u\|_{L^\infty(B_r(x))} + \|g\|_{L^{\frac{n}{2}}(B_r(x))}). \end{aligned}$$

Now we can present our regularity result.

**Theorem 4.1.** *Let $f$, $g$, $u$ as above with $q > \frac{n}{2}$. Then $u$ is locally bounded. Furthermore, $u \in H^1_{\mathrm{loc}}(\Omega)$, and for any $B_{2R} \subset \Omega$ with $R \le 1$, we have*

$$R^{2-n} \int_{B_{\frac{R}{2}}} |\nabla u|^2\, dx \le c \sup_{B_R} u\Big(\sup_{B_R} u + \|g\|_{L^{\frac{n}{2}}(B_R)}\Big),$$

*where $c = c(n)$.*

**Proof.** Again, we consider the mollified equation

$$\triangle u_\varepsilon = f_\varepsilon + g_\varepsilon.$$

Then we have

$$\triangle u_\varepsilon^2 = 2\left|\nabla u_\varepsilon\right|^2 + 2u_\varepsilon\left(f_\varepsilon + g_\varepsilon\right).$$

Since $u_\varepsilon^2$ is locally bounded, applying Lemma 4.2, we have

$$\begin{aligned} R^{2-n}\int_{B_{\frac{R}{2}}(x)}(\left|\nabla u_\varepsilon\right|^2 + u_\varepsilon f_\varepsilon)dy &\le c\Big(\sup_{B_R} u_\varepsilon^2 + \left\|u_\varepsilon g_\varepsilon\right\|_{L^{\frac{n}{2}}(B_R)}\Big) \\ &\le c\sup_{B_R} u_\varepsilon\Big(\sup_{B_R} u_\varepsilon + \left\|g_\varepsilon\right\|_{L^{\frac{n}{2}}(B_R)}\Big). \end{aligned}$$

The theorem is proved by letting $\varepsilon \to 0$.

**Proof of Theorem 1.2.** Take $f = u^{-\alpha}$ and $g = -h$. Then Theorem 4.1 says $u \in H^1_{\text{loc}}(\Omega)$. Hence, the result follows from Theorem 1.1 since we also have $u^{-\alpha} \in L^1(\Omega)$.

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