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SPECTRAL GAP SOLUTIONS OF THE DISSIPATIVE KIRCHHOFF EQUATION**

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Abstract

The author proves that for initial data in a set $S \subset (H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$, unbounded in $H_0^1(\Omega) \times L^2(\Omega)$, the solutions of the Cauchy-Dirichlet problem for the dissipative Kirchhoff equation

$$\partial_t^2 u - \Big(\nu + L\int_\Omega |\bigtriangledown_x u|^2 dx\Big) \bigtriangleup_x u + \delta \partial_t u = 0 \qquad (x \in \Omega,\, t > 0),$$

are global in $[0, +\infty)$ and decay exponentially. The functions in S do not satisfy any additional regularity assumption, instead they must satisfy a condition relating their energy with the largest lacuna in their Fourier expansion. The larger is the lacuna the larger is the energy allowed.

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§ 1 . Introduction

This paper is devoted to the question of the global solvability of the dissipative Kirchhoff equation

$$\begin{aligned} \partial_t^2 u - m\Big(\int_\Omega |\bigtriangledown_x u|^2 dx\Big) \bigtriangleup_x u + \delta \partial_t u &= 0 \qquad (x \in \Omega,\, t > 0), \\ u(x,t) &= 0 \qquad (x \in \partial\Omega,\, t > 0), \end{aligned} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbf{R}^n$, m is a positive function of one real variable and $\delta > 0$.

The non dissipative case ($\delta = 0$) has been studied starting from the pioneering paper of S. Bernstein [1] in 1940, who proved in the case $n = 1$ and m an affine function, the local existence for solutions having sufficiently high Sobolev regularity and the global existence for real analytic solutions. From then on, many authors [2–6] have generalized and improved the results of [1], but the global solvability for general $(H^2(\Omega) \cap H_0^1(\Omega)) \times H_0^1(\Omega)$

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(or even C^∞) initial data still remains as the main open question (see the surveys [7, 8] for more mathematical references and physical motivations). In particular, in [2, 4] the global solvability was proved for C^∞ initial data satisfying an additional regularity condition (in fact for quasi-analytic classes in the sense of Denjoy-Carleman). In the same direction of research, in a recent and remarkable paper [9], R. Manfrin proved the global solvability for solutions in the class $\mathcal{B}^2_\Delta \times \mathcal{B}^1_\Delta$.

The functions f belonging to the spaces $\mathcal{B}^s_\Delta$ are well described in terms of their Fourier expansion

$$f = \sum_{k \geq 1} \hat{f}(k)\phi_k,$$

where ϕ_k are the eigenvectors of the Laplace operator with corresponding eigenvalues μ_k^2, and $\hat{f}(k)$ are the Fourier coefficients of f with respect to the basis $\{\phi_k\}$. Following [9], we say that $f \in \mathcal{B}^s_\Delta$ if there exists a sequence $\{\rho_j\}_{j\in\mathbf{N}}$, such that $\rho_j \to +\infty$, and there exists $\eta > 0$, such that

$$\sum_1^\infty \mu_k^{2s}|\hat{f}(k)|^2 < \infty, \qquad \sup_j \sum_{\rho_j < \mu_k < \rho_j^2} \mu_k^{2s}|\hat{f}(k)|^2 \exp(\eta\rho_j^2/\mu_k) < \infty.$$

The most interesting feature of these spaces is that a function in $\mathcal{B}^s_\Delta$ has, in general, only H^s-Sobolev regularity. On the other hand, the behaviour of the Fourier coefficients in the sequence of 'gaps' (ρ_j, ρ_j^2), renders these spaces quasi-analytic classes, in the sense that functions having compact support must vanish everywhere.

For the dissipative equation ($\delta > 0$), in addition to the regularity assumptions mentioned above, global existence theorems are also available for small initial data (see [10–12]). In this case it is assumed that the size of the data in the $H^2 \times H^1$ norm does not exceed a certain bound depending on the parameters of the equation.

This note represents an attempt to introduce some of the ideas of [9] in the study of the dissipative equation. By estimating an energy functional introduced in [9] for $\delta = 0$, we obtain the global existence of solutions under hypotheses which relate certain spectral properties of the initial data to their size in the $H^2 \times H^1$ norm. The typical situation is that of initial data having at least one large lacuna in their spectra, i.e. $\hat{f}(j) = 0$ for $n_0 < j < n_1$ with $\mu_{n_1} \sim \mu_{n_0}^2$. If this is the case, the bound we need to prescribe on the initial energy increases with the ratio $\mu_{n_1}/\mu_{n_0}^2$. Roughly speaking, the combination of the damping with the spectral gap prevents the energy from moving from the low to the high frequencies.

We stress the fact that our initial data are not lacunary in the sense of Hadamard or similar, for by 'gap solutions' we simply mean that the Fourier coefficients vanish on a sufficiently large interval. The precise statement of the result, being rather technical, is postponed to Section 3 (Theorem 3.1 or Proposition 3.2 for a more explicit version). Here we limit ourselves to give a simple example.

Let us consider the equation

$$\begin{gathered} u_{tt} - \Big(1 + L\int_0^\pi u_x^2\,dx\Big)u_{xx} + \delta u_t = 0 \qquad (x \in]0,\pi[\,) \quad (t > 0), \\ u(0,t) = u(\pi,t) = 0, \end{gathered} \tag{1.2}$$

where $L > 0$ and $0 < \delta < 1$. We have

$$\mu_j = j, \quad \phi_j(x) = \sqrt{2/\pi}\sin(jx) \qquad (j \in \mathbf{N}).$$

To avoid needless complications, assume that

$$u(\cdot,0)=0, \quad u_t(\cdot,0)=u_1\in H_0^1(\,]0,\pi[\,),$$

where u_1 is a perturbation of the monochromatic datum $A\phi_1(x)$, i.e, for $k>1$, let

$$u_1(x)=A\phi_1(x)+g_k(x), \quad g_k(x)=\sum_{j=k}^{\infty}\beta_j\phi_j(x).$$

It is well known that in the case $g_k\equiv 0$, Equation (1.2) reduces to a globally solvable ODE of Duffing's type. We ask under which conditions on the H_0^1-perturbed initial data we can ensure the global solvability of Equation (1.2). The usual smallness conditions (see [10, 11]) would require

$$A^2+\|g_k\|^2_{H_0^1}<C,$$

where C is a suitable constant independent of k. Thus, according to this requirement, perturbations are allowed only for a bounded interval of amplitudes A. On the other hand, as an application of Proposition 3.2 (in the form of Remark 3.3), we get that, if

$$\|g_k\|^2_{H_0^1}<A^2+\|g_k\|^2_{L^2}<\frac{\delta\ln 2}{9L}k,$$

then the problem is globally solvable and the solution decays exponentially. Thus, on condition that the spectral gap (in this case the integer k) be large enough, we can perturb any monochromatic initial datum with a H_0^1-function and still have global existence.

This is the plan of the paper. In Section 2 we introduce the abstract setting in which we study the Kirchhoff equation, we recall the exponential decay of the Hamiltonian, and state a global existence theorem for small initial data in $H^{3/2}\times H^{1/2}$. This result extends other known global existence theorems for small data in $H^2\times H^1$. In Section 3 we state and prove our main result (Theorem 3.1). The statement is rather implicit and a set of sufficient conditions is given in Proposition 3.2.

A general warning is in order. Since our main purpose is that of illustrating a method, which hopefully may be improved, we do not provide the result in its full generality. This is the reason why we have chosen to keep things as simple as possible: the function m is assumed to be affine, the damping parameter δ is sufficiently small (non over-damping regime).

§ 2. Preliminary Results

We reformulate the Kirchhoff equation (1.1) as an evolution equation in a Hilbert space. Let V and H be real Hilbert spaces, normed respectively by $\|\cdot\|$ and $|\cdot|, V\subseteq H$. If V' denotes the dual of V, we have then $V\subseteq H\subseteq V'$, in the sense that the duality bracket $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{V\times V'}$ coincides with the inner product $(\cdot,\cdot):=(\cdot,\cdot)_H$ on $H\times V$. Let $A: V\longrightarrow V'$ be a linear bounded operator, symmetric in the sense that

$$\langle Av,w\rangle=\langle Aw,v\rangle \qquad (w\in V,\ v\in V),$$

and such that for some $\eta>0$,

$$\langle Av,v\rangle\geq\eta\|v\|^2 \qquad (v\in V).$$

The operator A turns out to be an isomorphism of V onto V'. Moreover, if we denote $D(A) := \{v \in V : Av \in H\}$, the operator

$$A : D \longrightarrow H$$

turns out to be a self-adjoint positive definite operator on H. Under these hypotheses (see, e.g. [13]), we may consider the power operators A^s of A for $s \in \mathbf{R}$ and the spaces $D(A^s)$ are Hilbert spaces for the scalar product $(A^s u, A^s v)$. In particular, we have $V = D(A^{1/2})$ and

$$\langle Au, u \rangle = |A^{1/2}u|^2 \qquad (u \in V).$$

Finally, since our main tool will be Fourier expansion, we assume that $A^{-1} : H \to H$ is compact. In this case there exists an orthonormal basis $\{\phi_j : j \in \mathbf{N}\}$ of H consisting of eigenvectors of $A^{1/2}$,

$$A^{1/2}\phi_j = \mu_j \phi_j \qquad (j \in \mathbf{N}),$$

where the eigenvalues satisfy

$$0 < \mu_1 \le \mu_2 \le \cdots, \qquad \mu_j \to \infty \quad \text{as} \quad j \to \infty.$$

In this paper we consider the Cauchy problem for the abstract evolution equation (dots denote time derivatives),

$$\ddot{u} + m(|A^{1/2}u|^2)Au + \delta \dot{u} = 0 \qquad (t > 0) \tag{2.1}$$

with initial data

$$u(0) = u_0, \qquad \dot{u}(0) = u_1. \tag{2.2}$$

To keep our arguments as clear as possible and to provide precise computations of the constants involved, we assume that m is a positive affine function, that is,

$$m(r) = \nu + Lr \tag{2.3}$$

with $\nu > 0$ and $L \ge 0$. Actually, in what follows, the assumption (2.3) might be replaced by the weaker condition: $m \in C^2$, positive and $m(\rho)\rho \ge CM(\rho)$ $(C > 0)$, where

$$M(\rho) := \int_0^\rho m(r)dr. \tag{2.4}$$

The starting point of our argument is the following simple remark. Let $a(t)$ be a positive function, and let v be a solution of the linear equation

$$\ddot{v} + a(t)Av + \delta \dot{v} = 0 \qquad (t > 0). \tag{2.5}$$

Since Equation (2.5) may be seen as an infinite system of uncoupled linear oscillators, if for some integer i, we have

$$(v(0), \phi_i) = (\dot{v}(0), \phi_i) = 0,$$

then

$$(v(t), \phi_i) = (\dot{v}(t), \phi_i) = 0 \qquad \text{for all} \quad t > 0.$$

The same considerations apply to the Kirchhoff equation (2.1), where

$$a(t) = m(|A^{1/2}u|^2).$$

Therefore if, for a given solution $u(\cdot)$ of Equation (2.1), $\{\mu_{n_k}\}$ is the sequence of frequencies which are 'on' at time zero, we shall write

$$\lambda_k = \mu_{n_k} = \mu_{n_k}(u), \qquad y_k(t) = (u(t), \phi_{n_k}),$$

and, for notations consistency,

$$\lambda_0 = 0, \qquad \phi_{n_0} = 0, \qquad y_0(t) \equiv 0,$$

in such a way that, for initial data (u_0, u_1) given by

$$u_p = \sum_{j=1}^{\infty} (u_p, \phi_{n_j})\phi_{n_j} \qquad (p = 0, 1), \tag{2.6}$$

we have

$$u(t) = \sum_{j=0}^{\infty} y_j(t)\phi_{n_j}, \qquad Au(t) = \sum_{j=0}^{\infty} \lambda_j^2 y_j(t)\phi_{n_j}. \tag{2.7}$$

We recall that the Hamiltonian for Equation (2.1) is given by the following functional (M is defined in (2.4))

$$H(t, u) := |\dot{u}|^2 + M(|A^{1/2}u|^2).$$

If the solution is global in time, the Hamiltonian decays exponentially. This is a result due to Biler [14].

Proposition 2.1. *Assume that m satisfies condition* (2.3) *and that the initial data $(u_0, u_1) \in D(A^{3/4}) \times D(A^{1/4})$ are given by* (2.6)*. Then, as long as $u(t)$ exists in the phase space $D(A^{3/4}) \times D(A^{1/4})$, $H(t)$ is decreasing*[1]*. Moreover, there exist constants*

$$C = C(\delta, \nu, L, \lambda_1, H(0))$$

such that

(i) *If $\delta < 2\sqrt{\nu}\lambda_1$ then $H(t) \le Ce^{-\delta t}$;*
(ii) *If $\delta = 2\sqrt{\nu}\lambda_1$ then $H(t) \le C(1 + t^2)e^{-\delta t}$;*
(iii) *If $\delta > 2\sqrt{\nu}\lambda_1$ then $H(t) \le Ce^{-(\delta-\sigma)t}$, where $\sigma^2 = \delta^2 - 4\nu\lambda_1^2$.*

Proof. We prove only the easy case (i). Deriving the Hamiltonian we have

$$\dot{H}(t) = -2\delta|\dot{u}|^2,$$

thus $H(t)$ is decreasing. To prove the exponential decay, we introduce the following modified energy functional

$$V(t) := H(t) + \delta(u, \dot{u}).$$

Since $|(u, \dot{u})| \le H(t)/2\lambda_1\sqrt{\nu}$, we have

$$0 < \Big(1 - \frac{\delta}{2\lambda_1\sqrt{\nu}}\Big)H(t) \le V(t) \le \Big(1 + \frac{\delta}{2\lambda_1\sqrt{\nu}}\Big).$$

Deriving $V(t)$ with respect to time, we get

$$\begin{aligned}\dot{V}(t) &= 2(\ddot{u}, \dot{u}) + 2m(|A^{1/2}u|^2)(Au, \dot{u}) + \delta|\dot{u}|^2 + \delta(u, \ddot{u})\\ &= -\delta|\dot{u}|^2 - \delta m(|A^{1/2}u|^2)|A^{1/2}u|^2 - \delta^2(u, \dot{u})\\ &\le -\delta V(t),\end{aligned}$$

[1]Hereafter, in our notations, we shall drop the dependence on the function u.

since in our case $m(\rho)\rho \geq M(\rho)$. Integrating this differential inequality, we get $V(t) \leq e^{-\delta t}V(0)$, thus

$$H(t) \leq Ce^{-\delta t}H(0),$$

where

$$C = (1+\delta/2\lambda_1\sqrt{\nu})(1-\delta/2\lambda_1\sqrt{\nu})^{-1}.$$

Remark 2.1. In particular, if

$$\delta < \sqrt{\nu}\lambda_1, \tag{2.8}$$

we have

$$H(t) \leq 3e^{-\delta t}H(0). \tag{2.9}$$

This estimate will be used later in the proof of the main result.

We recall that the Kirchhoff equation (2.1) is globally solvable for small initial data in $D \times V$. In the next theorem, by generalizing the estimates introduced in [11], we extend this result to data which lie in $D(A^{3/4}) \times D(A^{1/4})$. This is the weaker space in which a local existence theorem is available (see, e.g. [15]). For the sake of completeness we prove the exponential decay of the energy for initial data satisfying an additional assumption. The exponential decay for solutions satisfying only condition (2.10) below is an open problem.

We define the following energies, for $\alpha > 0$,

$$\mathcal{G}_\alpha(t,u) := \frac{|A^{(\alpha-1)/2}\dot{u}|^2}{m(|A^{1/2}u|^2)} + |A^{\alpha/2}u|^2.$$

Theorem 2.1. *Assume that m satisfies the condition* (2.3) *and that the initial data*

$$(u_0,u_1) \in D(A^{\alpha/2}) \times D(A^{(\alpha-1)/2}), \qquad 3/2 \leq \alpha \leq 2.$$

If

$$\frac{|A^{(\alpha-1)/2}u_1|^2}{\nu} + |A^{\alpha/2}u_0|^2 < \frac{((1\wedge\nu^{1/2})\delta)^{2(\alpha-1)}}{L}, \tag{2.10}$$

then there exists a unique global solution

$$(u,\dot{u}) \in C^0([0,+\infty[, D(A^{\alpha/2}) \times D(A^{(\alpha-1)/2}))$$

to (2.1), (2.2). *Moreover, if*

$$\frac{|A^{1/4}u_1|^2}{\nu} + |A^{3/4}u_0|^2 < \frac{(1\wedge\nu^{1/2})\delta}{18L(1+\delta^2/4\nu\lambda_1^2)}, \tag{2.11}$$

then $\mathcal{G}_{3/2}(t)$ decays exponentially.

Proof. We limit ourselves to provide the a priori estimates for regular solutions. The construction and the uniqueness of the solution are given, for instance, in [15].

Deriving $\mathcal{G}_\alpha$ with respect to time, we obtain

$$\dot{\mathcal{G}}_\alpha(t) = -\Big(\frac{\dot{m}}{m} + 2\delta\Big)\frac{|A^{\alpha-1)/2}\dot{u}|^2}{m}. \tag{2.12}$$

We have

$$\begin{aligned}|\dot{m}| &= 2L|(A^{1/2}u, A^{1/2}\dot{u})| \\ &= 2L|(A^{(3-\alpha)/2}u, A^{(\alpha-1)/2}\dot{u})| \\ &\leq 2L|A^{1/2}u|^\theta|A^{\alpha/2}u|^{1-\theta}|A^{(\alpha-1)/2}\dot{u}|,\end{aligned}$$

where θ is defined by

$$\theta + \alpha(1-\theta) = 3 - \alpha.$$

It follows that ($\rho = |A^{1/2}|^2$)

$$\frac{|\dot{m}|}{m} \le 2\Big(\frac{(L\rho)^\theta}{\nu + L\rho}\Big)^{1/2}(L\mathcal{G}_\alpha(t))^{1-\theta/2} \le 2(1 \vee \nu^{-1/2})(L\mathcal{G}_\alpha(t))^{1-\theta/2}. \tag{2.13}$$

Inserting this last inequality into (2.12), we get

$$\dot{\mathcal{G}}_\alpha(t) \le 2((1 \vee \nu^{-1/2})(L\mathcal{G}_\alpha(t))^{1-\theta/2} - \delta)\frac{|A^{\alpha-1)/2}\dot{u}|^2}{m}. \tag{2.14}$$

Note that

$$(1-\theta/2)^{-1} = 2(\alpha - 1)$$

and set

$$T := \sup\Big\{t > 0 : \ \mathcal{G}_\alpha(s) < \frac{((1 \wedge \nu^{1/2})\delta)^{2(\alpha-1)}}{L} \ \text{ for } \ 0 \le s \le t\Big\}.$$

Under the assumption (2.10), we have

$$\mathcal{G}_\alpha(0) < \frac{((1 \wedge \nu^{1/2})\delta)^{2(\alpha-1)}}{L}.$$

If, by absurd, $T < \infty$, thanks to (2.14), we would get $\dot{\mathcal{G}}_\alpha(t) \le 0$ for $0 \le t < T$. Thus

$$\mathcal{G}_\alpha(T) < \mathcal{G}_\alpha(0),$$

which is a contradiction.

To prove the exponential decay for initial data satisfying (2.11), let us set

$$\beta := \delta(1 + \delta^2/4\nu\lambda_1^2)^{-1}$$

and introduce the functional

$$\mathcal{V}_{3/2}(t) := \mathcal{G}_{3/2}(t) + \frac{\beta(A^{1/4}u, A^{1/4}\dot{u})}{m(|A^{1/2}u|^2)}.$$

Since, for any $\delta > 0$, $\beta \le \sqrt{\nu}\lambda_1$ and

$$\frac{|(A^{1/4}u, A^{1/4}\dot{u})|}{m(|A^{1/2}u|^2)} \le \frac{\mathcal{G}_{3/2}(t)}{2\sqrt{\nu}\lambda_1},$$

we have

$$\frac{1}{2}\mathcal{G}_{3/2}(t) \le \mathcal{V}_{3/2}(t) \le \frac{3}{2}\mathcal{G}_{3/2}(t). \tag{2.15}$$

Deriving $\mathcal{V}_{3/2}$ with respect to time, we obtain

$$\begin{aligned}
\dot{\mathcal{V}}_{3/2}(t) = &-\Big(\frac{\dot{m}}{m} + 2\delta\Big)\frac{|A^{1/4}\dot{u}|^2}{m} - \frac{\beta\dot{m}}{m^2}(A^{1/4}u, A^{1/4}\dot{u}) \\
&+ \beta\frac{|A^{1/4}\dot{u}|^2}{m} - \beta|A^{3/4}u|^2 - \frac{\beta\delta}{m}(A^{1/4}u, A^{1/4}\dot{u}) \\
= &-\frac{\dot{m}}{m}\Big(\frac{|A^{1/4}\dot{u}|^2}{m} + \frac{\beta}{m}(A^{1/4}u, A^{1/4}\dot{u})\Big) \\
&- (2\delta - \beta)\frac{|A^{1/4}\dot{u}|^2}{m} - \beta|A^{3/4}u|^2 - \frac{\beta\delta}{m}(A^{1/4}u, A^{1/4}\dot{u}) \\
\le &\frac{|\dot{m}|}{m}\mathcal{V}_{3/2}(t) - (2\delta - \beta)\frac{|A^{1/4}\dot{u}|^2}{m} - \beta|A^{3/4}u|^2 - \frac{\beta\delta}{m}(A^{1/4}u, A^{1/4}\dot{u}).
\end{aligned} \tag{2.16}$$

Let us set for a moment $x^2 = |A^{3/4}u|^2$ and $y^2 = |A^{1/4}\dot{u}|^2/m$. The lowest eigenvalue of the positive quadratic form

$$(2\delta - \beta)y^2 + \beta x^2 + (\beta\delta/\sqrt{\nu}\lambda_1)xy,$$

is given by

$$\frac{\delta}{1+\delta^2/4\nu\lambda_1^2}(1 + (1 - (1+\delta^2/4\nu\lambda_1^2)^{-1})^{1/2})^{-1} \geq \frac{\delta}{2(1+\delta^2/4\nu\lambda_1^2)} = \beta/2.$$

Therefore, from (2.16), (2.15) and (2.13) for $\theta = 0$, it follows that

$$\dot{\mathcal{V}}_{3/2}(t) \leq \Big(\frac{|\dot{m}|}{m} - \frac{\beta}{3}\Big)\mathcal{V}_{3/2}(t) \leq \Big(4(1 \vee \nu^{-1/2})L\mathcal{V}_{3/2}(t) - \frac{\beta}{3}\Big)\mathcal{V}_{3/2}(t).$$

Thus, by a standard comparison argument, we get

$$\mathcal{V}_{3/2}(t) \leq \frac{(\beta/3)\mathcal{V}_{3/2}(0)}{(\beta/3 - 4(1 \vee \nu^{-1/2})L\mathcal{V}_{3/2}(0))e^{(\beta/3)t} + 4(1 \vee \nu^{-1/2})L\mathcal{V}_{3/2}(0)}.$$

The thesis now follows from (2.15) and hypothesis (2.11).

Remark 2.2. An immediate consequence of Theorem 2.1 is that, for initial data in $D(A^{\alpha/2}) \times D(A^{(\alpha-1)/2})$, $\alpha \geq 3/2$, with Fourier expansion (2.6), Equation (2.1) is globally solvable under the assumption

$$\frac{|A^{(\alpha-1)/2}u_1|^2}{\nu} + |A^{\alpha/2}u_0|^2 < \frac{(1 \wedge \nu^{1/2})\delta\lambda_1^{2\alpha-3}}{L}. \tag{2.17}$$

In particular, for $\alpha = 2$, from (2.10) and (2.17), we obtain the smallness condition

$$\frac{|A^{1/2}u_1|^2}{\nu} + |Au_0|^2 < \frac{(1 \wedge \nu^{1/2})\delta}{L} \max\{(1 \wedge \nu^{1/2})\delta, \lambda_1\}. \tag{2.18}$$

§ 3. Main Result

Let us introduce a few functionals we shall use hereafter. Let $a \in C^2([0,T[)$ be a positive function. If $u \in C^0([0,T[, D) \cap C^1([0,T[, V)$ is given by (2.7), we define

$$\begin{aligned}
\gamma(t) &:= \frac{\dot{a}(t)}{2a^{3/2}(t)},\\
e_j(t,u) &:= (a(t))^{-1/2}\lambda_j^2\dot{y}_j^2 + (a(t))^{1/2}\lambda_j^4 y_j^2,\\
f_j(t,u) &:= \lambda_j^2 y_j\dot{y}_j,\\
w_j(t,u) &:= e_j(t,u) + [\delta(a(t))^{-1/2} + \gamma(t)]f_j(t,u),\\
E(t,u) &:= \sum_0^\infty e_j(t,u) = (a(t))^{-1/2}|A^{1/2}\dot{u}|^2 + (a(t))^{1/2}|Au|^2.
\end{aligned}$$

Moreover, we shall adopt the following notations: for a given integer k, we shall write

$$E^{\langle k \rangle}(t,u) := \sum_k^\infty e_j(t,u), \qquad E_{\langle k \rangle}(t,u) := \sum_0^{k-1} e_j(t,u).$$

The proof of Theorem 3.1 is based on the following energy identity for the linear ODE

$$\ddot{y}_j + a(t)\lambda_j^2 y_j + \delta\dot{y}_j = 0. \tag{3.1}$$

In the non dissipative case ($\delta = 0$), the identity is due to Manfrin [9].

Proposition 3.1. *Let $a(t)$ be a positive $C^2([0,T[)$ function. Then the solutions of the Equation* (3.1) *satisfy the following energy identity*

$$\dot{w}_j(t) = (\dot{\gamma}(t) - \delta\gamma(t))f_j(t) - \delta w_j(t). \tag{3.2}$$

Proof. The identity (3.2) is equivalent to

$$\dot{e}_j(t) + \Big(\frac{\delta}{\sqrt{a(t)}} + \gamma(t)\Big)\dot{f}_j(t) = -\delta w_j(t). \tag{3.3}$$

Using Equation (3.1), we have

$$\begin{aligned}\dot{f}_j(t) &= \lambda_j^2\dot{y}_j^2 - a(t)\lambda_j^4 y_j^2 - \delta f_j(t),\\ \dot{e}_j(t) &= -2\delta\frac{\lambda_j^2\dot{y}_j^2}{\sqrt{a(t)}} - \gamma(t)\lambda_j^2\dot{y}_j^2 + \frac{\dot{a}(t)}{2\sqrt{a(t)}}\lambda_j^4 y_j^2.\end{aligned}$$

Now (3.3) follows by straightforward computations.

For a given local solution u of the Kirchhoff equation (2.1), we set

$$a(t) = m(|A^{1/2}u|^2).$$

Lemma 3.1. *Assume that m satisfies the condition* (2.3)*. Let the initial data $(u_0,u_1) \in D\times V$ be given by* (2.6)*. Let $u \in C^0([0,T[,D)\cap C^1([0,T[,V)$ be the solution of the problem* (2.1), (2.2)*. Then the following estimates hold true*

$$|\dot{\gamma}(t)| \le \frac{L}{\nu}E(t) + \delta|\gamma(t)|, \tag{3.4}$$

$$|\gamma(t)| \le \frac{L}{2\nu^{3/2}\lambda_1}E(t), \tag{3.5}$$

$$|\gamma(t)| \le \frac{L}{\nu^{7/4}}(H(t)E(t))^{1/2}. \tag{3.6}$$

Proof. Under the assumption (2.3), we simply have $\dot{a}(t) = 2L(Au,\dot{u})$, thus

$$\gamma(t) = \frac{L(Au,\dot{u})}{m^{3/2}},$$

and, using the equation, we get

$$\ddot{a}(t) = 2L(|A^{1/2}\dot{u}|^2 - m(|A^{1/2}u|^2)|Au|^2) - 2\delta L(Au,\dot{u}). \tag{3.7}$$

Since

$$\dot{\gamma}(t) = \frac{\ddot{a}}{2a^{3/2}} - \frac{3\dot{a}^2}{4a^{5/2}},$$

by (3.7), we obtain

$$\begin{aligned}\dot{\gamma}(t) &= -3\Big(\frac{L}{m}\Big)^2\frac{(Au,\dot{u})^2}{\sqrt{m}} + \frac{L}{m^{3/2}}(|A^{1/2}\dot{u}|^2 - m|Au|^2) - \frac{\delta L}{m^{3/2}}(Au,\dot{u})\\ &= \Big(\frac{L}{m} - 3\theta(t)\Big(\frac{L}{m}\Big)^2|A^{1/2}u|^2\Big)\frac{|A^{1/2}\dot{u}|^2}{\sqrt{m}} - \frac{L}{m}\sqrt{m}|Au|^2 - \delta\gamma(t),\end{aligned} \tag{3.8}$$

where, by the Cauchy-Schwartz inequality, $0 \le \theta(t) \le 1$. Since

$$\begin{aligned}&\Big(\frac{L}{m} - 3\theta(t)\Big(\frac{L}{m}\Big)^2 |A^{1/2}u|^2\Big)\frac{|A^{1/2}\dot u|^2}{\sqrt{m}} - \frac{L}{m}\sqrt{m}|Au|^2\\ &\le \max\Big\{3\Big(\frac{L}{m}\Big)^2|A^{1/2}u|^2 - \frac{L}{m},\ \frac{L}{m}\Big\}E(t) = \frac{L}{m}E(t),\end{aligned}$$

thanks to (3.8), we have proved (3.4).

The inequality (3.5) follows at once by

$$|\gamma(t)| = \frac{L}{m^{3/2}}|(A^{1/2}u, A^{1/2}\dot u)| \le \frac{L}{\lambda_1 \nu^{3/2}}|(Au, A^{1/2}\dot u)|,$$

and by the Cauchy-Schwartz inequality. On the other hand, we have

$$|(Au, \dot u)| \le (H(t))^{1/2}\frac{m^{1/4}|Au|}{\nu^{1/4}} \le \nu^{-1/4}(H(t)E(t))^{1/2},$$

from which (3.6) follows immediately.

In order to state our main result, we need two more functions. Let $(u_0, u_1) \in D \times V$ be given by (2.6). For any $k \ge 1$, we define

$$\begin{aligned}G_k(x) &:= \frac{6\lambda_{k-1}^2}{\sqrt{\nu}}H(0) + 3\sum_{j=k}^{\infty} e_j(0)\exp\Big(\frac{3L}{2\nu\lambda_j}x\Big) \qquad (x \in \mathbf{R}),\\ \psi_k(x) &:= \int_0^x \frac{dy}{G_k(y)} \qquad (x \in \mathbf{R}).\end{aligned}$$

Theorem 3.1. *Assume that the conditions* (2.3) *and* (2.8) *hold true. Let the initial data* $(u_0, u_1) \in D \times V$ *be given by* (2.6).

If for some integer $k \ge 1$, *such that*

$$\lambda_k \ge \frac{2\delta}{\sqrt{\nu}}, \tag{3.9}$$

(u_0, u_1) *satisfies the following assumptions*

$$\lim_{x \to +\infty} \psi_k(x) > 1/\delta, \tag{I}$$

$$(G_k \circ \psi_k^{-1})(1/\delta) < \frac{\nu^{7/2}}{4L^2 H(0)}\lambda_k^2, \tag{II}$$

then the problem (2.1), (2.2) *has a unique global solution*

$$u \in C^0([0,\infty[; D) \cap C^1([0,\infty[; V).$$

Moreover the following estimate holds true

$$E(t) \le (G_k \circ \psi_k^{-1})(1/\delta)e^{-\delta t} \qquad (t > 0). \tag{3.10}$$

Proof. Let us set

$$C_1 := \frac{L}{\nu^{7/4}}, \qquad C_2 := \frac{3L}{2\nu}. \tag{3.11}$$

Under the hypothesis (2.8), and thanks to (3.4)–(3.6) of Lemma 3.1, we have

$$|\dot{\gamma}(t) - \delta\gamma(t)| \leq C_2 E(t,u). \tag{3.12}$$

Thanks to the inequality

$$|f_j| \leq \frac{e_j}{2\lambda_j}, \tag{3.13}$$

and by (3.6) and definition (3.11), we get that, for any $j \in \mathbf{N}$,

$$\Big|\Big(\frac{\delta}{\sqrt{a}} + \gamma\Big) f_j\Big| \leq \Big(\frac{\delta}{\sqrt{\nu}} + C_1 (H(t)E(t))^{1/2}\Big)\frac{e_j}{2\lambda_j}. \tag{3.14}$$

Note that

$$E(t) = E_{\langle\, k\,\rangle}(t) + E^{\langle\, k\,\rangle}(t) \leq \frac{2\lambda_{k-1}^2}{\sqrt{\nu}} H(t) + E^{\langle\, k\,\rangle}(t). \tag{3.15}$$

Therefore, thanks to the hypotheses (I), (II) and by the monotonicity of G_k, we have

$$E(0) < G_k(0) < (G_k \circ \psi_k^{-1})(1/\delta) < \frac{\lambda_k^2}{4C_1^2 H(0)}. \tag{3.16}$$

Let T_k be defined as follows

$$T_k := \sup\Big\{ t > 0 :\ E(s) < \frac{\lambda_k^2}{4C_1^2 H(0)}, \quad s \in [0,t] \Big\}.$$

The inequality (3.16) implies that $T_k > 0$. Thanks to (3.14), (3.9) and Proposition 2.1 we have

$$\Big|\Big(\frac{\delta}{\sqrt{a}} + \gamma\Big) f_j\Big| \leq \frac{e_j}{2} \qquad (j \geq k) \ \ (0 \leq t < T_k),$$

thus

$$\frac{e_j(t)}{2} \leq w_j(t) \leq \frac{3e_j(t)}{2} \qquad (j \geq k) \ \ (0 \leq t < T_k). \tag{3.17}$$

Now we use the energy identity (3.2) and thanks to (3.12), (3.13), (3.17), for $j \geq k$ and $0 < t < T_k$, we have

$$\dot{w}_j(t) = (\dot{\gamma}(t) - \delta\gamma(t)) f_j(t) - \delta w_j(t) \leq \Big(\frac{C_2 E(t)}{\lambda_j} - \delta\Big) w_j(t). \tag{3.18}$$

An application of the comparison principle to (3.18) yields

$$w_j(t) \leq w_j(0) e^{-\delta t} \exp\Big[\frac{C_2}{\lambda_j}\int_0^t E(s)ds\Big] \qquad (j \geq k) \ \ (0 \leq t < T_k).$$

Thus, by (3.17), we have

$$e_j(t) \leq 3e_j(0) e^{-\delta t} \exp\Big[\frac{C_2}{\lambda_j}\int_0^t E(s)ds\Big] \qquad (j \geq k) \ \ (0 \leq t < T_k). \tag{3.19}$$

Summing up in (3.19) with respect to j, we get

$$E^{\langle\, k\,\rangle}(t) \leq 3e^{-\delta t} \sum_{j=k}^{\infty} e_j(0) \exp\Big[\frac{C_2}{\lambda_j}\int_0^t E(s)ds\Big] \qquad (0 \leq t < T_k). \tag{3.20}$$

On the other hand, by (2.8) and (2.9), we have

$$H(t) \le 3e^{-\delta t} H(0). \tag{3.21}$$

Putting together (3.15), (3.20) and (3.21), it follows that the function $\int_0^t E(s)ds$ satisfies the following differential inequality

$$E(t) \le G_k\Big(\int_0^t E(s)ds\Big)e^{-\delta t} \qquad (0 < t < T_k).$$

Let $\varphi : [0, T_\varphi[\to \mathbf{R}$ be the maximal solution of the Cauchy problem

$$\dot{\varphi}(t) = G_k(\varphi(t))e^{-\delta t}, \qquad \varphi(0) = 0. \tag{3.22}$$

Since G_k is increasing, by a standard comparison argument, we have, as long as $t < \min(T_k, T_\varphi)$,

$$E(t) \le G_k\Big(\int_0^t E(s)ds\Big)e^{-\delta t} \le G_k(\varphi(t))e^{-\delta t}. \tag{3.23}$$

The solution φ of the Cauchy problem (3.22) is easily computed. In fact, by the definition of ψ_k, we have

$$\varphi(t) = \psi_k^{-1}((1 - e^{-\delta t})/\delta).$$

Thanks to the assumption (I) we have $T_\varphi = +\infty$. Thus, by (3.23),

$$E(t) \le (G_k \circ \psi_k^{-1})(1/\delta)e^{-\delta t} < \frac{\lambda_k^2}{4C_1^2 H(0)} \qquad (0 < t < T_k),$$

thanks to the assumption (II). We have then that $T_k = +\infty$ and the estimate (3.10).

Remark 3.1. The assumption $\delta < \sqrt{\nu}$ was made only to keep the statement of the theorem at a reasonable length. In the general case, we should replace the definition of G_k with

$$G_k(x) = \frac{2\lambda_{k-1}^2 C}{\sqrt{\nu}} + 3\sum_{j=k}^{\infty} e_j(0)\exp\Big(\frac{C_2 x}{\lambda_j}\Big),$$

where $C = C(\delta, \nu, L, \lambda_1, H(0))$ is the constant in Proposition 2.1, $C_2 = L/\nu + L/(2\nu^{3/2}\lambda_1)$ and replace (II) with $\lim\limits_{x\to\infty} \psi_k(x) > 1/(\delta - \sigma)$.

It is clear that for a fixed amount of initial energy $H(0)$, (I), (II) of Theorem 3.1 may be interpreted as conditions relating the high-frequencies energy $E^{\langle k \rangle}(0)$ with the gap between the eigenvalues λ_{k-1}, λ_k. The larger is the gap, actually the ratio $\lambda_k/\lambda_{k-1}^2$, the larger is the energy $E^{\langle k \rangle}(0)$ allowed. In the next proposition we provide a set of sufficient conditions which make this argument more explicit.

Proposition 3.2. *Assume that the conditions* (2.3) *and* (2.8) *hold true. Let the initial data* $(u_0, u_1) \in D \times V$ *be given by* (2.6). *If for some integer* $k \ge 1$, *such that* (3.9) *holds true,* (u_0, u_1) *satisfies the following conditions:*

$$H(0) < \frac{\nu^{3/2}\delta}{9L}\frac{\lambda_k}{\lambda_{k-1}^2}\ln\Big(1 + \frac{\lambda_{k-1}^2 H(0)}{\sqrt{\nu}E^{\langle k \rangle}(0)}\Big), \tag{Ia}$$

$$\frac{6\lambda_{k-1}^2}{\sqrt{\nu}}H(0)^2 + 3E^{\langle k \rangle}(0)H(0) < \frac{\nu^{7/2}}{8L^2}\lambda_k^2, \tag{IIa}$$

then the problem (2.1), (2.2) *has a unique global solution*

$$u \in C^0([0,\infty[;D) \cap C^1([0,\infty[;V).$$

Moreover, the following estimate holds true

$$E(t) \leq \Big(\frac{12\lambda_{k-1}^2 H(0)}{\sqrt{\nu}} + 6E^{\langle k \rangle}(0)\Big)e^{-\delta t}. \tag{3.24}$$

Proof. We have to prove that (Ia) and (IIa) imply conditions (I) and (II). For brevity, let us set

$$A_k = \frac{6\lambda_{k-1}^2 H(0)}{\sqrt{\nu}}, \quad B_k = 3E^{\langle k \rangle}(0), \quad \varepsilon_k = \frac{C_2}{\lambda_k}.$$

Since

$$G_k(x) < \widetilde{G}_k(x) := A_k + B_k e^{\varepsilon_k x},$$

we have

$$\widetilde{\psi}_k(x) := \int_0^x \frac{dy}{\widetilde{G}_k(y)} < \psi_k(x), \quad \psi_k^{-1}(x) < \widetilde{\psi}_k^{-1}(x). \tag{3.25}$$

It follows that

$$(G_k \circ \psi_k^{-1})(x) < (\widetilde{G}_k \circ \widetilde{\psi}_k^{-1})(x). \tag{3.26}$$

By a simple computation we obtain

$$\lim_{x\to\infty} \widetilde{\psi}_k(x) = \frac{1}{\varepsilon_k A_k} \ln\Big(1 + \frac{A_k}{B_k}\Big). \tag{3.27}$$

Since the condition (Ia) may be written as follows

$$\frac{1}{\varepsilon_k A_k} \ln\Big(1 + \frac{A_k}{2B_k}\Big) > 1/\delta,$$

we have that, thanks to (3.25) and (3.27), (I) is satisfied.

On the other hand, by the condition (Ia) we have

$$(\widetilde{G}_k \circ \widetilde{\psi}_k^{-1})(1/\delta) = A_k + \frac{B_k}{\big(1 + \frac{B_k}{A_k}\big)e^{-(\varepsilon_k A_k)/\delta} - \frac{B_k}{A_k}} \leq 2(A_k + B_k). \tag{3.28}$$

The condition (II) follows by (IIa), (3.26) and (3.28) whereas the estimate (3.24) follows by (3.26), (3.28) and (3.10).

Remark 3.2. By letting $\lambda_{k-1} \to 0$, the conditions (Ia), (IIa) have a meaning even for $k = 1$. Thus, in addition to (2.18), we obtain another sufficient condition for initial data in $D \times V$: $\lambda_1 \geq 2\delta/\sqrt{\nu}$, and

$$\frac{|A^{1/2}u_1|^2}{\nu} + |Au_0|^2 < (m(|A^{1/2}u_0|^2))^{-1/2} \min\Big\{\frac{\delta\nu\lambda_1}{9L}, \frac{\lambda_1^2\nu^{7/2}}{24L^2H(0)}\Big\}.$$

Remark 3.3. Other sets of sufficient conditions may be derived by (Ia) and (IIa). This is the version we used in the Introduction:

$$E^{\langle k \rangle}(0) < \frac{\lambda_{k-1}^2 H(0)}{\sqrt{\nu}},$$

$$H(0) < \min\Big\{\frac{\nu^{3/2}\delta \ln 2}{9L\lambda_{k-1}}, \frac{\nu^2}{6\sqrt{2}L}\Big\}\frac{\lambda_k}{\lambda_{k-1}}$$

for some $k \geq 1$.

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