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GRAPHS CHARACTERIZED BY LAPLACIAN EIGENVALUES**

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Abstract

This paper characterizes all connected graphs with exactly two Laplacian eigenvalues greater than two and all connected graphs with exactly one Laplacian eigenvalue greater than three.

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§ 1 . Introduction

Let $G = (V, E)$ be a simple graph. The Laplacian matrix of G is $L(G) = D(G) - A(G)$, where $D(G) = \text{diag}\,(d_u, u \in V(G))$ (d_u is the degree of a vertex u) and $A(G)$ are the degree diagonal and the adjacency matrices of G. The eigenvalues of $L(G)$ are called the Laplacian eigenvalues and denoted by

$$\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G) = 0$$

or for short

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0.$$

The Laplacian matrix of a simple graph has been extensively investigated for a long time, in particular, for the past twenty years. It has been established that there are a lot of relations between its spectrum and numerous graph invariants, including connectivity, diameter, isoperimetric number, maximum cut, expanding property of a graph (see, for example, [10, 11] and the references therein). There may be two reasons for stimulating us to determine graphs with a small number of Laplacian eigenvalues exceeding a given value. On one hand, there are close relations between the number of Laplacian eigenvalues and some graph invariants, such as matchings, etc. (for instance, see [5, 9]). On the other hand, Getman, Babic and Gineityte in [6] and Getman, Gineityte, Lepovic and Petrovic in [7] discovered some connections between photoelectron spectra of saturated hydrocarbons (alkanes) and the Laplacian eigenvalues of the underlying molecular graphs. Recently, Petrovic, Getman,

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Lepovic and Milekic determined all bipartite connected graphs with exactly two Laplacian eigenvalues greater than two and all connected bipartite graphs with exactly one Laplacian eigenvalue greater than three. In this paper, we characterize all connected graphs with exactly two Laplacian eigenvalues greater than two and all connected graphs with exactly one Laplacian eigenvalue greater than three.

§ 2. Graphs with Exactly Two Laplacian Eigenvalues Greater than Two

Denote by $H_7 = H_7(p,q)$ $(p \geq 1, q \geq 0)$, $H_8 = H_8(p,q,m)$ $(p \geq 0, q \geq 0, m \geq 1)$, $H_9 = H_9(p,q,m)$ $(p \geq 0, q \geq 0, m \geq 1)$.

Theorem 2.1. *The graphs $H_1 - H_9$ in Fig. 1 have the property $\lambda_3(H_i) \leq 2$ for $i = 1, \cdots, 9$.*

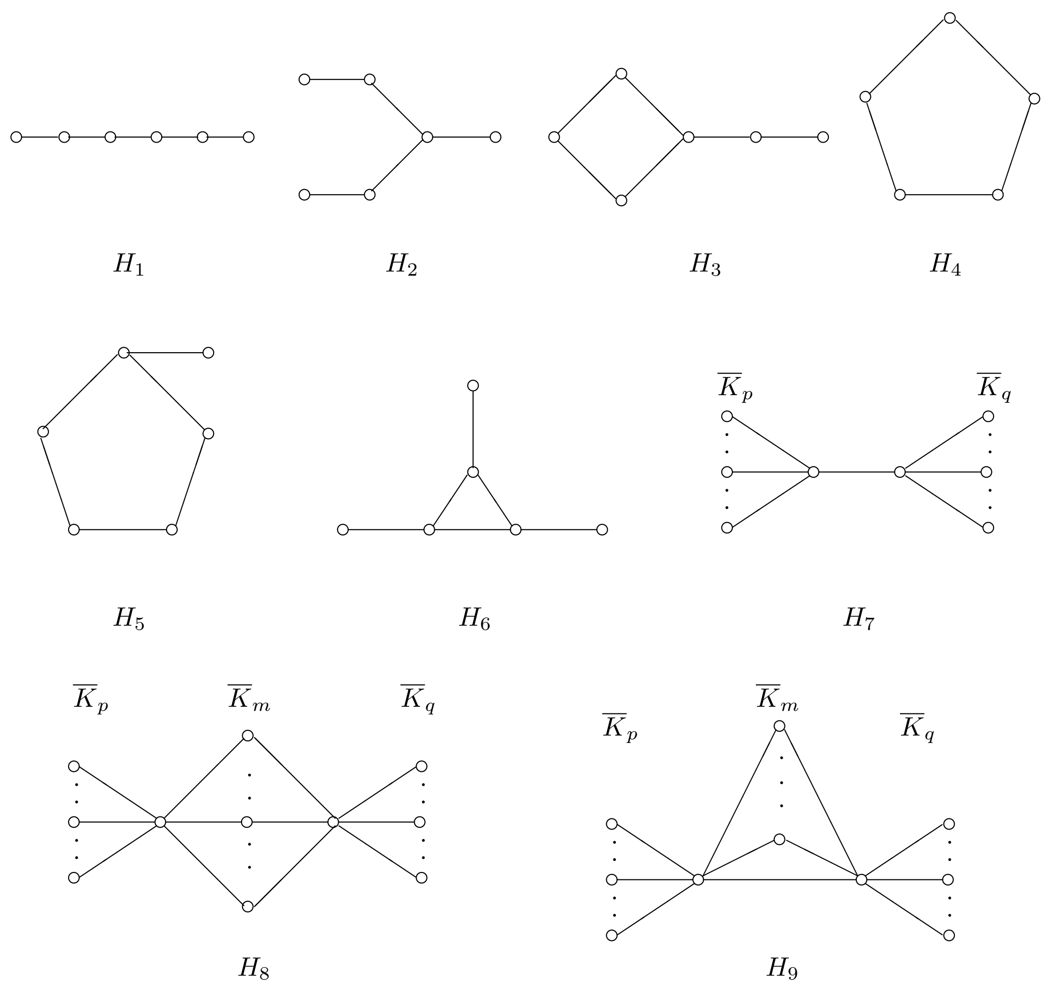


Fig. 1

Proof. It follows from Theorem 5 in [12] that

$$\lambda_3(H_i) \leq 2 \qquad \text{for} \ \ i = 1, 2, 3, 7, 8.$$

Moreover, by a direct calculation,

$$\lambda_3(H_i) \leq 2 \qquad \text{for} \ \ i = 4, 5, 6.$$

Now we show that

$$\lambda_3(H_9) \leq 2.$$

It is not difficult to see that the characterization polynomial of $L(H_9)$ is

$$\begin{aligned} &\lambda(\lambda-2)^{m-1}(\lambda-1)^{p+q-2}\{\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda + d\} \\ &= \lambda(\lambda-2)^{m-1}(\lambda-1)^{p+q-2}F(\lambda), \end{aligned}$$

where

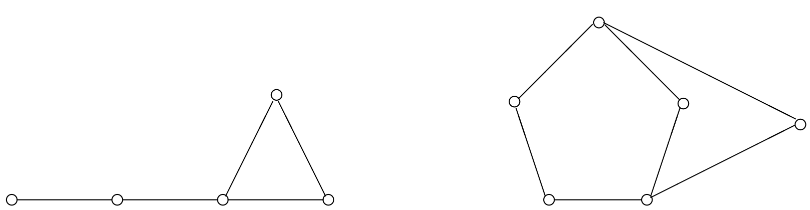
$$\begin{aligned} a &= -(2m+p+q+6), \\ b &= m^2 + (p+q+8)m + 4(p+q) + pq + 13, \\ c &= -\{2m^2 + (2p+2q+10)m + 5(p+q) + 2pq + 12\}, \\ d &= m^2 + (p+q+4)m + 2(p+q) + 4. \end{aligned}$$

If $p \geq q \geq 1$, then there are only two eigenvalues which are greater than two, since $F(0) > 0$, $F(1) = -pq < 0$, $F(2) = m^2 + (p+q)m > 0$, $F(p+m+2) = -pq < 0$ and $F(\infty) > 0$. Hence $\lambda_3(H_9) \leq 2$. If $p = 1, q = 0$, or $p = q = 0$, then by a similar argument, we may show that the assertion holds.

Now we consider all connected graphs G with the property

$$\lambda_3(G) \leq 2. \tag{2.1}$$

The property (2.1) is hereditary, because it follows from a directed consequence of the interlacing theorem (see [4]). Hence there are minimal graphs that do not obey (2.1). Such graphs are called forbidden subgraphs. By a direct calculation, we have the following subgraphs: F_1 and F_2 are forbidden subgraphs.



F_1 F_2

Fig. 2

A set of pairwise independent edges in G is called a match in G. The cardinality of a maximum matching of G is known as its matching number and denoted by $\mu(G)$. We may slightly generalize the result of Theorem 4 in [5] as follows.

Lemma 2.1. *Let G be a connected graph of order n with the matching number k.*
(i) *If $n > 2k$, then $\lambda_k > 2$.*
(ii) *If $n = 2k$, then $\lambda_{k-1} > 2$.*

Proof. (i) follows from Theorem 4 in [5]. For (ii), since G is connected, there exists a block which has only one cut vertex (for instance, see [2]). Hence there exists a vertex v such that $G_1 = G\backslash\{v\}$ is still connected. Thus G_1 has the matching number $k-1$, since $n = 2k$. Moreover the cardinality of G_1 is $n-1 > 2(k-1)$. Thus $\lambda_{k-1} > 2$ by (i). Hence the assertion of (ii) follows from the interlacing theorem.

Corollary 2.1. *Let G be a connected graph of order $n > 6$. If the matching number of G is $k \geq 3$, then $\lambda_3 > 2$.*

Proof. It is a direct consequence of Lemma 2.1.

Lemma 2.2. *Let G be a connected non-bipartite graph of order n. If $\lambda_3 \leq 2$, then G must be one of the following graphs H_4, H_5, H_6 and H_9.*

Proof. Since G is a non-bipartite graph, there exists an odd cycle C_{2p+1}. By Corollary 2.1, G does not contain odd cycle of order greater than 6, since the matching number of C_{2p+1} is p. Hence we consider the following two cases.

Case 1. G contains an odd cycle C_5 of order 5. If $|V(G)| > 6$, then G contains H_5 as a subgraph, since G is connected. Hence by Corollary 2.1, $\lambda_3 > 2$, since the matching number is at least three. So it is a contradiction. Hence $|V(G)| = 6$ or 5. Since F_1 and F_2 are forbidden subgraphs, G has to be either H_5 or H_4.

Case 2. G contains an odd cycle $C_3 = v_1v_2v_3$ of order 3. If there are greater than one common neighbor for two vertices of v_1, v_2, v_3, then we may assume that

$$|N(v_1) \cap N(v_2)| > 1,$$

where $|N(v_1)|$ is the cardinality of set of neighbors of v_1. Hence $N(v_3) = \{v_1, v_2\}$, since F_1 and K_4 (complete graph of order 4) are forbidden subgraphs. Therefore it is easy to see that G has to be H_9. If there are only one common neighbor for any two vertices in $\{v_1, v_2, v_3\}$, then we consider the number of $N(v_i)$ for $i = 1, 2, 3$. If $|N(v_i)| \geq 3$ for $i = 1, 2, 3$, then $|V(G)| \leq 6$. Otherwise the matching number of G is at least 3 and $|V(G)| \geq 7$, then by Corollary 2.1, $\lambda_3 > 2$. It is a contradiction. Hence G has to be H_6. If there exists a vertex in $\{v_1, v_2, v_3\}$ whose neighbor number is just two. Since F_1 is forbidden subgraphs, G must be H_9.

We are ready to present the characterization of all connected graph with exactly two eigenvalues greater than two.

Theorem 2.2. *A connected graph G has exactly two Laplacian eigenvalues greater than two if and only if G is one of the following graphs $H_1 - H_9$ in Fig.*1 *except $H_7(p, 0)$, $p \geq 1$, $H_7(1, 1)$, $H_8(0, 0, 1)$, $H_8(0, 0, 2)$ and $H_8(1, 0, 1) = H_8(0, 1, 1)$.*

Proof. Sufficiency. By a direct calculation, the following graphs $H_1 - H_6$ in Fig. 1 have exactly two Laplacian eigenvalues greater than two. Further by Theorem 4 in [12] and Theorem 2.1, the graphs H_7, H_8, H_9 in Fig. 1, except $H_7(p, 0)$, $p \geq 1$, $H_7(1, 1)$, $H_8(0, 0, 1)$, $H_8(0, 0, 2)$ and $H_8(1, 0, 1) = H_8(0, 1, 1)$, have exactly two Laplacian eigenvalues greater than two.

Necessity. If G is bipartite, then by Theorem 4 in [12], G is one of the following graphs $H_1 - H_3$ and H_7, H_8 except $H_7(p, 0)$, $p \geq 1$, $H_7(1, 1)$, $H_8(0, 0, 1)$, $H_8(0, 0, 2)$ and $H_8(1, 0, 1) = H_8(0, 1, 1)$. If G is non-bipartite, then by Lemma 2.2, G is one of the following graphs H_4, H_5, H_6 and H_9 in Fig. 1.

§ 3 . Graphs with Exactly One Laplacian Eigenvalue Greater than Three

In this section, we determine all connected graphs with exactly one Laplacian eigenvalue greater than three.

Theorem 3.1. *The following graphs $W_1 - W_5$ in Fig. 3 have property*

$$\lambda_2(W_i) \leq 3, \qquad i = 1, \cdots, 5. \tag{3.1}$$

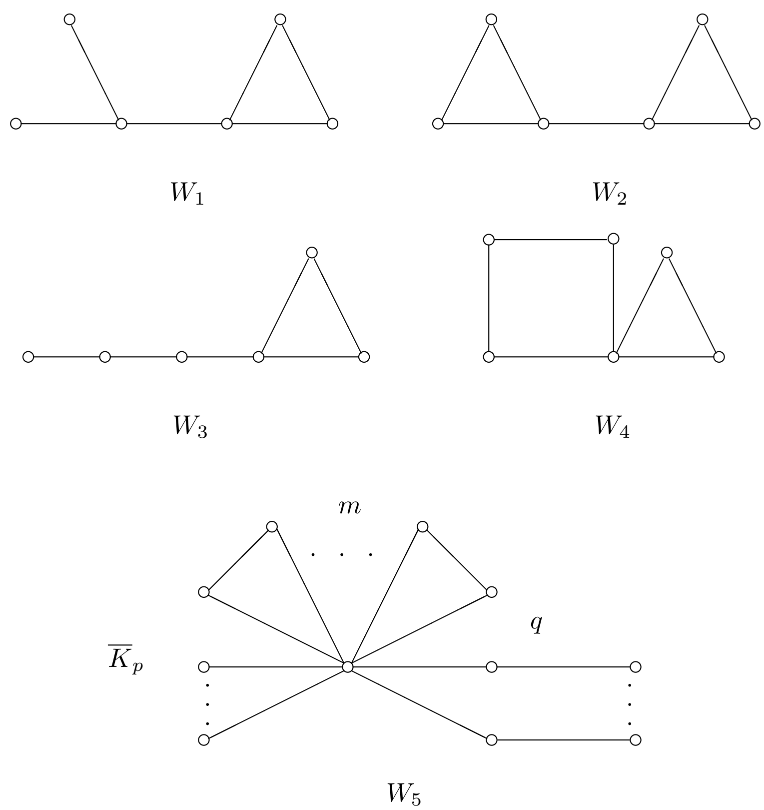


Fig. 3

Proof. Proof Denote by $W_5 = W_5(p, q, m)$, $p \geq 0, q \geq 0, m \geq 1$. By a direct calculation, we show that $\lambda_2(W_i) \leq 3$, $i = 1, \cdots, 4$. For W_5, we consider the submatrix M of $L(W_5)$ by deleting the row and column corresponding to the vertex with the largest degree. It is easy to see that M is direct sum of submatrices of order 1 and 2. Moreover the largest eigenvalues of these submatrices of order 1 and 2 are no greater than 3. Hence $\lambda_2(W_5) \leq 3$ follows from interlacing theorem.

Let $\mathcal{G}$ be the set of all connected graphs of order n with the property (3.1). Note that the Laplacian eigenvalues of cycle of order n are $2 + 2\cos\frac{2\pi j}{n}$ for $j = 0, 1, \cdots, n-1$. By a direct calculation, we have $\lambda_1 > \lambda_2 > 3$ for $n = 5$ and $n \geq 7$. Hence odd cycles of order $n \geq 5$ are forbidden subgraphs of $G \in \mathcal{G}$. Moreover, it is easy to see that the following graphs $F_3 - F_8$ are forbidden subgraphs of G in $\mathcal{G}$.

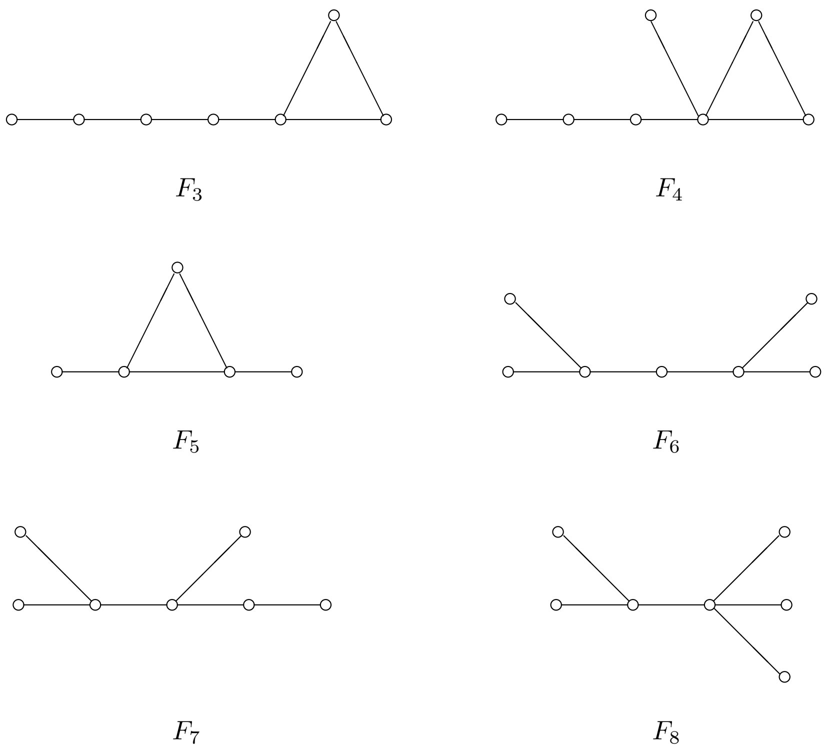


Fig. 4

Lemma 3.1. *Let $G \in \mathcal{G}$ contain at least a triangle. If the largest and second largest degrees are d_1 and $d_2 = 2$, respectively, then G is one of the following graphs W_3, W_4, W_5 in Fig.* 3.

Proof. Let $v_1v_2v_3$ be a triangle of G. Since G is connected and $d_2 = 2$, we may assume that $d_1 = d(v_1) \geq 3$ and $d_2 = d(v_2) = 2$ (otherwise $G = W_5(0,0,1)$). We consider the following two cases.

Case 1. Suppose that the order of graph G is $n \geq 7$. If there exists a vertex v such that the length of a path from u to v_1 is greater than 3 or equal to 3 respectively, then G contains F_3 or F_4 as a subgraph, respectively. It is a contradiction to $G \in \mathcal{G}$. Hence the lengths of any paths from $v \in G$ to v_1 are no greater than 2. Hence, G must be W_5, since any other vertices, except vertices v_1, v_2, v_3, of G are not adjacent to v_1, v_2.

Case 2. $n \leq 6$. If $n = 4$, then G is just $W_5(1,0,1)$. If $n = 5$, then it is easy to see that G must be $W_5(0,1,1)$, $W_5(0,0,2)$ or $W_5(2,0,1)$. If $n = 6$, it is not difficult to see that G must be one of the following graphs $W_3, W_4, W_5(1,0,2)$ and $W_5(1,1,1)$.

Lemma 3.2. *Let $G \in \mathcal{G}$ contain at least a triangle. If the largest and second largest degrees are d_1 and d_2 with $d_1 \geq d_2 \geq 3$, then G is one of the following graphs W_1, W_2 in Fig.* 3.

Proof. By Theorem 4 in [8], $\lambda_2 \geq d_2 \geq 3$. Since $G \in \mathcal{G}$, we have $d_2 = 3$. We now assume that $v_1v_2v_3$ is a triangle with $d(v_1) \geq d(v_2) \geq d(v_3)$, where $d(v_i)$ is the degree of vertex v_i, $i = 1, 2, 3$. We consider the following two cases.

Case 1. $d(v_2) \geq 3$. Then there exists a vertex v_4 such that v_4 is adjacent to v_2. Since F_5 is a forbidden subgraph of G and $d(v_1) \geq 3$, v_4 is adjacent to v_1. Let H be the induced subgraph of G by vertices v_1, v_2, v_3, v_4. Clearly, $\lambda_2(H) > 3$ which implies $\lambda_2(G) > 3$. It is a contradiction.

Case 2. $d(v_2) = 2$. Since $d_1 \geq d_2 = 3$, there exists a vertex v_4 such that $d(v_4) \geq 3$. Since F_3 and F_6 are forbidden subgraphs of G, v_4 must be adjacent to v_1. On the other hand, there exist two vertices v_5 and v_6 such that v_5 and v_6 are adjacent to v_4. Since F_7 and F_8 are forbidden subgraphs of G, the number of vertices of G is 6. Moreover, v_5 and v_6 are not adjacent to v_1, since F_5 is a forbidden subgraph of G. Hence G is one of the graphs W_1 and W_2.

We are ready to present our main result in this section.

Theorem 3.2. *A connected graph G has exactly one Laplacian eigenvalue greater than three if and only if*

(i) *G is a connected bipartite graph with four or five vertices.*

(ii) *G is a connected spanning subgraph of one of the graphs W_6, W_7 in Fig. 5.*

(iii) *G is one of the graphs $W_1 - W_5$ in Fig. 3 and W_8 in Fig. 5 except $W_5(0,0,1)$.*

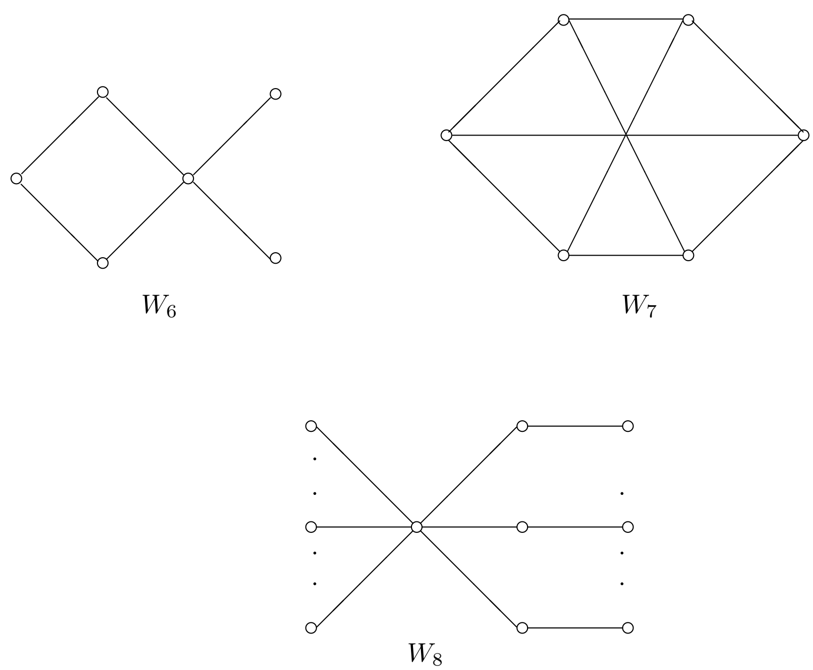


Fig. 5

Proof. Sufficiency. By the direct calculation, it is easy to see that all graphs in (i), (ii) and $W_1 - W_4, W_6, W_7$ have only one Laplacian eigenvalue greater than three. Further W_5 and W_8 have only one Laplacian eigenvalue greater than three by Theorem 3.1 and Theorem 7 in [12].

Necessity. If G is a bipartite graph, then by Theorem 7 in [12], G is one of the graphs in (i), (ii) and W_8. If G is a non-bipartite graph, then G contains a triangle, since odd cycle of order greater than 3 is forbidden subgraph. Hence by Lemmas 3.1, 3.2, G is one of the graphs $W_1 - W_5$ in Fig. 2 except $W_5(0, 0, 1)$.

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