Chin. Ann. Math.
25B:1(2004),111–118.

TRANSFERENCE OF MAXIMAL MULTIPLIER OPERATORS ON LOCAL HARDY-LORENTZ SPACES

CHEN Daning*

Abstract

The author establishes a deLeeuw-type theorem on maximal multiplier operators on local Hardy-Lorentz space.

Keywords Hardy-Lorentz spaces, Multiplier operator, deLeeuw's theorem
2000 MR Subject Classification 42B15, 42B25, 42B30
Chinese Library Classification O174
Document Code A
Article ID 0252-9599(2004)01-0111-08

§ 1 . Introduction

Let $H^p(R^n),\ 0<p<\infty$ be the Hardy spaces defined by

$$H^p(R^n)=\Big\{f\in S'(R^n):\Big\|\sup_{t>0}|\varphi_t*f|\Big\|_{L^p(R^n)}<\infty\Big\},$$

where $\varphi\in S(R^n),\ \int\varphi=1$ and $\varphi_t(x)=t^{-n}\varphi\left(\frac{x}{t}\right)$.

For convenience, we fix such a φ with supp$\varphi\subset\left\{x:|x|\le\frac{1}{2}\right\}$ once for all in the following.

The corresponding periodic Hardy spaces are

$$H^p(T^n)=\Big\{f\in S'(T^n):\Big\|\sup_{t>0}|\widetilde{\varphi}_t*f|\Big\|_{L^p(T^n)}<\infty\Big\},$$

where

$$\widetilde{\varphi}_t*f(x)=\sum_{k\in Z^n}\widehat{\varphi}(tx)a_k(f)e^{2\pi ik\cdot x},\quad f(x)=\sum_{k\in Z^n}a_k(f)e^{2\pi ik\cdot x},$$

and

$$\widehat{\varphi}(u)=\int_{R^n}\varphi(x)e^{-2\pi iu\cdot x}dx$$

is the Fourier transform of φ.

Manuscript received January 22, 2002. Revised October 8, 2002.
*Department of Mathematics, Jackson State University, Jackson, MS 39157, USA.
E-mail: chendn@yahoo.com

Let $\lambda \in L^\infty(R^n)$. For each $\varepsilon > 0$, define

$$\widehat{(T_\varepsilon f)(u)} = \lambda(\varepsilon u)\widehat{f}(u), \qquad f \in L^2(R^n) \cap H^p(R^n),$$
$$\widetilde{T}_\varepsilon f(x) = \sum_{k\in Z^n} \lambda(\varepsilon k) a_k(f) e^{2\pi i k\cdot x}, \qquad f \in L^2(T^n) \cap H^p(T^n).$$

We call λ a maximal multiplier on $H^p(R^n)$ if $T^* f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$ can be extended to a bounded operator from $H^p(R^n)$ to $L^p(R^n)$.

Similarly, λ is a maximal multiplier on $H^p(T^n)$ if $\widetilde{T}^* f(x) = \sup_{\varepsilon>0} |\widetilde{T}_\varepsilon f(x)|$ can be extended to a bounded operator from $H^p(T^n)$ to $L^p(T^n)$.

The transference relation between maximal multilpiers on $H^p(R^n)$ to $H^p(T^n)$ was studied by Kenig and Tomas [4] in the case $1 < p < \infty$, and by Liu and Lu [5] in the case $0 < p \le 1$.

In [5], Liu and Lu proved the following theorem.

Theorem A. *Let $0 < p \le 1, \lambda \in L^\infty(R^n)$ be a continuous function on R^n and satisfy*

$$\lim_{|x|\to\infty} \lambda(x) = \alpha. \tag{1.1}$$

Suppose that λ is a maximal multiplier on $H^p(R^n)$. Then λ is a maximal multiplier on $H^p(T^n)$. (*See also* [3]*, in which the authors proved that the above condition* (1.1) *in Theorem* A *is superflous*).

The main purpose of this paper is to extend Theorem A to the local Hardy-Lorentz space $h(p, q)$, $0 < p < \infty$, $0 < q < \infty$. The definition of $h(p, q)$ will be reviewed in the second section. But we point out here that $h(p, q) = H^p$ if $p = q > 1$. It should also be noted that, unlike H^p, there is no a standard atomic decomposition for $h(p, q)$, if $p \ne q$, $0 < q \le 1$.

The following is the main result in the present paper.

Theorem 1.1. *Let $\lambda \in L^\infty(R^n)$ be a continuous function. For $0 < p, q < \infty$, if*

$$\|T^* f\|_{L^{p,q}(R^n)} \le C\|f\|_{h(p,q,R^n)} \qquad \text{for all } \ f \in h(p, q, R^n),$$

then

$$\|\widetilde{T}^* f\|_{L^{p,q}(T^n)} \le C\|f\|_{h(p,q,T^n)} \qquad \text{for all } \ f \in h(p, q, T^n),$$

where $L^{p,q}$ is the Lorentz space and $h(p, q)$ is the local Hardy-Lorentz space. Their definitions will be reviewed in Section 2.

The proof of the theorem will be in Section 3.

The converse part of the theorem is proved in [1].

§ 2 . Basic Notation and Lemmas

Let (X, μ) be a measure space. For a measurable function f, its distribution function m is defined by

$$m(\alpha) = \mu\{x \in X : |f(x)| > \alpha\}, \qquad \alpha > 0. \tag{2.1}$$

The non-increasing rearrangement of f, f_* is defined by

$$f_*(t) = \inf_{\alpha>0}\{\alpha : m(\alpha) \le t\}, \qquad t > 0.$$

Definition 2.1. *The Lorentz space $L^{p,q}(X)$, $0<p,q<\infty$, is the set of all measurable functions f on X with $\|f\|_{L^{p,q}(X)}<\infty$, where*

$$\|f\|_{L^{p,q}(X)}=\Big[\frac{q}{p}\int_0^\infty [t^{\frac{1}{p}}f_*(t)]^q\frac{dt}{t}\Big]^{\frac{1}{q}}.$$

A well-known result of the Lorentz space is $L^{p,p}=L^p$ (see [6]). In this paper, we are interested in $X=\mathbf{R}^n$ and $X=\mathbf{T}^n$ with the Lebesgue measure $\mu(E)=|E|$.

Definition 2.2. *Local Lorentz-Hardy space $h(p,q,\mathbf{R}^n)$ is the set of all $f\in S'(\mathbf{R}^n)$ such that*

$$\|f\|_{h(p,q,\mathbf{R}^n)}\equiv\Big\|\sup_{0<t\le 1}|\varphi_t*f|\Big\|_{L^{p,q}(\mathbf{R}^n)}<\infty.$$

Similarly, the space $h(p,q,\mathbf{T}^n)$ is the set of all $\tilde f\in S'(\mathbf{T}^n)$ such that

$$\|\tilde f\|_{h(p,q,\mathbf{T}^n)}=\Big\|\sup_{0<t\le 1}|\tilde f*\widetilde{\varphi}_t|\Big\|_{L^{p,q}(\mathbf{T}^n)}<\infty.$$

We introduce the following known lemmas.

Lemma 2.1. (cf. [2]) *Suppose that $\{f_n\}$ is a sequence of nonnegative functions on the measure space (X,μ) and that f is a nonnegative function on a measure space (Y,ν). If $\{a_n\}$ is a positive sequence such that*

$$\liminf_{n\to\infty} a_n\mu\{x\in X: f_n(x)>\alpha\}\ge\nu\{y\in Y: f(y)>\alpha\}\qquad \text{for all }\ \alpha>0, \tag{2.2}$$

then we have

$$f_*(t)\le\liminf_{n\to\infty}(f_n)_*\Big(\frac{t}{a_n}\Big)\qquad \text{for all }\ t>0.$$

The following lemma can be found in [6, p.190].

Lemma 2.2. *Suppose $\{f_n\}$ is a sequence of measurable functions such that for all $x\in X$,*

$$|f_n(x)|\le|f_{n+1}(x)|,\qquad n=1,2,\cdots.$$

If f is a measurable function satisfying

$$|f(x)|=\lim_{n\to\infty}|f_n(x)|\qquad \text{for all }\ x\in X,$$

then for each $t>0$, $(f_n)_*(t)$ increases monotonically to $f_*(t)$.

Also, it is easy to check

Lemma 2.3. *If for any $\alpha>0$,*

$$\lim_{n\to\infty}\mu\{x\in X,|f_n(x)|>\alpha\}=0,$$

then for any $t>0$,

$$\lim_{n\to\infty}(f_n)_*(t)=0.$$

Lemma 2.4. *Suppose that $\psi(x)$ is a continuous function with compact support. Let $\lambda(x)$ be a bounded and continuous function on $\mathbf{R}^n$ and let T_ε and $\widetilde{T}_\varepsilon$ be the families of operators on $\mathbf{R}^n$ and $\mathbf{T}^n$, respectively, associated to the function λ.*

Take

$$\psi^{\frac{1}{N}}(\xi) = \psi\Big(\frac{\xi}{N}\Big). \tag{2.3}$$

If ψ satisfies $\psi(0) = 1$ and $\widehat{\psi} \in L^1(\mathbf{R}^n)$, then for any $g(x) = \sum C_k e^{2\pi i k\cdot x} \in C^\infty(T^n)$ and any positive integer N, we have

$$\psi\Big(\frac{y}{N}\Big)(\widetilde{T}_\varepsilon g)(y) = T_\varepsilon(g\psi^{\frac{1}{N}})(y) + J_{N,\varepsilon}(y), \tag{2.4}$$

where

$$J_{N,\varepsilon}(y) = \sum C_k e^{2\pi i k\cdot x} N^n \int_{\mathbf{R}^n} \widehat{\psi}(Nx)\{\lambda(\varepsilon k) - \lambda(\varepsilon k + \varepsilon x)\} e^{2\pi i y\cdot x} dx \to 0$$

uniformly on y as $N \to \infty$.

This lemma can be found in [2].

§ 3. Proof of Theorem 1.1

Suppose now that λ is a maximal multiplier on $h(p, q, \mathbf{R}^n)$. Since the class of trigonometric polynomials forms a dense subset of $h(p, q, \mathbf{T}^n)$ and $L^{p,q}(\mathbf{T}^n)$, we need only prove that for every trigonometric polynomial $f(x) = \sum a_k e^{2\pi i k\cdot x}$,

$$\|\widetilde{T}^* f\|_{L^{p,q}(T^n)} \le C\|f\|_{h^{p,q}(T^n)}. \tag{3.1}$$

Define

$$\widetilde{T}^*_R f(x) = \sup_{0<\varepsilon\le R} |\widetilde{T}_\varepsilon f(x)|.$$

By Lemma 2.2, we only need prove

$$\|\widetilde{T}^*_R f\|_{L^{p,q}(T^n)} \le C\|f\|_{h^{p,q}(T^n)}, \tag{3.2}$$

where C is a constant independent of $R > 0$ and all trigonometric polynomials f.

For positive integers M and N, we denote the cube

$$\Big[-\frac{N}{2M}, \frac{N}{2M}\Big)^n \qquad \text{by} \quad \frac{N\mathbf{Q}}{M}.$$

Let $\psi \in D(\mathbf{R}^n)$ be a radial function that satisfies

$$\operatorname{supp}(\psi) \subset \mathbf{Q}, \qquad 0 \le \psi(x) \le 1 \quad \text{and} \quad \psi(x) = 1 \qquad \text{if} \ \ x \in \frac{\mathbf{Q}}{2}.$$

Since $\widetilde{T}_\varepsilon \tilde{f}(x)$ is a periodic function, for any positive even number N,

$$|\{x \in \mathbf{Q} : |\widetilde{T}^*_R \tilde{f}(x)| > \alpha\}| \le \Big(\frac{2}{N}\Big)^n \Big|\Big\{x \in \frac{N\mathbf{Q}}{2} : \sup_{\mathbf{0}<\varepsilon\le\mathbf{R}} \Big|\widetilde{T}_\varepsilon \tilde{f}(x)\psi\Big(\frac{x}{N}\Big)\Big| > \alpha\psi\Big(\frac{x}{N}\Big)\Big\}\Big|.$$

Thus by Lemma 2.4, we have

$$|\{x \in \mathbf{Q} : |\widetilde{T}^*_R \tilde{f}(x)| > \alpha\}| \leq \Big(\frac{2}{N}\Big)^n \Big|\Big\{x \in \frac{N\mathbf{Q}}{2} : \sup_{\mathbf{0}<\varepsilon\leq \mathbf{R}} |T_\varepsilon(\tilde{f}\psi^{\frac{1}{N}})(x)| > \frac{\alpha}{2}\Big\}\Big| + \Big(\frac{2}{N}\Big)^n \Big|\Big\{x \in \frac{N\mathbf{Q}}{2} : \sup_{\mathbf{0}<\varepsilon\leq \mathbf{R}} |J_{N,\varepsilon}(x)| > \frac{\alpha}{2}\Big\}\Big|.$$

Since λ is continuous, by Lemma 2.3 we know that $J_{N,\varepsilon}(x) \to 0$ uniformly for $x \in \mathbf{R}^n$ and $0 \leq \varepsilon \leq R$ ($R > 0$ is any fixed positive integer), as $N \to \infty$. So we have

$$\lim_{N\to\infty} \Big(\frac{2}{N}\Big)^n \Big|\Big\{x \in \frac{N\mathbf{Q}}{2} : \sup_{\mathbf{0}<\varepsilon\leq \mathbf{R}} |J_{N,\varepsilon}(x)| > \frac{\alpha}{2}\Big\}\Big| = 0.$$

This shows that

$$|\{x \in \mathbf{Q} : |\widetilde{T}^*_R \tilde{f}(x)| > \alpha\}| \leq 2^n \liminf_{N\to\infty} N^{-n} \Big|\Big\{x \in \mathbf{R}^n : 2\sup_{\varepsilon>\mathbf{0}} |T_\varepsilon(\tilde{f}\psi^{\frac{1}{N}})(x)| > \alpha\Big\}\Big|.$$

By Lemma 2.1, we have

$$(\widetilde{T}^*_R \tilde{f})_*(t) \leq C \lim_{N\to\infty} \Big\{\Big|(T^*(\tilde{f}\psi^{\frac{1}{N}}))_*\Big(\frac{tN^n}{2^n}\Big)\Big|\Big\}.$$

Therefore, by Fatou's Lemma and changing variables, we have

$$\begin{aligned}
\|\widetilde{T}^*_R \tilde{f}\|_{L^{p,q}(\mathbf{T}^n)} &= \Big[\frac{q}{p}\int_0^\infty [t^{\frac{1}{p}}(\widetilde{T}^*_R \tilde{f})_*(t)]^q \frac{dt}{t}\Big]^{\frac{1}{q}} \\
&\leq C \lim_{N\to\infty} N^{-\frac{n}{p}} \Big[\int_0^\infty [t^{\frac{1}{p}}\{|T^*(\tilde{f}\psi^{\frac{1}{N}})|\}_*(t)]^q \frac{dt}{t}\Big]^{\frac{1}{q}} \\
&= C \lim_{N\to\infty} N^{-\frac{n}{p}} \|T^*(\tilde{f}\psi^{\frac{1}{N}})\|_{L^{p,q}(\mathbf{R}^n)}.
\end{aligned}$$

By the assumption of the theorem, we now have

$$\|\widetilde{T}^*_R \tilde{f}\|_{L^{p,q}(\mathbf{T}^n)} \leq C \lim_{N\to\infty} N^{-\frac{n}{p}} \|\tilde{f}\psi^{\frac{1}{N}}\|_{h(p,q,\mathbf{R}^n)}. \tag{3.3}$$

By Lemma 2.4 again, we get

$$\sup_{\mathbf{0}<\delta\leq\mathbf{1}} |\varphi_\delta * (\tilde{f}\psi^{\frac{1}{N}})(y)| \leq \psi\Big(\frac{y}{N}\Big) \sup_{\mathbf{0}<\delta\leq\mathbf{1}} |\widetilde{\varphi}_\delta * \tilde{f}(y)| + \sup_{\mathbf{0}<\delta\leq\mathbf{1}} |J_{N,\delta}(y)|.$$

Thus

$$N^{-\frac{n}{p}} \|\tilde{f}\psi^{\frac{1}{N}}\|_{h(p,q,\mathbf{R}^n)} \leq CN^{-\frac{n}{p}} \Big\|\psi\Big(\frac{\cdot}{N}\Big) \sup_{\mathbf{0}<\delta\leq\mathbf{1}} |\widetilde{\varphi}_\delta * \tilde{f}(\cdot)|\Big\|_{L^{p,q}(\mathbf{R}^n)} + CN^{-\frac{n}{p}} \Big\|\sup_{\mathbf{0}<\delta\leq\mathbf{1}} |J_{N,\delta}(\cdot)|\Big\|_{L^{p,q}(\mathbf{R}^n)},$$

and finally we only need to show

$$\lim_{N\to\infty} N^{-\frac{n}{p}} \Big\|\psi\Big(\frac{\cdot}{N}\Big) \sup_{\mathbf{0}<\delta\leq\mathbf{1}} |\widetilde{\varphi}_\delta * \tilde{f}(\cdot)|\Big\|_{L^{p,q}(\mathbf{R}^n)} \leq C \Big\|\sup_{\mathbf{0}<\delta\leq\mathbf{1}} |\widetilde{\varphi}_\delta * \tilde{f}|\Big\|_{L^{p,q}(\mathbf{T}^n)}, \tag{3.4}$$

$$\lim_{N\to\infty} N^{-\frac{n}{p}} \Big\|\sup_{\mathbf{0}<\delta\leq\mathbf{1}} |J_{N,\delta}(\cdot)|\Big\|_{L^{p,q}(\mathbf{R}^n)} = 0. \tag{3.5}$$

To prove (3.4), by the support condition of ψ, we have

$$\Big|\Big\{x\in\mathbf{R}^n:\ \psi\Big(\frac{x}{N}\Big)\sup_{0<\delta\le 1}|\widetilde{\varphi}_\delta*\tilde{f}(x)|>\alpha\Big\}\Big|$$
$$\le\Big|\Big\{x\in\mathbf{NQ}:\sup_{0<\delta\le 1}|\widetilde{\varphi}_\delta*\tilde{f}(x)|>\alpha\Big\}\Big|=N^n\Big|\Big\{x\in\mathbf{Q}:\ \sup_{0<\delta\le 1}|\widetilde{\varphi}_\delta*\tilde{f}(x)|>\alpha\Big\}\Big|$$

since $\sup_{0<\delta\le 1}|\widetilde{\varphi}_\delta*\tilde{f}(x)|$ is a periodic function.

By Lemma 2.2 and the definition, we now obtain

$$\begin{aligned}&N^{-\frac{n}{p}}\Big\|\psi\Big(\frac{\cdot}{N}\Big)\sup_{\mathbf{0}<\delta\le\mathbf{1}}|\widetilde{\varphi}_\delta*\tilde{f}(\cdot)|\Big\|_{L^{p,q}(\mathbf{R}^n)}\\
&\le N^{-\frac{n}{q}}\Big[\int_0^\infty\Big[t^{\frac{1}{p}}\Big(\sup_{0<\delta\le 1}\|\widetilde{\varphi}_\delta*\tilde{f}|\Big)_*\Big(\frac{t}{N^n}\Big)\Big]^q\frac{dt}{t}\Big]^{\frac{1}{q}}\\
&=C\Big[\int_0^\infty\Big[t^{\frac{1}{p}}\Big(\sup_{0<\delta\le 1}|\widetilde{\varphi}_\delta*\tilde{f}|\Big)_*(t)\Big]^q\frac{dt}{t}\Big]^{\frac{1}{q}}\\
&=C\Big\|\sup_{0<\delta\le 1}|\widetilde{\varphi}_\delta*\tilde{f}|\Big\|_{L^{p,q}(\mathbf{T}^n)}.\end{aligned}$$

So (3.4) is proved. We return to prove (3.5).

First, it is easy to see that

$$\sup_{0<\delta\le 1}|J_{N,\delta}(y)|\le\int_{R^n}\Big|\widehat{\psi}(x)\Big[\widehat{\phi}\Big(\delta k+\frac{\delta x}{N}\Big)-\widehat{\phi}(\delta k)\Big]\Big|dx=O(N^{-1})$$

uniformly on $y\in\mathbf{R}^n$.

On the other hand, for each y there is a y_j such that $|y_j|\ge\frac{|y|}{n}$. Without loss of generality, we assume $|y_1|\ge\frac{|y|}{n}$.

Let

$$x=(x_1,\bar{x}),\quad y=(y_1,\bar{y}),$$

where $\bar{x}=(x_2,\cdots,x_n)$ and $\bar{y}=(y_2,\cdots,y_n)$.

Then

$$|J_{N,\delta}(y)|=\int_{\mathbf{R}^{n-1}}\Big\{\int_R N^n\widehat{\psi}(Nx)[\widehat{\phi}(\delta k+\delta x)-\widehat{\phi}(\delta k)]e^{2\pi iy_1\cdot x_1}dx_1\Big\}e^{2\pi i\bar{y}\cdot\bar{x}}d\bar{x}.$$

We use integration by parts with respect to x_1 to obtain

$$\begin{aligned}|J_{N,\delta}(y)|\le{}&O\Big(\int_{\mathbf{R}^n}N^{n+1}\frac{1}{|y|}\frac{\partial}{\partial x_1}\widehat{\psi}(Nx)e^{2\pi ix\cdot y}[\widehat{\phi}(\delta k+\delta x)-\widehat{\phi}(\delta k)]dx\Big)\\
&+O\Big(\int_{\mathbf{R}^n}\frac{N^n}{|y|}\widehat{\psi}(Nx)e^{2\pi ix\cdot y}\delta\Big(\frac{\partial}{\partial x_1}\widehat{\phi}\Big)(\delta k+\delta x)dx\Big).\end{aligned}$$

So

$$\sup_{0<\delta\le 1}|J_{N,\delta}(y)|\le C\frac{1}{|y|},$$

where C is independent of N.

It is easy to see that if we use integration by parts for m times, then

$$|J_{N,\delta}(y)|\le C\Big|\frac{N^{m-1}}{y^m}\Big|.$$

Since the constant C is independent of $0 < \delta \le 1$, we have

$$\sup_{0<\delta\le 1} |J_{N,\delta}(y)| \le \begin{cases} \dfrac{C}{N} & \text{if } |y| \le N, \\ C\Big|\dfrac{N^{m-1}}{y^m}\Big| & \text{if } |y| > N. \end{cases}$$

So we have

$$\begin{aligned}
&\Big|\Big\{x \in \mathbf{R}^n : \sup_{0<\delta\le 1} |J_{N,\delta}(x)| > \alpha\Big\}\Big| \\
&= \Big|\Big\{|x| < N : \sup_{0<\delta\le 1} |J_{N,\delta}(x)| > \alpha\Big\} \bigcup \Big\{|x| > N : \sup_{0<\delta\le 1} |J_{N,\delta}(x)| > \alpha\Big\}\Big| \\
&\le \Big|\Big\{|x| < N : \sup_{0<\delta\le 1} |J_{N,\delta}(x)| > \alpha\Big\}\Big| + \Big|\Big\{|x| > N : \sup_{0<\delta\le 1} |J_{N,\delta}(x)| > \alpha\Big\}\Big| \\
&\le \Big|\Big\{|x| < N : \Big|\frac{C}{N}\Big| > \alpha\Big\}\Big| + \Big|\Big\{|x| > N : C\frac{N^{m-1}}{|x|^m} > \alpha\Big\}\Big| \\
&= \Big|\Big\{|x| < N : N < \frac{C}{\alpha}\Big\}\Big| + \Big|\Big\{|x| > N : |x| < C\Big(\frac{N^{m-1}}{\alpha}\Big)^{\frac{1}{m}}\Big\}\Big|.
\end{aligned}$$

The first measure is equal to

$$\int_{|x|<N} dx \approx CN^n,$$

where $N < \frac{C}{\alpha}$.

For the second measure, since

$$N < |x| < C\Big(\frac{N^{m-1}}{\alpha}\Big)^{\frac{1}{m}},$$

we have $N < \frac{C}{\alpha}$, and

$$\begin{aligned}
&\Big|\Big\{|x| > N : |x| < C\Big(\frac{N^{m-1}}{\alpha}\Big)^{\frac{1}{m}}\Big\}\Big| \\
&= \int_{N<|x|<C\left(\frac{N^{m-1}}{\alpha}\right)^{\frac{1}{m}}} dx \approx \Big\{\Big[\frac{N^{m-1}}{\alpha}\Big]^{\frac{n}{m}} - N^n\Big\}.
\end{aligned}$$

Thus, for large N, we obtain some constants C and A independent of N, such that

$$\Big|\Big\{x \in \mathbf{R}^n : \sup_{0<\delta\le 1} |J_{N,\delta}(x)| > \alpha\Big\}\Big| \le \begin{cases} 0, & \alpha \ge \frac{C}{N}, \\ A\Big(\dfrac{N^{m-1}}{\alpha}\Big)^{\frac{n}{m}}, & \alpha < \frac{C}{N}. \end{cases}$$

This shows that

$$\begin{aligned}
&\inf_{\alpha>0}\Big\{\alpha : \Big|\Big\{x \in \mathbf{R}^n : \sup_{0<\delta\le 1} |J_{N,\delta}(x)| > \alpha\Big\}\Big| \le t\Big\} \\
&\le \inf_{\alpha>0}\Big\{\alpha : \Big|A\Big(\frac{N^{m-1}}{\alpha}\Big)^{\frac{n}{m}}\Big| \le t\Big\} \\
&= \inf_{\alpha>0}\Big\{\alpha : A\Big(\frac{N^{m-1}}{\alpha}\Big) \le t^{\frac{m}{n}}\Big\} \\
&= \inf_{\alpha>0}\Big\{\alpha : \alpha \ge A\Big(\frac{N^{m-1}}{t^{\frac{m}{n}}}\Big)\Big\} = A\Big(\frac{N^{m-1}}{t^{\frac{m}{n}}}\Big).
\end{aligned}$$

Thus

$$\Big(\sup_{0<\delta\le 1}|J_{N,\delta}|\Big)_*(t)\le\begin{cases}A\Big(\dfrac{N^{m-1}}{t^{\frac{m}{n}}}\Big), & t\ge N^n,\\ \dfrac{A}{N}, & t<N^n.\end{cases}$$

Now choose m such that

$$\frac{m}{n}>\frac{1}{p}.$$

Then

$$\begin{aligned}
&C\liminf_{N\to\infty}\Big(\frac{2}{N}\Big)^{\frac{n}{p}}\Big[\frac{q}{p}\int_0^\infty\Big\{t^{\frac{1}{p}}\Big(\sup_{0<\delta\le 1}|J_{N,\delta}|\Big)_*(t)\Big\}^q\frac{dt}{t}\Big]^{\frac{1}{q}}\\
&=C\liminf_{N\to\infty}\Big(\frac{1}{N}\Big)^{\frac{n}{p}}\Big[\int_0^{N^n}\Big\{t^{\frac{1}{p}}\Big(\sup_{0<\delta\le 1}|J_{N,\delta}|\Big)_*(t)\Big\}^q\frac{dt}{t}\Big]^{\frac{1}{q}}\\
&\quad+C\liminf_{N\to\infty}\Big(\frac{1}{N}\Big)^{\frac{n}{p}}\Big[\int_{N^n}^\infty\Big\{t^{\frac{1}{p}}\Big(\sup_{0<\delta\le 1}|J_{N,\delta}|\Big)_*(t)\Big\}^q\frac{dt}{t}\Big]^{\frac{1}{q}}\\
&\le C\liminf_{N\to\infty}\Big(\frac{1}{N}\Big)^{\frac{n}{p}}\Big[\int_0^{N^n}\Big\{t^{\frac{1}{p}}\frac{1}{N}\Big\}^q\frac{dt}{t}\Big]^{\frac{1}{q}}\\
&\quad+C\liminf_{N\to\infty}\Big(\frac{1}{N}\Big)^{\frac{n}{p}}\Big[\int_{N^n}^\infty\Big\{t^{\frac{1}{p}}\Big(\frac{N^{m-1}}{t^{\frac{m}{n}}}\Big)\Big\}^q\frac{dt}{t}\Big]^{\frac{1}{q}}\\
&=C\liminf_{N\to\infty}\Big(\frac{1}{N}\Big)^{\frac{n}{p}}\cdot\frac{C}{N}\Big[\int_0^{N^n}t^{\frac{q}{p}-1}dt\Big]^{\frac{1}{q}}\\
&\quad+C\liminf_{N\to\infty}\Big(\frac{1}{N}\Big)^{\frac{n}{p}}\cdot N^{m-1}\Big[\int_{N^n}^\infty t^{\frac{q}{p}-\frac{mq}{n}-1}dt\Big]^{\frac{1}{q}}\\
&=C\liminf_{N\to\infty}\frac{1}{N}=0.
\end{aligned}$$

The theorem is proved.

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