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EXPLICIT CONSTRUCTION FOR HARMONIC SURFACES IN $U(N)$ VIA ADDING UNITONS***

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Abstract

The authors give an algebraic method to add uniton numbers for harmonic maps from a simply connected domain $\Omega \subseteq R^2 \cup \{\infty\}$ into the unitary group $U(N)$ with finite uniton number. So, it is proved that any n-uniton can be obtained from a 0-uniton by purely algebraic operations and integral transforms to solve the $\bar{\partial}$-problem via two different ways.

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§ 0. Introduction

In [1–4] it has been proved that any harmonic map $\varphi : \Omega \to U(N)$ from a simply connected domain $\Omega \subseteq R^2 \cup \{\infty\}$ to the unitary group $U(N)$ with finite uniton number can be factorized into a product of a finite number of flag factors (called unitons in [1]). In [6–9], by using Darboux transformations, a purely algebraic explicit algorithm to construct unitons from a known one was given. Particularly, the singular Darboux transformation realizes the singular Bäcklund transformation via purely algebraic operations. These have been generalized to harmonic maps into the symplectic group in [10, 11].

However, flag factors given as in [4–9] are used to substract or to preserve uniton numbers. So far as known, it is an open problem to construct flag factors adding uniton numbers via purely algebraic operations (see [8, Remark 3.1]). In this direction, K. Uhlenbeck firstly proposed a criterion of such flag factors in [1]. Introducing the so-called basic flag factors, an explicit construction of basic flag factors was given in [2], where the Cauchy's integral transform to solve $\bar{\partial}$-problem is required. The basic flag factors may be either to add, to substract, or to preserve uniton numbers, but it is very hard to find out basic flag factors adding uniton numbers (see Theorem 2.4 and Theorem 2.5 of [2]) because the condition (iii) of Theorem 2.4 in [2] can not be checked.

The purpose of this paper is to give a concrete algebraic method (including integral transforms) to construct flag factors adding uniton numbers for harmonic maps $\varphi : \Omega \to$

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$U(N)$ with finite uniton number. Firstly, we introduce the notion on the standard extended solution of type two, and give two kinds of criterions of flag factors adding uniton numbers, which are more simplified than that of [1]. Next, by using the theory due to J. C. Wood in [2], we give a concrete algebraic method to construct flag factors adding uniton numbers, which are called AUN-flag factors. By using AUN-flag factors, any n-uniton can be obtained from a 0-uniton by purely algebraic operations together with integral transforms to solve $\bar{\partial}$-problem. Finally, for some special unitons, we give more simplified algebraic methods to construct AUN-flag factors. The notations used here will follow those in [1, 2].

§ 1. Preliminaries

Let $U(N)$ be the unitary group of order N, whose Lie algebra is $u(N)$. Let $\Omega \subseteq R^2 \cup \{\infty\} = \mathbb{C} \cup \{\infty\}$ be a simply connected domain, and z the complex coordinate on Ω. Consider a smooth map $\varphi : \Omega \to U(N)$. On putting

$$A^\varphi = \frac{1}{2}\varphi^{-1}d\varphi = A^\varphi_z dz + A^\varphi_{\bar{z}} d\bar{z},$$

we have

$$\varphi^* = \varphi, \qquad (A^\varphi_{\bar{z}})^* = -A^\varphi_z, \tag{1.1}$$

$$\bar{\partial}A^\varphi_z - \partial A^\varphi_{\bar{z}} - 2[A^\varphi_z, A^\varphi_{\bar{z}}] = 0, \tag{1.2}$$

where $\partial = \partial/\partial z$, $\bar{\partial} = \partial/\partial\bar{z}$. The map φ is harmonic if and only if

$$\bar{\partial}A^\varphi_z + \partial A^\varphi_{\bar{z}} = 0. \tag{1.3}$$

The Lax pair of the harmonic maps is

$$\bar{\partial}\Phi_\lambda = (1-\lambda)\Phi_\lambda A^\varphi_{\bar{z}}, \quad \partial\Phi_\lambda = (1-\lambda^{-1})\Phi_\lambda A^\varphi_z, \tag{1.4}$$

of which the integrability condition is (1.2) and (1.3). A nondegenerated $N \times N$ matrix solution Φ_λ to (1.4) is called an extended solution of the harmonic map φ. Moreover, Φ_λ can be normalized as

$$\Phi_1 = I, \quad (\Phi_{\sigma(\lambda)})^* = \Phi_\lambda^{-1}, \quad \sigma(\lambda) = (\bar{\lambda})^{-1}, \tag{1.5}$$

so that $\Phi_{-1} = Q\varphi$ for some constant $Q \in U(N)$.

A harmonic map $\varphi : \Omega \to U(N)$ is called an n-uniton if φ has the extended solution of the following form:

$$\Phi_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha, \quad T_\alpha : \Omega \to gl(N, \mathbb{C}). \tag{1.6}$$

Such Φ_λ is also called the extended n-uniton. In general, the extended solution of φ is not unique. The minimal uniton number of φ is defined as (see [1])

$$\mathfrak{m}(\varphi) = \min\Big\{n \,\Big|\, \Phi_\lambda = \sum_{\alpha=0}^{n} T_\alpha\lambda^\alpha \text{ is the extended solution of } \varphi\Big\}.$$

From (1.4) and the reality condition (1.5) it follows that

$$T_n T_0^* = 0, \qquad T_n^* T_0 = 0, \tag{1.7}$$

$$\bar{\partial}T_\alpha = (T_\alpha - T_{\alpha-1})A^\varphi_{\bar{z}}, \qquad \partial T_\alpha = (T_\alpha - T_{\alpha+1})A^\varphi_z, \tag{1.8}$$

where $\alpha = 1, \cdots, n+1$, $T_{-2} = T_{-1} = T_{n+1} = T_{n+2} = 0$.

Set

$$V_0(\Phi_\lambda) = \mathrm{span}\{\mathrm{Im}T_0(z),\ z \in \Omega\}, \quad V_n(\Phi_\lambda) = \mathrm{span}\{\mathrm{Im}T_n(z),\ z \in \Omega\}. \tag{1.9}$$

Definition 1.1. *Φ_λ is called the standard extended solution of type one if $V_0(\Phi_\lambda) = \mathbb{C}^N$; and is called the standard extended solution of type two if $V_n(\Phi_\lambda) = \mathbb{C}^N$.*

It is proved in [1, Theorems 13.2 and 13.3] that there exists uniquely the standard extended solution of type one for a uniton φ with minimal uniton number. Moreover, by its proof, it is seen that the standard extended solution of type one for φ can be obtained by any extended solution of φ with minimal uniton number via purely algebraic constructions. By using the similar way, we may prove the following lemma, whose proof is omitted.

Lemma 1.1. *There exists uniquely the standard extended solution of type two for a uniton φ with minimal uniton number. Moreover, it can be obtained by any extended solution of φ with minimal uniton number via purely algebraic constructions.*

Let $\underline{\mathbb{C}}^N = \Omega \times \mathbb{C}^N$ be the trivial bundle over Ω, and $\pi : \underline{\mathbb{C}}^N \to \underline{\mathbb{C}}^N$ a Hermitian projection. We write $\underline{\mathrm{Im}}\pi = \Omega \times \{\mathrm{Im}\pi\} \subset \underline{\mathbb{C}}^N$, or simply, $\underline{\pi} = \Omega \times \{\mathrm{Im}\pi\}$. Similarly, for $T \in \mathrm{Hom}(\underline{\mathbb{C}}^N, \underline{\mathbb{C}}^N)$, we write $\underline{\ker}T = \Omega \times \{\ker T\} \subset \underline{\mathbb{C}}^N$. Let Φ_λ be an extended solution of a harmonic map $\varphi : \Omega \to U(N)$. If π is a Hermitian projection such that $\widetilde{\Phi}_\lambda = \Phi_\lambda(\pi + \lambda\pi^\perp)$ is the extended solution of $\widetilde{\varphi} = \widetilde{\Phi}_{-1}$, then $\pi - \pi^\perp$ (resp. $\underline{\pi}$ or π) is called a flag factor of φ. For $A^\varphi = \frac{1}{2}\varphi^{-1}d\varphi$, $D^\varphi = d + A^\varphi$ is a unitary connection on $\underline{\mathbb{C}}^N$, so that there are the Koszul-Malgrange holomorphic structures determined by D^φ and the trivial connection d respectively. If $\underline{\alpha} \subset \underline{\mathbb{C}}^N$ is a smooth subbundle of $\underline{\mathbb{C}}^N$, then the connection (denoted still by D^φ) on $\underline{\alpha}$ induced from D^φ determines the Koszul-Malgrange holomorphic structure on $\underline{\alpha}$. If $\underline{\alpha}$ is a holomorphic (resp. anti-holomorphic) subbundle of $\underline{\mathbb{C}}^N$, then any holomorphic (resp. anti-holomorphic) section of $(\underline{\alpha}, D^\varphi)$ is also the holomorphic (resp. anti-holomorphic) section of $(\underline{\mathbb{C}}^N, D^\varphi)$. It is known in [1, 2] that a Hermitian projection $\pi : \underline{\mathbb{C}}^N \to \underline{\pi}$ is a flag factor of φ if and only if $\underline{\pi}$ is the holomorphic subbundle of $(\underline{\mathbb{C}}^N, D^\varphi)$ and $A^\varphi_{\bar{z}} : \underline{\pi} \to \underline{\pi}$. It follows from (1.2) and (1.3) that $\underline{\mathrm{Im}}A^\varphi_{\bar{z}}$ and $\underline{\ker}A^\varphi_{\bar{z}}$ are flag factors of φ. Moreover, the holomorphic subbundles of $(\underline{\mathbb{C}}^N, D^\varphi)$ containing $\underline{\mathrm{Im}}A^\varphi_{\bar{z}}$ and the holomorphic subbundles of $\underline{\ker}A^\varphi_{\bar{z}}$ all are flag factors of φ, called the flag factors of type one and type two, respectively, in [2]. The flag factors of type two are also called basic.

Lemma 1.2. (cf. [2]) *Let π be a flag factor of the harmonic map $\varphi : \Omega \to U(N)$. Then $\pi^\perp$ is the flag factor of $\widetilde{\varphi} = \varphi(\pi - \pi^\perp)$ and $\underline{\mathrm{Im}}A^\varphi_{\bar{z}} \subseteq \underline{\pi}$ (resp. $\underline{\pi} \subseteq \underline{\ker}A^\varphi_{\bar{z}}$) if and only if $\underline{\pi}^\perp \subseteq \underline{\ker}A^{\widetilde{\varphi}}_{\bar{z}}$ (resp. $\underline{\mathrm{Im}}A^{\widetilde{\varphi}}_{\bar{z}} \subseteq \underline{\pi}^\perp$).*

In the following, we shall use the following convention on the range of indeces unless otherwise stated: $1 \leq i, j, \cdots \leq N$. Set

$$\mathcal{F} = \{\text{the set of meromorphic functions on } S^2\},$$

so that $\mathcal{F}$ becomes a field with pointwise addition and multiplication. Let $\underline{\alpha}$ be a k-dimensional smooth subbundle of $\underline{\mathbb{C}}^N$. The sets of meromorphic sections of $(\underline{\alpha}, D^\varphi)$ and $(\underline{\mathbb{C}}^N, D^\varphi)$ become respectively the k-dimensional and N-dimensional vector spaces over the field $\mathcal{F}$, denoted by $\Gamma(\underline{\alpha}, D^\varphi)$ and $\Gamma(\underline{\mathbb{C}}^N, D^\varphi)$, whose bases are called the meromorphic bases on $(\underline{\alpha}, D^\varphi)$ and $(\underline{\mathbb{C}}^N, D^\varphi)$, respectively.

Lemma 1.3. (cf. [2]) *Let $\varphi : \Omega \to U(N)$ be a harmonic map and $\{e_i\}$ a meromorphic basis of $(\underline{\mathbb{C}}^N, D^\varphi)$. Then there are meromorphic functions λ_{ij} on Ω such that*

$$A_z^\varphi(e_i) = \sum_j \lambda_{ij} e_j.$$

Lemma 1.4. *Let $\Phi_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha$ be the extended solution of a harmonic map $\varphi : \Omega \to U(N)$, and $\{e_i\}$ and $\{e_i^0\}$ the meromorphic bases of $(\underline{\mathbb{C}}^N, D^\varphi)$ and $(\underline{\mathbb{C}}^N, d)$, respectively. Then there are meromorphic functions μ_{ij} and ν_{ij} on Ω so that*

$$T_0(e_i) = \sum_j \mu_{ij} e_j^0, \quad T_n^*(e_i^0) = \sum_j \nu_{ij} e_j.$$

Proof. It follows from (1.8) that $\bar\partial T_0 = T_0 A_{\bar z}^\varphi$ and $\partial T_n = T_n A_z^\varphi$. One may see easily that T_0 and T_n^* are the global holomorphic sections of $\mathrm{Hom}\{(\underline{\mathbb{C}}^N, D^\varphi), (\underline{\mathbb{C}}^N, d)\}$ and $\mathrm{Hom}\{(\underline{\mathbb{C}}^N, d), (\underline{\mathbb{C}}^N, D^\varphi)\}$, respectively. Thus, $T_0(e_i)$ and $T_n^*(e_i^0)$ are the meromorphic sections of $(\underline{\mathbb{C}}^N, d)$ and $(\underline{\mathbb{C}}^N, D^\varphi)$, respectively. Hence, μ_{ij} and ν_{ij} are meromorphic on Ω.

As is well known, there exists always the meromorphic basis $\{e_i^0\}$ of $(\underline{\mathbb{C}}^N, d)$, for example, the constant orthonormal basis in $\underline{\mathbb{C}}^N$. For the meromorphic basis of $(\underline{\mathbb{C}}^N, D^\varphi)$, we have the following

Lemma 1.5. (cf. [2]) *Let $\varphi : \Omega \to U(N)$ be a harmonic map and $\{e_i\}$ a meromorphic basis of $(\underline{\mathbb{C}}^N, D^\varphi)$. If π is the basic flag factor of φ and $\widetilde\varphi = \varphi(\pi - \pi^\perp)$, then the meromorphic basis of $(\underline{\mathbb{C}}^N, D^{\widetilde\varphi})$ can be obtained by algebraic operations and Cauchy's integral transforms to solve the $\bar\partial$-problem.*

Here the $\bar\partial$-problem is as follows. Let $\mathcal{D} \subseteq \mathbb{C} = R^2$ be a connected open subset with compact closure $\bar{\mathcal{D}}$. Let F be a C^∞ function on an open set containing $\bar{\mathcal{D}}$. Then the equation $\bar\partial\lambda = F$ has a C^∞ solution on $\mathcal{D}$ given by the Cauchy integral transform (see [12] for detail)

$$\lambda(z) = \frac{1}{2\pi i} \int_{\mathcal{D}} \frac{F(w)}{w - z} dw \wedge d\bar w.$$

By factorization theorems in [1, 2, 4], any uniton φ can be factorized into a product of the basic flag factors. So, we have

Lemma 1.6. *For any uniton $\varphi : \Omega \to U(N)$, there exists a meromorphic basis $\{e_i\}$ of $(\underline{\mathbb{C}}^N, D^\varphi)$ which can be obtained via a meromorphic basis $\{e_i^0\}$ of $(\underline{\mathbb{C}}^N, d)$ by finite algebraic operators and Cauchy's integral transforms to solve the $\bar\partial$-problem.*

§ 2. Adding One Uniton Number

Let $\Phi_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha$ be the extended solution of a uniton φ. It is shown in [1, 9] that $\underline{\ker} T_0$ and $\underline{\mathrm{Im}} T_n^*$ are flag factors such that the uniton number is subtracted one. The factorization theorems imply that any n-uniton may be obtained from a constant map (i.e., 0-uniton) by n flag transforms. However, the key of realizing this construction is to find out the flag factor adding uniton numbers. In [1] such flag factor was characterized firstly (see [1, Proposition 14.5]).

Theorem A. (cf. [1]) *Let $\Phi_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha$ be the standard extended n-uniton of type one, $n > 0$. Then $\widetilde{\Phi}_\lambda = \Phi_\lambda(\pi + \lambda\pi^\perp)$ is the standard extended $(n+1)$-uniton of type one satisfying $\underline{\ker}\widetilde{T}_0 = \underline{\pi}^\perp$ if and only if the Hermitian projection $\pi : \underline{\mathbb{C}}^N \to \underline{\mathbb{C}}^N$ satisfies*

(a) *π is holomorphic in $(\underline{\mathbb{C}}^N, D^\varphi)$;*
(b) *$\underline{\pi} \subset \underline{\ker}A^\varphi_z$;*
(c) *$\underline{\pi} \cap \underline{\ker}T_0 = 0$;*
(d) *$\underline{\pi} \notin \underline{\ker}\rho T_0$ for $\rho \in \mathbb{C}P^N$.*

However, it is very difficult to check the condition (d). We now reduce these conditions as follows.

Theorem 2.1. *Let $\Phi_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha$ be the standard extended n-uniton of type two, $n > 0$. Then $\widetilde{\Phi}_\lambda = \Phi_\lambda(\pi + \lambda\pi^\perp)$ is the standard extended $(n+1)$-uniton of type two satisfying $\underline{\ker}\widetilde{T}_0 = \underline{\pi}^\perp$ if and only if the Hermitian projection $\pi : \underline{\mathbb{C}}^N \to \underline{\mathbb{C}}^N$ satisfies*

(a) *$\underline{\pi}$ is a holomorphic subbundle of $(\underline{\ker}A^\varphi_z, D^\varphi)$;*
(b) *$\underline{\pi} \cap \underline{\ker}T_0 = 0$.*

Proof. Consider the sufficiency. The condition (a) implies that π is a basic flag factor of φ. Set $\widetilde{\Phi}_\lambda = \sum_{\alpha=0}^{n+1} \widetilde{T}_\alpha \lambda^\alpha$, so that $\widetilde{T}_0 = T_0\pi$ and $\widetilde{T}_{n+1} = T_n\pi^\perp$. It follows from the condition (b) that $T_0\pi \neq 0$ and $\underline{\ker}\widetilde{T}_0 = \underline{\pi}^\perp$. If $T_n\pi^\perp = 0$, then $\underline{\mathrm{Im}}T_n^* \subseteq \underline{\pi}$ and $\underline{\mathrm{Im}}T_n^* \subseteq \underline{\ker}T_0$ according to (1.7). This contradicts the condition (b). Hence, $\widetilde{\Phi}_\lambda$ is an extended $(n+1)$-uniton.

On the other hand, it follows from the condition (b) that $\underline{\ker}\widetilde{T}^*_{n+1} = \underline{\ker}(\pi^\perp T_n^*) = \underline{\ker}T_n^*$. Thus, when $V_n(\Phi_\lambda) = \mathbb{C}^N$, we have

$$\bigcap_{z\in\Omega} \underline{\ker}\widetilde{T}^*_{n+1}(z) = \bigcap_{z\in\Omega} \underline{\ker}T^*_n(z) = 0.$$

This implies that $V_n(\widetilde{\Phi}_\lambda) = \mathbb{C}^N$, i.e., $\widetilde{\Phi}_\lambda$ is a standard extended $(n+1)$-uniton of type two.

The necessity follows from Theorem A directly.

By the similar way, we have

Theorem 2.2. *Let $\Phi_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha$ be a standard extended n-uniton of type one, $n > 0$. Then $\widetilde{\Phi}_\lambda = \Phi_\lambda(\pi + \lambda\pi^\perp)$ is a standard extended $(n+1)$-uniton of type one satisfying $\underline{\mathrm{Im}}\widetilde{T}^*_{n+1} = \underline{\pi}^\perp$ if and only if the Hermitian projection $\pi^\perp : \underline{\mathbb{C}}^N \to \underline{\mathbb{C}}^N$ satisfies*

(a) *$\underline{\pi}^\perp$ is an anti-holomorphic subbundle of* $(\underline{\ker}A^\varphi_{\bar z}, D^\varphi)$;
(b) $\underline{\pi}^\perp \cap \underline{\ker}T_n = 0$.

Definition 2.1. *The flag factors satisfying the conditions in Theorem* 2.2 (*resp.* 2.1) *are called AUN-flag factors of type one* (*resp. two*). *The corresponding flag transforms are called AUN-flag transforms of type one* (*resp. two*).

Clearly, the necessary condition that there exist the AUN-flag factors of type one (resp. two) is that $\underline{\ker}A^\varphi_{\bar z} \setminus \underline{\ker}T_n \neq \emptyset$ (resp. $\underline{\ker}A^\varphi_z \setminus \underline{\ker}T_0 \neq \emptyset$). The unitons (resp. extended solutions) satisfying this necessary condition are called AUN-unitons (resp. extended solutions) of type one (resp. two); otherwise, are called the End-unitons. It follows from [1, 9] that any n-uniton can be obtained from an AUN-$(n-1)$-uniton of type one (resp. two) by the AUN-flag transform of type one (resp. two). This implies that there is a semi-ordering relation on the set

$$\mathcal{U} = \{\text{unitons}\} = \{\text{harmonic maps with finite uniton number}\},$$

of which the maximum elements are the End-unitons. The 0-uniton is its minimum element. Thus, if one can construct all of AUN-flag factors for every AUN-uniton φ, then any element of $\mathcal{U}$ can be obtained from a 0-uniton by a finite number of AUN-flag transforms. We now give a purely algebraic method to construct AUN-flag factors.

Lemma 2.1. *Let $\Phi_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha$ be the extended solution of a uniton $\varphi : \Omega \to U(N)$. Assume that $\mathrm{rank} A_{\bar z}^{\varphi} = r$. Then a meromorphic basis $\{b_1, \cdots, b_{N-r}\}$ of $(\underline{\ker} A_{\bar z}^{\varphi}, D^{\varphi})$ can be obtained by solving the system of linear algebraic equations, so that $\{b_1, \cdots, b_s\}$ $(1 \le s \le N-r)$ is the local meromorphic basis of $(\underline{\ker} A_{\bar z}^{\varphi} \cap \underline{\ker} T_0, D^{\varphi})$. If φ is the AUN-uniton of type two, i.e., $s < N-r$, then*

$$\underline{\pi} = \mathrm{span}\{b_{s+1}, \cdots, b_{s+k}\} \qquad (k \le N-r-s)$$

is an AUN-flag factor of type two with rank k.

Proof. Noting that $\underline{\mathrm{Im}} A_{\bar z}^{\varphi}$, $\underline{\ker} A_{\bar z}^{\varphi}$ and $\underline{\ker} T_0$ are global holomorphic subbundles of $(\underline{\mathbb{C}}^N, D^{\varphi})$, we see that r and s are constants. Since $\underline{\mathrm{Im}} T_n^* \subset \underline{\ker} A_{\bar z}^{\varphi} \cap \underline{\ker} T_0$, then $s > 0$. Let $\{e_i\}$ be a meromorphic basis of $(\underline{\mathbb{C}}^N, D^{\varphi})$ (see Lemma 1.6), and $v = \sum_j x_j e_j$. By Lemma 1.2 and Lemma 1.3, we see that v is a meromorphic section of $(\underline{\ker} A_{\bar z}^{\varphi} \cap \underline{\ker} T_0, D^{\varphi})$ if and only if $\{x_i\}$ is a meromorphic solution to the system of linear algebraic equations on the field $\mathcal{F}$:

$$\sum_i \lambda_{ij} x_i = 0,$$
$$\sum_i \mu_{ij} x_i = 0.$$

Thus, solving the system gives the required meromorphic basis $\{b_1, \cdots, b_{N-r}\}$. If $s < N-r$, then the projection π corresponding to $\underline{\pi} = \mathrm{span}\{b_{s+1}, \cdots, b_{s+k}\}$ satisfies the conditions (a) and (b) of Theorem 2.1.

Let $GL(n, \mathcal{F})$ be the group of $n \times n$ invertible matrices on the field $\mathcal{F}$. Set

$$G(N-r; s) = \left\{ \begin{pmatrix} P_1 & * \\ 0 & P_2 \end{pmatrix} \in GL(N-r, \mathcal{F}) : P_1 \in GL(s, \mathcal{F}) \right\},$$
$$K_1(k) = \left\{ \begin{pmatrix} P_1 & 0 & * \\ 0 & P_2 & * \\ 0 & 0 & P_3 \end{pmatrix} \in GL(N-r, \mathcal{F}) : P_1 \in GL(s, \mathcal{F}), P_2 \in GL(k, \mathcal{F}) \right\}$$

for $k \le N-r-s$. From Lemma 2.1 we have immediately

Proposition 2.1. *Let $\Phi_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha$ be the extended solution of a uniton $\varphi : \Omega \to U(N)$. Then there is a bijection from the set*

$$\mathbf{f}_2(\Phi_\lambda; k) = \{\textit{AUN-flag factors of } \varphi \textit{ of type two with rank } k\}$$

to $GL(N-r, s)/K_1(k)$.

By the similar way, we can construct AUN-flag factors of type one with rank k.

Lemma 2.2. *Let $\Phi_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha$ be the extended solution of a uniton $\varphi : \Omega \to U(N)$. Assume that* $\mathrm{rank} A_z^\varphi = r$. *Then another meromorphic basis $\{b_i\}$ of $(\underline{\mathbb{C}}^N, D^\varphi)$ can be obtained by algebraic operations, so that $\{b_1, \cdots, b_r\}$ is a meromorphic basis of $(\underline{\mathrm{Im}} A_z^\varphi, D^\varphi)$ and $\{b_1, \cdots, b_{r+t}\}$ $(0 \le t < N - r)$ is a meromorphic basis of $(\underline{\mathrm{Im}} A_z^\varphi + \underline{\mathrm{Im}} T_n^*, D^\varphi)$. If φ is an AUN-uniton of type one, i.e., $t > 0$, then*

$$\underline{\pi} = \mathrm{span}\{b_1, \cdots, b_{r+t+k-N}, b_{r+t+1}, \cdots, b_N\} \qquad (N - t \le k < N)$$

is an AUN-flag factor of type one with rank k.

Proof. Since r is constant and $\underline{\mathrm{Im}} A_z^\varphi + \underline{\mathrm{Im}} T_n^*$ is a global holomorphic subbundle of $(\underline{\mathbb{C}}^N, D^\varphi)$, t is constant. Since $\underline{\mathrm{Im}} T_0^* \subseteq \underline{\ker} A_{\bar z}^\varphi \cap \underline{\ker} T_n$, $r + t = \dim(\underline{\mathrm{Im}} A_z^\varphi + \underline{\mathrm{Im}} T_n^*) < N$. Let $\{e_i^0\}$ be a meromorphic basis of $(\underline{\mathbb{C}}^N, d)$, from which we can construct a meromorphic basis $\{e_i\}$ of $(\underline{\mathbb{C}}^N, D^\varphi)$ according to Lemma 1.6. By Lemmas 1.2 and 1.3, we see that

$$a_i = A_z^\varphi(e_i) = \sum_j \lambda_{ij} e_j \quad \text{and} \quad d_i = T_n^*(e_i^0) = \sum_j \gamma_{ij} e_j$$

are meromorphic sections of $(\underline{\mathbb{C}}^N, D^\varphi)$. In $\{a_i\}$ and $\{d_i\}$ we choose r linearly independent sections $\{b_1, \cdots, b_r\}$ and t linearly independent sections $\{b_{r+1}, \cdots, b_{r+t}\}$ respectively, so that $\{b_1, \cdots, b_{r+t}\}$ are linearly independent. Thus, by complementing $\{b_{r+t+1}, \cdots, b_N\}$, we can obtain another meromorphic basis $\{b_i\}$ of $(\underline{\mathbb{C}}^N, D^\varphi)$ as required. If $t > 0$ and $N - t \le k < N$ for some k, then

$$\underline{\pi} = \mathrm{span}\{b_1, \cdots, b_{r+t+k-N}, b_{r+t+1}, \cdots, b_N\}$$

is a holomorphic subbundle of $(\underline{\mathbb{C}}^N, D^\varphi)$, which contains $\underline{\mathrm{Im}} A_z^\varphi$ and satisfies $\underline{\pi} + \underline{\mathrm{Im}} T_n^* = \underline{\mathbb{C}}^N$. Hence, π satisfies the conditions (a) and (b) of Theorem 2.2.

Set

$$G(N; r, t) = \left\{ \begin{pmatrix} P_1 & * & * \\ 0 & P_2 & * \\ 0 & 0 & P_3 \end{pmatrix} \in GL(N, \mathcal{F}) : P_1 \in GL(r, \mathcal{F}), P_2 \in GL(t, \mathcal{F}) \right\},$$

$$K_2(k) = \left\{ \begin{pmatrix} P_1 & * & * & * \\ 0 & P_2 & * & * \\ 0 & 0 & P_3 & 0 \\ 0 & 0 & 0 & P_4 \end{pmatrix} \in GL(N, \mathcal{F}) : \begin{array}{l} P_1 \in GL(r, \mathcal{F}), \\ P_2 \in GL(k + t - N, \mathcal{F}), \\ P_3 \in GL(N - k, \mathcal{F}) \end{array} \right\}$$

for $N - t \le k < N$. By Lemma 2.2, we have

Proposition 2.2. *Let $\Phi_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha$ be the extended solution of a uniton $\varphi : \Omega \to U(N)$. Then there is a bijection from the set*

$$\mathbf{f}_1(\Phi_\lambda; k) = \{\textit{AUN-flag factors of } \varphi \textit{ of type one with rank } k\}$$

to $G(N; r, t)/K_2(k)$.

If φ is a 0-uniton, then there is a bijection from the set of all flag factors of φ with rank k, i.e., the set of all k-dimensional holomorphic subbundles of $(\underline{\mathbb{C}}^N, d)$ to

$$G(N-r;s)/K_1(k) = G(N;r,t)/K_2(k) = GL(N;\mathcal{F})/G(N;k).$$

Also, there is a bijection from the set of AUN-1-unitons of type one (resp. two) to the set

$$\{\underline{\beta} : \ \underline{\beta} \text{ is the full holomorphic subbundle of } (\underline{\mathbb{C}}^N, d), \ \operatorname{rank}(\partial\beta) < \operatorname{rank}\beta^\perp\}$$

(resp. the set

$$\{\underline{\beta}^\perp : \ \underline{\beta}^\perp \text{ is the full anti-holomorphic subbundle of } (\underline{\mathbb{C}}^N, d), \ \operatorname{rank}(\partial\beta) < \operatorname{rank}\beta\}).$$

By Lemmas 1.1 and 1.6, Theorems 2.1 and 2.2, Lemmas 2.1 and 2.2, we have proved the following

Theorem 2.3. *Let $\varphi : \Omega \to U(N)$ be an AUN-uniton and $\mathfrak{m}(\varphi) = n \geq 0$. Then all of two kinds of AUN-flag factors π can be constructed by finite algebraic operations and Cauchy's integral transforms to solve the $\bar{\partial}$-problem, so that $\widetilde{\varphi} = \varphi(\pi - \pi^\perp)$ is a minimal $(n+1)$-uniton. Hence, any n-uniton can be obtained from a 0-uniton by the purely algebraic algorithms and Cauchy's integral transforms to solve the $\bar{\partial}$-problem via two different ways.*

§ 3 . Commutative Extended Solutions

Let $\Phi_\lambda = \sum\limits_{\alpha=0}^{n} T_\alpha \lambda^\alpha$ be the extended solution of a harmonic map $\varphi : \Omega \to U(N)$. It follows from (1.7) and (1.8) that

$$T_0 T_n^* = 0, \quad A_z^\varphi T_n^* = 0, \quad T_0 A_z^\varphi = 0.$$

Set

$$\alpha : \underline{\mathbb{C}}^N \to \underline{\ker} A_z^\varphi, \quad \rho : \underline{\mathbb{C}}^N \to \underline{\mathrm{Im}} A_z^\varphi, \quad \beta : \underline{\mathbb{C}}^N \to \underline{\mathrm{Im}} T_0^*, \quad \sigma : \underline{\mathbb{C}}^N \to \underline{\mathrm{Im}} T_n^*, \tag{3.1}$$

which are Hermitian projections. Thus, we have

$$\alpha\sigma = \sigma, \quad \rho\beta = 0, \quad \beta\sigma = 0. \tag{3.2}$$

Definition 3.1. *Φ_λ is called the $\underline{\ker} A_z^\varphi$-commutative extended solution if $\alpha\beta = \beta\alpha$; the $\underline{\mathrm{Im}} A_z^\varphi$-commutative extended solution if $\rho\sigma = \sigma\rho$.*

For any extended 1-uniton $\Phi_\lambda = \pi + \lambda\pi^\perp$ where π is a Hermitian projection, we have

$$\begin{aligned} \underline{\rho} &= \underline{\mathrm{Im}} A_z^\varphi = \underline{\mathrm{Im}} \partial\pi^\perp \subset \underline{\pi}^\perp = \underline{\sigma}, \\ \underline{\alpha}^\perp &= \underline{\mathrm{Im}} A_{\bar{z}}^\varphi = \underline{\mathrm{Im}} \bar{\partial}\pi \subset \underline{\pi} = \underline{\beta}, \end{aligned}$$

from which it follows that

$$\alpha\beta = \beta\alpha, \quad \rho\sigma = \sigma\rho.$$

Hence, we have the following

Lemma 3.1. *Any extended 1-uniton is both $\underline{\ker} A_z^\varphi$-commutative and $\underline{\mathrm{Im}} A_z^\varphi$-commutative.*

Next, we may give more succinct global AUN-flag factors for commutative extended solutions.

Lemma 3.2. *If* $\Phi_\lambda = \sum_{\alpha=0}^{n} T_\alpha \lambda^\alpha$ *is the* $\underline{\ker} A_z^\varphi$*-commutative AUN-extended solution of type two, then* $\underline{\pi} = \underline{\ker} A_{\bar z}^\varphi \cap \underline{\mathrm{Im}} T_0^*$ *is an AUN-flag factor of type two.*

Proof. It is sufficient to prove that $\underline{\pi}^\perp = \underline{\mathrm{Im}} A_{\bar z}^\varphi + \underline{\ker} A_z^\varphi \cap \underline{\ker} T_0$ is an anti-holomorphic subbundle of $(\underline{\mathbb{C}}^N, D^\varphi)$. Since $\underline{\mathrm{Im}} A_{\bar z}^\varphi$ is an anti-holomorphic subbundle of $(\underline{\mathbb{C}}^N, D^\varphi)$, it is sufficient to prove that ∂v is a section of $\underline{\pi}^\perp$ when v is a section of $\underline{\ker} A_z^\varphi \cap \underline{\ker} T_0$. Assume that $A_z^\varphi v = 0$ and $T_0 v = 0$. Then we have

$$0 = \partial(T_0 v) = -T_1 A_z^\varphi v + T_0 \partial v = T_0 \partial v,$$

which implies that ∂v is a section of $\underline{\ker} T_0 \subset \underline{\pi}^\perp$.

In the same way we can prove the following

Lemma 3.3. *If* Φ_λ *is the* $\underline{\mathrm{Im}} A_z^\varphi$*-commutative AUN-extended solution of type one, then* $\underline{\pi} = \underline{\mathrm{Im}} A_z^\varphi + \underline{\ker} A_{\bar z}^\varphi \cap \underline{\ker} T_n$ *is an AUN-flag factor of type one.*

Finally, we have

Theorem 3.1. *Let* Φ_λ *be the* $\underline{\mathrm{Im}} A_z^\varphi$*-commutative* (*resp.* $\underline{\ker} A_z^\varphi$*-commutative*) *AUN-extended n-uniton. Assume π is given as in Lemma* 3.3 (*resp. Lemma* 3.2). *Then* $\widetilde{\Phi}_\lambda = \Phi_\lambda(\pi + \lambda\pi^\perp)$ *is the* $\underline{\mathrm{Im}} A_z^\varphi$*-commutative* (*resp.* $\underline{\ker} A_z^\varphi$*-commutative*) *extended* $(n+1)$*-uniton. Moreover, if* $\mathrm{rank}(\partial\pi) <$ *rank*$\pi^\perp$ (*resp.* $\mathrm{rank}\pi$), *then* $\widetilde{\Phi}_\lambda$ *is an AUN-extended* $(n+1)$*-unoton.*

Proof. Let Φ_λ be the $\underline{\mathrm{Im}} A_z^\varphi$-commutative extended n-unoton of a harmonic map $\varphi : \Omega \to U(N)$, and $\pi = \rho + \rho^\perp \sigma^\perp$. Then we see from Lemma 3.3 and Theorem 2.2 that $\widetilde{\Phi}_\lambda$ is an extended $(n+1)$-unoton and $\underline{\pi}^\perp$ is an anti-holomorphic subbundle of $(\underline{\mathbb{C}}^N, D^\varphi)$, i.e.,

$$\pi(A_z^\varphi \pi^\perp + \partial\pi^\perp) = 0, \quad \underline{\pi}^\perp = \underline{\rho}^\perp \cap \underline{\sigma} \subset \underline{\sigma}.$$

Thus, it follows from (3.1) and (3.2) that $A_z^\varphi \pi^\perp = 0$, i.e., $\pi\partial\pi^\perp = 0$. Set

$$\underline{\rho}' = \underline{\mathrm{Im}} A_z^{\widetilde{\varphi}} = \underline{\mathrm{Im}}(A_z^\varphi + \partial\pi^\perp), \quad \underline{\sigma}' = \underline{\mathrm{Im}} \widetilde{T}_{n+1}^* = \underline{\pi}^\perp.$$

Since $\underline{\sigma}^\perp$ is an anti-holomorphic subbundle of $(\underline{\mathbb{C}}^N, D^\varphi)$, then $\sigma(A_z^\varphi \sigma^\perp + \partial\sigma^\perp) = 0$, i.e., $\underline{\mathrm{Im}}\partial\sigma^\perp \subset \underline{\rho^\perp\sigma^\perp} + \underline{\rho} = \underline{\pi}$, which implies that $\pi^\perp\partial\sigma^\perp = 0$. Thus, we have

$$A_z^{\widetilde{\varphi}}\sigma = (\partial\pi^\perp)\sigma = \partial(\pi^\perp\sigma) = \partial\pi^\perp, \quad A_z^{\widetilde{\varphi}}\sigma^\perp = A_z^\varphi\sigma^\perp + (\partial\pi^\perp)\sigma^\perp = A_z^\varphi,$$

which implies that $\underline{\rho}' = \underline{\mathrm{Im}} A_z^\varphi + \underline{\mathrm{Im}}\partial\pi^\perp$. Since $\underline{\mathrm{Im}}\partial\pi^\perp \subset \underline{\pi}^\perp = \underline{\sigma}'$ and $\underline{\mathrm{Im}} A_z^\varphi \subset \underline{\pi}$, it is clear that $\rho'\sigma' = \sigma'\rho'$, i.e., $\widetilde{\Phi}_\lambda$ is $\underline{\mathrm{Im}} A_z^\varphi$-commutative. Moreover, if $\mathrm{rank}(\partial\pi^\perp) < \mathrm{rank}\pi^\perp$, then $\underline{\ker} A_z^{\widetilde{\varphi}} \setminus \underline{\ker} \widetilde{T}_{n+1}^* \neq \emptyset$, i.e., $\widetilde{\Phi}_\lambda$ is an AUN-extended solution.

Remark. By using Lemma 3.1 and Theorem 3.1 and repeating the flag transforms given as in Lemma 3.2 (resp. Lemma 3.3), we can obtain a series of unitons $\varphi = \varphi_1, \varphi_2, \cdots, \varphi_n$ $(1 < n \leq N-1)$ from any given AUN-1-uniton φ, such that $\mathfrak{m}(\varphi_k) = k$ $(k = 1, 2, \cdots, n)$ and φ_n is the End-uniton.

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