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INJECTIVE PRECOVERS AND MODULES OF GENERALIZED INVERSE POLYNOMIALS**

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Abstract

This paper is motivated by S. Park [10] in which the injective cover of left $R[x]$-module $M[x^{-1}]$ of inverse polynomials over a left R-module M was discussed. The author considers the $\mathbf{\Omega}$-covers of modules and shows that if $\eta : P \longrightarrow M$ is an $\mathbf{\Omega}$-cover of M, then $[\eta^{S,\leq}] : [P^{S,\leq}] \longrightarrow [M^{S,\leq}]$ is an $[\mathbf{\Omega}^{S,\leq}]$-cover of left $[[R^{S,\leq}]]$-module $[M^{S,\leq}]$, where $\mathbf{\Omega}$ is a class of left R-modules and $[M^{S,\leq}]$ is the left $[[R^{S,\leq}]]$-module of generalized inverse polynomials over a left R-module M. Also some properties of the injective cover of left $[[R^{S,\leq}]]$-module $[M^{S,\leq}]$ are discussed.

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Let $(S, \leq)$ be an ordered set. Recall that $(S, \leq)$ is artinian if every strictly decreasing sequence of elements of S is finite, and that $(S, \leq)$ is narrow if every subset of pairwise order-incomparable elements of S is finite. Let S be a commutative monoid. Unless stated otherwise, the operation of S shall be denoted additively, and the neutral element by 0. The following definition is due to [11].

Let $(S, \leq)$ be a strictly ordered monoid (that is, $(S, \leq)$ is an ordered monoid satisfying the condition that, if $s, s', t \in S$ and $s < s'$, then $s + t < s' + t$), and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f : S \longrightarrow R$ such that

$$\operatorname{supp}(f) = \{s \in S \mid f(s) \neq 0\}$$

is artinian and narrow. With pointwise addition, $[[R^{S,\leq}]]$ is an abelian additive group. For every $s \in S$ and $f, g \in [[R^{S,\leq}]]$, let

$$X_s(f, g) = \{(u, v) \in S \times S \mid s = u + v, f(u) \neq 0, g(v) \neq 0\}.$$

It follows from [11, 4.1] that $X_s(f, g)$ is finite. This fact allows to define the operation of convolution:

$$(fg)(s) = \sum_{(u,v) \in X_s(f,g)} f(u)g(v).$$

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With this operation, and pointwise addition, $[[R^{S,\leq}]]$ becomes a ring, which is called the ring of generalized power series. The elements of $[[R^{S,\leq}]]$ are called generalized power series with coefficients in R and exponents in S.

For example, if $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[[R^{\mathbb{N}\cup\{0\},\leq}]] \cong R[[x]]$, the usual ring of power series. If S is a commutative monoid and $\leq$ is the trivial order, then $[[R^{S,\leq}]] = R[S]$, the monoid-ring of S over R. Further examples are given in [12, 13]. Some results on the rings of generalized power series are given in [11–14, 8].

If M is a left R-module, we let $[M^{S,\leq}]$ be the set of all maps $\phi : S \longrightarrow M$ such that the set

$$\mathrm{supp}(\phi) = \{s \in S \mid \phi(s) \neq 0\}$$

is finite. Now $[M^{S,\leq}]$ can be turned into a left $[[R^{S,\leq}]]$-module under some additional conditions. The addition in $[M^{S,\leq}]$ is componentwise and the scalar multiplication is defined as follows:

$$(f\phi)(s) = \sum_{t\in S} f(t)\phi(s+t) \qquad \text{for every} \quad s \in S,$$

where $f \in [[R^{S,\leq}]]$, and $\phi \in [M^{S,\leq}]$. From [8] it follows that if $(S, \leq)$ is a strictly totally ordered monoid and $\leq$ is also artinian, then $[M^{S,\leq}]$ becomes a left $[[R^{S,\leq}]]$-module.

For example, if $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual natural order, then

$$[M^{\mathbb{N}\cup\{0\},\leq}] \cong M[x^{-1}],$$

the usual left $R[[x]]$-module discussed in [10].

Let $r \in R$. Define a mapping $c_r \in [[R^{S,\leq}]]$ as follows:

$$c_r(0) = r, \quad c_r(s) = 0, \quad 0 \neq s \in S.$$

Let $s \in S$. Define a mapping $e_s \in [[R^{S,\leq}]]$ as follows:

$$e_s(s) = 1, \quad e_s(t) = 0, \quad s \neq t \in S.$$

Let $\mathbf{\Omega}$ be a class of left R-modules. We assume that $\mathbf{\Omega}$ is closed under isomorphisms. According to [4] (or [15], or [3]), an $\mathbf{\Omega}$-cover of a left R-module M is a linear map $\eta : P \longrightarrow M$ with P in $\mathbf{\Omega}$ such that

(1) any diagram with $Q \in \mathbf{\Omega}$,

$$\begin{tikzcd} Q \arrow[d, "\delta"] \arrow[dr] & \\ P \arrow[r, "\eta"] & M \end{tikzcd}$$

can be completed by a linear map $\delta : Q \longrightarrow P$ and

(2) the diagram

$$\begin{tikzcd} P \arrow[d] \arrow[dr, "\eta"] & \\ P \arrow[r, "\eta"] & M \end{tikzcd}$$

can only be completed by an automorphism of P.

If (1) holds (and perhaps not (2)), then $\eta : P \longrightarrow M$ is called an $\mathbf{\Omega}$-precover of M.

Note that if an $\mathbf{\Omega}$-cover exists, then it is unique up to isomorphism.

If $\mathbf{\Omega}$ is the class of all injective left R-modules (the class of all flat left R-modules), then an $\mathbf{\Omega}$-cover is called an injective cover (a flat cover, respectively) and an $\mathbf{\Omega}$-precover is called an injective precover (a flat precover, respectively). If $\mathbf{\Omega}$ is the class of all torsion-free injective left R-modules, then an $\mathbf{\Omega}$-cover is called a torsion-free injective cover (see [1]). If $\mathbf{\Omega}$ is the class of all left R-modules of finite injective dimension, or equivalently, the class of all left R-modules of finite projective dimension (see [7]), then $\mathbf{\Omega}$-covers exist by [5]. Enochs and Jenda in [4], Enochs and Xu in [6] and Auslander and Buchweitz in [2] studied $\mathbf{\Omega}$-covers when $\mathbf{\Omega}$ is other classes of left R-modules.

We shall henceforth assume that $(S, \leq)$ is a strictly totally ordered monoid which is also artinian. Then, from [9], for any $s \in S$, we have $0 \leq s$. This result will be often used throughout the rest of this paper.

Let M be a left R-module. For any $t \in S$ and $m \in M$, define $\phi_{tm} \in [M^{S,\leq}]$ as follows:

$$\phi_{tm}(x) = \begin{cases} m, & x = t, \\ 0, & x \neq t. \end{cases}$$

Denote $G_t = \{\phi_{tm} \,|\, m \in M\}$. Then G_t is a left R-module by the left R-action: $r \cdot \phi_{tm} = c_r \phi_{tm}$. Clearly there exists an isomorphism of left R-modules $\lambda_t : M \longrightarrow G_t$ via $\lambda_t(m) = \phi_{tm}$. The following lemma appeared in [8, Lemma 2.3].

Lemma 1. *Let M, N be left R-modules. Then there exists an isomorphism of abelian groups*

$$F : \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}], [N^{S,\leq}]) \cong [[\mathrm{Hom}_R(M, N)^{S,\leq}]].$$

We note that F is defined via $F(\alpha) : S \longrightarrow \mathrm{Hom}_R(M, N)$ as

$$F(\alpha)(s) = \beta\alpha\lambda_s$$

for any $\alpha \in \mathrm{Hom}_{[[R^{S,\leq}]]}([M^{S,\leq}], [N^{S,\leq}])$, where $\beta : [N^{S,\leq}] \longrightarrow N$ is an R-homomorphism defined via $\beta(\phi) = \phi(0)$ for any $\phi \in [N^{S,\leq}]$. By the proof of Lemma 1 (see [8, Lemma 2.3]), for any $h \in [[\mathrm{Hom}_R(M, N)^{S,\leq}]]$, $F^{-1}(h) : [M^{S,\leq}] \longrightarrow [N^{S,\leq}]$ is defined via

$$\begin{aligned} F^{-1}(h)(\phi) : S &\longrightarrow N \\ s &\longmapsto \sum_{u \in S} h(u)(\phi(s + u)) \end{aligned}$$

for every $\phi \in [M^{S,\leq}]$.

Let P, M be left R-modules. If $\eta \in \mathrm{Hom}_R(P, M)$, then we define $g : S \longrightarrow \mathrm{Hom}_R(P, M)$ via

$$g(s) = \begin{cases} \eta, & s = 0, \\ 0, & s \neq 0. \end{cases}$$

By Lemma 1, there exists $\alpha \in \mathrm{Hom}_{[[R^{S,\leq}]]}([P^{S,\leq}], [M^{S,\leq}])$ such that $F(\alpha) = g$. We denote α by $[\eta^{S,\leq}]$.

Set $[\mathbf{\Omega}^{S,\leq}] = \{[N^{S,\leq}] | N \in \mathbf{\Omega}\}$.

Proposition 1. *Let M be a left R-module. Suppose that $\eta : P \longrightarrow M$ is an $\mathbf{\Omega}$-precover. Then $[\eta^{S,\leq}] : [P^{S,\leq}] \longrightarrow [M^{S,\leq}]$ is an $[\mathbf{\Omega}^{S,\leq}]$-precover of left $[[R^{S,\leq}]]$-module $[M^{S,\leq}]$.*

Proof. By Lemma 1, $[\eta^{S,\leq}]$ is an $[[R^{S,\leq}]]$-homomorphism. Let Q be in $\mathbf{\Omega}$ and $\alpha : [Q^{S,\leq}] \longrightarrow [M^{S,\leq}]$ an $[[R^{S,\leq}]]$-homomorphism. For every $s \in S$, $F(\alpha)(s) \in \mathrm{Hom}_R(Q, M)$.

Since $\eta : P \longrightarrow M$ is an $\mathbf{\Omega}$-precover, there exists an R-homomorphism $\psi_s : Q \longrightarrow P$ such that

$$\eta\psi_s = F(\alpha)(s).$$

Define $h : S \longrightarrow \mathrm{Hom}_R(Q, P)$ via $h(s) = \psi_s$. Clearly $h \in [[\mathrm{Hom}_R(Q, P)^{S,\leq}]]$. By Lemma 1,

$$F^{-1}(h) \in \mathrm{Hom}_{[[R^{S,\leq}]]}([Q^{S,\leq}], [P^{S,\leq}]).$$

Now for any $\phi \in [Q^{S,\leq}]$ and any $s \in S$,

$$\begin{aligned}
&(([\eta^{S,\leq}]F^{-1}(h))(\phi))(s)\\
&= ([\eta^{S,\leq}](F^{-1}(h)(\phi)))(s)\\
&= (F^{-1}(g)(F^{-1}(h)(\phi)))(s)\\
&= \sum_{u\in S} g(u)((F^{-1}(h)(\phi))(s+u))\\
&= \sum_{u\in S} g(u)\Big(\sum_{v\in S} h(v)(\phi(s+u+v))\Big)\\
&= \sum_{u\in S} g(u)\Big(\sum_{v\in S} \psi_v(\phi(s+u+v))\Big)\\
&= \eta\Big(\sum_{v\in S} \psi_v(\phi(s+v))\Big) = \sum_{v\in S} \eta\psi_v(\phi(s+v))\\
&= \sum_{v\in S} F(\alpha)(v)(\phi(s+v)) = (F^{-1}(F(\alpha)))(\phi)(s)\\
&= \alpha(\phi)(s)
\end{aligned}$$

which implies that the following diagram

$$\begin{array}{ccc}
[Q^{S,\leq}] & & \\
\downarrow^{F^{-1}(h)} & \searrow^{\alpha} & \\
[P^{S,\leq}] & \xrightarrow{[\eta^{S,\leq}]} & [M^{S,\leq}]
\end{array}$$

commutes, and thus the result follows.

For every $0 \neq \phi \in [M^{S,\leq}]$, we denote by $\sigma(\phi)$ the maximal element in $\mathrm{supp}(\phi)$.

Theorem 1. *Let $\eta : P \longrightarrow M$ be an $\mathbf{\Omega}$-cover. Then $[\eta^{S,\leq}] : [P^{S,\leq}] \longrightarrow [M^{S,\leq}]$ is an $[\mathbf{\Omega}^{S,\leq}]$-cover of left $[[R^{S,\leq}]]$-module $[M^{S,\leq}]$.*

Proof. By Proposition 1, $[\eta^{S,\leq}] : [P^{S,\leq}] \longrightarrow [M^{S,\leq}]$ is an $[\mathbf{\Omega}^{S,\leq}]$-precover of $[M^{S,\leq}]$. Suppose that $\alpha : [P^{S,\leq}] \longrightarrow [P^{S,\leq}]$ is an $[[R^{S,\leq}]]$-homomorphism which makes the diagram

$$\begin{array}{ccc}
[P^{S,\leq}] & & \\
\downarrow^{\alpha} & \searrow^{[\eta^{S,\leq}]} & \\
[P^{S,\leq}] & \xrightarrow{[\eta^{S,\leq}]} & [M^{S,\leq}]
\end{array}$$

commutative. We will show that α is an automorphism. Suppose that $h \in [[\mathrm{Hom}_R(P, P)^{S,\leq}]]$

is such that $F(\alpha) = h$. For any $m \in P$,

$$\begin{aligned}
(\eta h(0))(m) &= \eta(h(0)(m)) = \eta\Big(\sum_{v\in S} h(v)(\phi_{0m}(v))\Big)\\
&= \eta(\alpha(\phi_{0m})(0)) = \sum_{u\in S} F([\eta^{S,\leq}])(u)(\alpha(\phi_{0m})(u))\\
&= ([\eta^{S,\leq}](\alpha(\phi_{0m})))(0) = ([\eta^{S,\leq}]\alpha(\phi_{0m}))(0)\\
&= [\eta^{S,\leq}](\phi_{0m})(0) = \sum_{x\in S} F([\eta^{S,\leq}])(x)(\phi_{0m}(x))\\
&= \eta(\phi_{0m}(0)) = \eta(m).
\end{aligned}$$

Thus $\eta h(0) = \eta$. Since $\eta : P \longrightarrow M$ is an $\mathbf{\Omega}$-cover, it follows that $h(0)$ is an automorphism.

Let $\phi \in$Ker(α). If $\phi \neq 0$, then supp(ϕ) is not empty. Now suppose that $\sigma(\phi) = u$. Then

$$h(0)(\phi(u)) = \sum_{x\in S} h(x)(\phi(u+x)) = \alpha(\phi)(u) = 0,$$

which implies that $\phi(u) = 0$, a contradiction. Thus α is one to one.

Let $\phi \in [P^{S,\leq}]$ and $\phi \neq 0$. Suppose $\sigma(\phi) = 0$. Then $\phi(u) = 0$ when $u > 0$. Set $m = h(0)^{-1}(\phi(0))$, and $\psi = \phi_{0m} \in [P^{S,\leq}]$. Then

$$\begin{aligned}
\alpha(\psi)(s) &= \sum_{u\in S} h(u)(\psi(s+u))\\
&= \begin{cases} h(0)(\psi(0)), & s = 0,\\ 0, & s \neq 0\end{cases}\\
&= \phi(s),
\end{aligned}$$

which implies that $\alpha(\psi) = \phi$.

Suppose that $u \in S$ is such that, for any $\phi \in [P^{S,\leq}]$ with $\sigma(\phi) < u$, there exists $\psi \in [P^{S,\leq}]$ such that $\alpha(\psi) = \phi$. Now suppose that ϕ is in $[P^{S,\leq}]$ such that $\sigma(\phi) = u$. We will show that there exists $\psi \in [P^{S,\leq}]$ such that $\alpha(\psi) = \phi$. Set $m = h(0)^{-1}(\phi(u))$, and $\psi_1 = \phi_{um} \in [P^{S,\leq}]$. Then for any $s \geq u$, we have

$$\alpha(\psi_1)(s) = \sum_{v\in S} h(v)(\psi_1(s+v)) = \begin{cases} h(0)(\psi_1(u)) = \phi(u), & s = u,\\ 0, & s > u.\end{cases}$$

Thus $(\phi - \alpha(\psi_1))(s) = 0$ when $s \geq u$. If $\phi \neq \alpha(\psi_1)$, then supp$(\phi - \alpha(\psi_1))$ is not empty. Clearly $\sigma(\phi - \alpha(\psi_1)) < u$. Thus, by hypothesis, there exists $\psi_2 \in [P^{S,\leq}]$ such that $\alpha(\psi_2) = \phi - \alpha(\psi_1)$. Now $\phi = \alpha(\psi_1 + \psi_2)$. This means that α is onto. Hence α is an automorphism.

If $g : M_1 \longrightarrow M_2$ is a morphism and $\eta_1 : F_1 \longrightarrow M_1$, $\eta_2 : F_2 \longrightarrow M_2$ are $\mathbf{\Omega}$-precovers, then the diagram

$$\begin{tikzcd}
F_1 \arrow[r, "h"] \arrow[d, "\eta_1"'] & F_2 \arrow[d, "\eta_2"'] \\
M_1 \arrow[r, "g"] & M_2
\end{tikzcd}$$

can be completed to a commutative diagram. According to [3], h is called a lifting of g (relative to the two precovers). A morphism $g : M_1 \longrightarrow M_2$ is said to be an $\mathbf{\Omega}$-covering if

M_1 and M_2 have $\mathbf{\Omega}$-covers $\eta_1 : F_1 \longrightarrow M_1$ and $\eta_2 : F_2 \longrightarrow M_2$ and some lifting $h : F_1 \longrightarrow F_2$ is an isomorphism. Equivalently, $g : M_1 \longrightarrow M_2$ is an $\mathbf{\Omega}$-covering morphism if and only if M_1 and M_2 have $\mathbf{\Omega}$-covers $\eta_1 : F_1 \longrightarrow M_1$ and $\eta_2 : F_2 \longrightarrow M_2$ and every lifting $h : F_1 \longrightarrow F_2$ is an isomorphism.

Corollary 1. *If $g : M_1 \longrightarrow M_2$ is an $\mathbf{\Omega}$-covering morphism, then $[g^{S,\leq}] : [M_1^{S,\leq}] \longrightarrow [M_2^{S,\leq}]$ is an $[\mathbf{\Omega}^{S,\leq}]$-covering $[[R^{S,\leq}]]$-morphism.*

Proof. It was proved in [9, Lemma 5] that the functor $[(-)^{S,\leq}]$ is exact. Now the result follows from Theorem 1.

It was showed in [10] that if $\eta : P \longrightarrow M$ is an injective cover of M, then $\eta[x^{-1}] : P[x^{-1}] \longrightarrow M[x^{-1}]$ is an $\mathbf{\Omega}[x^{-1}]$-cover of left $R[x]$-module $M[x^{-1}]$, where

$$\begin{aligned}\eta[x^{-1}] &= \eta + 0\cdot x + 0\cdot x^2 + \cdots \\ &\in \mathrm{Hom}_R(P, M)[[x]] \cong \mathrm{Hom}_{R[x]}(P[x^{-1}], M[x^{-1}]), \\ \mathbf{\Omega}[x^{-1}] &= \{P[x^{-1}] | P \in \mathbf{\Omega}\}.\end{aligned}$$

From Theorem 1, we have

Corollary 2. *Let $\eta : P \longrightarrow M$ be an $\mathbf{\Omega}$-cover. Then $\eta[x^{-1}] : P[x^{-1}] \longrightarrow M[x^{-1}]$ is an $\mathbf{\Omega}[x^{-1}]$-cover of left $R[[x]]$-module $M[x^{-1}]$, where $\mathbf{\Omega}[x^{-1}] = \{P[x^{-1}] \,|\, P \in \mathbf{\Omega}\}$.*

Corollary 3. *Let R be a ring. Denote*

$$\mathbf{\Gamma}_1 = \{[E^{S,\leq}] \,|\, E \text{ is a quasi-injective left } R\text{-module}\}.$$

If $P \longrightarrow M$ is a quasi-injective cover of M, then $[P^{S,\leq}] \longrightarrow [M^{S,\leq}]$ is a $\mathbf{\Gamma}_1$-cover of $[M^{S,\leq}]$.

Note from [8, Lemma 3.2] that a left R-module M is quasi-injective if and only if the left $[[R^{S,\leq}]]$-module $[M^{S,\leq}]$ is quasi-injective.

Denote

$$\mathbf{\Gamma}_0 = \{[E^{S,\leq}] \,|\, E \text{ is an injective left } R\text{-module}\}.$$

Corollary 4. *If $P \longrightarrow M$ is an injective cover of M, then $[P^{S,\leq}] \longrightarrow [M^{S,\leq}]$ is a $\mathbf{\Gamma}_0$-cover of $[M^{S,\leq}]$.*

Note from [9] that if S is a finitely generated monoid, R is a left noetherian ring and M a left R-module, then $[M^{S,\leq}]$ is an injective left $[[R^{S,\leq}]]$-module if and only if M is an injective left R-module.

In the following we will give a characterization of the elements of $\mathbf{\Gamma}_0$.

Lemma 2. *Let M be a left R-module. Then for any $\phi \in [M^{S,\leq}]$,*

$$|\{s \in S \,|\, e_s\phi \neq 0\}| < \infty.$$

Proof. Suppose that $D = \{s \in S \,|\, e_s\phi \neq 0\}$ is infinite. Then for any $s \in D$, there exists $x \in S$ such that

$$0 \neq (e_s\phi)(x) = \sum_{u\in S} e_s(u)\phi(u + x) = \phi(s + x).$$

Thus $s + x \in \mathrm{supp}(\phi)$. Since $\mathrm{supp}(\phi)$ is finite, there exists an infinite subset $\{s_j | j \in J\}$ such that $s_j + x_j = s_k + x_k$ for any $j, k \in J$. Since $(S, \leq)$ is artinian and narrow, by [11, 1.2], there exist indices $n_1 < n_2 < n_3 < \cdots$ such that $n_1, n_2, \cdots \in J$ and $x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \cdots$. Now it follows that

$$s_{n_1} \geq s_{n_2} \geq s_{n_3} \geq \cdots .$$

Since $(S, \leq)$ is artinian, we have $s_{n_m} = s_{n_{m+1}} = \cdots$ for some positive integer m, which is a contradiction.

Proposition 2. *Let A be an injective left $[[R^{S,\leq}]]$-module. Then the following conditions on A are equivalent:*

(i) *There exists a left R-module E such that $A \cong [E^{S,\leq}]$.*

(ii) *There exists an injective left R-module E such that $A \cong [E^{S,\leq}]$.*

(iii) *For every $z \in A$, $|\{s \in S \mid e_s z \neq 0\}| < \infty$.*

Proof. (i)$\Leftrightarrow$(ii). It was showed in the proof of [9, Theorem 6] that if $[E^{S,\leq}]$ is an injective left $[[R^{S,\leq}]]$-module then E is an injective left R-module.

(i)$\Rightarrow$(iii). It follows from Lemma 8.

(iii)$\Rightarrow$(i). Denote

$$E = \{z \in A \mid e_s z = 0 \text{ for any } 0 < s \in S\}.$$

Assume that $z \in A$. Then $\{s \in S | e_s z \neq 0\}$ is finite. Denote

$$t = \max\{s \in S \mid e_s z \neq 0\}.$$

Then $e_t z \neq 0$. But for any $0 < s \in S$,

$$e_s(e_t z) = e_{s+t} z = 0.$$

Thus $e_t z \in E$. This means that $E \neq 0$. Clearly E is an additive subgroup of A. For any $z \in E$, any $f \in [[R^{S,\leq}]]$ and any $0 < s \in S$, $e_s(fz) = f(e_s z) = 0$. Thus $fz \in E$. This means that E is an $[[R^{S,\leq}]]$-submodule of A and, so is an R-submodule of A by action of $r \cdot z = c_r z$ for any $r \in R$.

For any $z \in E$, define $\lambda_z = \phi_{0z} \in [E^{S,\leq}]$. Denote $G = \{\lambda_z | z \in E\}$. Then it is easy to see that there is an isomorphism of left R-modules $\lambda : E \longrightarrow G$, where the left action on G of R is defined by $r \cdot \lambda_z = c_r \lambda_z$. Now, by injectivity of A, there exists an $[[R^{S,\leq}]]$-homomorphism $\alpha : [E^{S,\leq}] \longrightarrow A$ such that the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\lambda"] \arrow[d, "\tau"] & {[E^{S,\leq}]} \arrow[dl, "\alpha"] \\ A & \end{tikzcd}$$

where τ is the natural inclusion map.

Suppose that $0 \neq \phi \in [E^{S,\leq}]$ is such that $\alpha(\phi) = 0$. Denote $t = \sigma(\phi)$. Then

$$(e_t\phi)(x) = \sum_{u \in S} e_t(u)\phi(u + x) = \phi(t + x) = \begin{cases} \phi(t), & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Thus $e_t\phi = \lambda_{\phi(t)} = \lambda(\phi(t))$. Hence we have $\phi(t) = \tau(\phi(t)) = \alpha\lambda(\phi(t)) = \alpha(e_t\phi) = e_t\alpha(\phi) = 0$, a contradiction. Therefore α is a monomorphism.

Suppose that $z \in A - \alpha([E^{S,\leq}])$. Then

$$|\{s \in S \mid e_s z \neq 0\}| < \infty.$$

Denote

$$s_1 = \max\{s \in S \,|\, e_s z \neq 0\}.$$

For any $0 < s \in S$,

$$e_s(e_{s_1} z) = (e_s e_{s_1})z = e_{s+s_1} z = 0.$$

Thus $e_{s_1} z \in E$. Denote $a_1 = e_{s_1} z$. Define $\psi_1 = \phi_{s_1 a_1} \in [E^{S,\leq}]$. Then $z - \alpha(\psi_1) \neq 0$.

Now assume that there exist $\psi_1, \psi_2, \cdots, \psi_{n-1} \in [E^{S,\leq}]$ such that

$$z - \alpha(\psi_1) - \cdots - \alpha(\psi_i) \neq 0, \qquad i = 1, 2, \cdots, n-1,$$
$$s_1 > s_2 > \cdots > s_n,$$

where $s_i = \max\{s \in S \,|\, e_s(z - \alpha(\psi_1) - \cdots - \alpha(\psi_{i-1})) \neq 0\}$, $i = 2, \cdots, n$. Since $z - \alpha(\psi_1) - \cdots - \alpha(\psi_{i-1}) \neq 0$ and $\{s \in S \,|\, e_s(z - \alpha(\psi_1) - \cdots - \alpha(\psi_{i-1})) \neq 0\}$ is finite, s_i is well defined. Denote

$$z_n = z - \alpha(\psi_1) - \cdots - \alpha(\psi_{n-1}).$$

For any $0 < s \in S$,

$$e_s(e_{s_n} z_n) = (e_s e_{s_n}) z_n = e_{s+s_n} z_n = 0.$$

Thus $e_{s_n} z_n \in E$. Denote $a_n = e_{s_n} z_n$. Define $\psi_n = \phi_{s_n a_n} \in [E^{S,\leq}]$. Then

$$(e_{s_n}\psi_n)(x) = \psi_n(x + s_n) = \begin{cases} e_{s_n} z_n = a_n, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

Thus $e_{s_n}\psi_n = \lambda_{a_n}$ and, so

$$e_{s_n}(z_n - \alpha(\psi_n)) = e_{s_n} z_n - \alpha(e_{s_n}\psi_n) = e_{s_n} z_n - \alpha(\lambda_{a_n}) = e_{s_n} z_n - \tau(e_{s_n} z_n) = 0.$$

For any $s_n < s \in S$,

$$e_s(z_n - \alpha(\psi_n)) = e_s z_n - \alpha(e_s\psi_n) = -\alpha(e_s\psi_n) = 0,$$

since $e_s\psi_n = 0$. Obviously

$$z_n - \alpha(\psi_n) = z - \alpha(\psi_1) - \cdots - \alpha(\psi_{n-1}) - \alpha(\psi_n) \neq 0$$

(otherwise, $z \in \alpha([E^{S,\leq}])$). Denote

$$s_{n+1} = \max\{s \in S \,|\, e_s(z_n - \alpha(\psi_n)) \neq 0\}.$$

Then clearly $s_{n+1} < s_n$.

By induction principle, we obtain a descending chain of elements of S:

$$s_1 > s_2 > \cdots > s_n > s_{n+1} > \cdots,$$

which is contradicted with the hypothesis that $(S, \leq)$ is artinian.

Hence $A = \alpha([E^{S,\leq}])$ and, so α is an isomorphism of left $[[R^{S,\leq}]]$-modules.

Proposition 3. *Let M be a left R-module and $\beta : A \longrightarrow [M^{S,\leq}]$ an injective precover of left $[[R^{S,\leq}]]$-module $[M^{S,\leq}]$. Denote*

$$E = \{z \in A \,|\, fz = c_{f(0)} z \text{ for all } f \in [[R^{S,\leq}]]\}.$$

Then E is an injective precover of M.

Proof. For any $m \in M$, define $\lambda_m = \phi_{0m} \in [M^{S,\leq}]$. Denote

$$G(M) = \{\lambda_m \mid m \in M\}.$$

Then it is easy to see that there is an isomorphism of left R-modules $\lambda : M \longrightarrow G(M)$, where the left action on $G(M)$ of R is defined by $r \cdot \lambda_m = c_r \lambda_m$.

For any $f \in [[R^{S,\leq}]]$ and any $m \in M$,

$$\begin{aligned}(f\lambda_m)(x) &= \sum_{u \in S} f(u)\lambda_m(u+x) \\ &= \begin{cases} f(0)m, & x = 0, \\ 0, & x \neq 0 \end{cases} \\ &= \lambda_{f(0)m}(x).\end{aligned}$$

Thus $f\lambda_m = \lambda_{f(0)m} \in G(M)$. This means that $G(M)$ is an $[[R^{S,\leq}]]$-submodule of $[M^{S,\leq}]$.

For any $f, g \in [[R^{S,\leq}]]$ and any $z \in E$,

$$\begin{aligned}f(gz) &= (fg)z = c_{(fg)(0)}z = c_{f(0)g(0)}z \\ &= (c_{f(0)}c_{g(0)})z = c_{f(0)}(c_{g(0)}z) = c_{f(0)}(gz).\end{aligned}$$

Thus $gz \in E$. This means that E is an $[[R^{S,\leq}]]$-submodule of A.

Let V be a left R-module and $W \leq V$. Suppose that $\alpha : W \longrightarrow E$ is an R-homomorphism. Define $\theta : G(W) \longrightarrow E$ via

$$\theta(\lambda_w) = \alpha(w), \qquad \forall w \in W.$$

For any $f \in [[R^{S,\leq}]]$ and any $w \in W$, we have

$$\begin{aligned}\theta(f\lambda_w) &= \theta(\lambda_{f(0)w}) = \alpha(f(0)w) = f(0)\alpha(w) \\ &= c_{f(0)}\alpha(w) = f\alpha(w) = f\theta(\lambda_w).\end{aligned}$$

Now it is easy to see that θ is an $[[R^{S,\leq}]]$-homomorphism. Since A is injective, there exists an $[[R^{S,\leq}]]$-homomorphism $h : G(V) \longrightarrow A$ such that the following diagram commutes:

$$\begin{array}{lll} G(W) & \longrightarrow & G(V) \\ \downarrow^{\theta} & \swarrow^{\alpha} & \\ A & & \end{array}$$

For any $v \in V$ and any $f \in [[R^{S,\leq}]]$, we have

$$fh(\lambda_v) = h(f\lambda_v) = h(\lambda_{f(0)v}) = h(c_{f(0)}\lambda_v) = c_{f(0)}h(\lambda_v).$$

Thus $h(\lambda_v) \in E$. So we can regard h as an $[[R^{S,\leq}]]$-homomorphism from $G(V)$ to E. Now for any $w \in W$,

$$h\lambda(w) = h(\lambda_w) = \theta(\lambda_w) = \alpha(w).$$

Thus $(h\lambda)|_W = \alpha$. This means that E is an injective left R-module.

For any $z \in E$, and any $0 < s \in S$, we have

$$\begin{aligned}\beta(z)(s) &= \sum_{u \in S} e_s(u)\beta(z)(u) = (e_s\beta(z))(0) \\ &= \beta(e_s z)(0) = \beta(c_{e_s(0)}z)(0) = \beta(c_0 z) = 0.\end{aligned}$$

Thus $\beta(z) = \lambda_{\beta(z)(0)}$. Hence $\lambda^{-1}\beta : E \longrightarrow M$ is an R-homomorphism.

Suppose that E_1 is an injective left R-module and $\eta : E_1 \longrightarrow M$ is an R-homomorphism. Then there exists an $[[R^{S,\leq}]]$-homomorphism δ such that the following diagram commutes:

$$\begin{array}{lcl} [E_1^{S,\leq}] & & \\ \downarrow^{\delta} & \searrow^{[\eta^{S,\leq}]} & \\ A & \xrightarrow{\beta} & [M^{S,\leq}] \end{array}$$

For any $e \in E_1$, and any $0 < s \in S$,

$$f\delta(\lambda_e) = \delta(f\lambda_e) = \delta(\lambda_{f(0)e}) = \delta(c_{f(0)}\lambda_e) = c_{f(0)}\delta(\lambda_e).$$

Thus $\delta(\lambda_e) \in E$. Now, by $\beta(\delta(\lambda_e)) = \lambda_{\beta(\delta(\lambda_e))(0)}$, it is easy to see that the following diagram

$$\begin{array}{lcl} E_1 & & \\ \downarrow^{\delta\lambda} & \searrow^{\eta} & \\ E & \xrightarrow{\lambda^{-1}\beta} & M \end{array}$$

commutes. This means that $\lambda^{-1}\beta : E \longrightarrow M$ is an injective precover of M.

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