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SYMMETRIC AND ASYMMETRIC DIOPHANTINE APPROXIMATION

TONG Jingcheng*

Abstract

Let ξ be an irrational number with simple continued fraction expansion

$$\xi = [a_0; a_1, \cdots, a_i, \cdots]$$

and $\frac{p_i}{q_i}$ be its ith convergent. Let C_i be defined by $\xi - \frac{p_i}{q_i} = (-1)^i/(C_i q_i q_{i+1})$. The author proves the following theorem:

Theorem. Let $r > 1, R > 1$ be two real numbers and

$$L = \tfrac{1}{r-1} + \tfrac{1}{R-1} + a_n a_{n+1} rR, \quad K = \tfrac{1}{2}\Big(L + \sqrt{L^2 - \tfrac{4}{(r-1)(R-1)}}\,\Big).$$

Then

(i) $C_{n-2} < r,\ C_n < R$ imply $C_{n-1} > K$;
(ii) $C_{n-2} > r,\ C_n > R$ imply $C_{n-1} < K$.

This theorem generalizes the main result in [1].

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§ 1 . Introduction

Let ξ be an irrational number with simple continued fraction expansion

$$\xi = [a_0; a_1, \cdots, a_i, \cdots]$$

and $\frac{p_i}{q_i}$ be its ith convergent. In [1], the present author considered the well-known inequality in the theory of continued fractions

$$\Big|\xi - \frac{p_i}{q_i}\Big| < 1/(q_i q_{i+1}). \tag{1.1}$$

It has been proved that among any three consecutive convergents $\frac{p_i}{q_i} (i = n-2, n-1, n)$, at least one satisfies $\big|\xi - \frac{p_i}{q_i}\big| < 1/(\alpha_n q_i q_{i+1})$ for some $\alpha_n > 1$, and that among any four consecutive convergents, at least one satisfies $-1/(\beta_n q_i q_{i+1}) < \xi - \frac{p_i}{q_i} < 1/(\beta_n q_i q_{i+1})$ for some $\beta_n > 1$. These results are consequences of the following theorem (cf. [1]):

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*Department of Mathematics and Statistics, University of North Florida, Jacksonville, FL 32224, USA.
E-mail: JTONG@unf.edu.

Theorem 1.1. *Let $\xi = [a_0; a_1, \cdots, a_i, \cdots]$ be an irrational number and*

$$D_n = [a_{n+1;} a_n, \cdots, a_1][a_{n+2}; a_{n+3}, \cdots].$$

If $r > 1$ is a real number and $L = \left(\sqrt{a_n a_{n+1}} + \frac{1}{\sqrt{r}}\right)/(\sqrt{r} - \sqrt{a_n a_{n+1}})$, then

(i) $D_{n-1} < r$ *implies* $\max(D_{n-2}, D_n) > L$;

(ii) $D_{n-1} > r$ *implies* $\min(D_{n-2}, D_n) < L$.

Theorem 1.1 can be restated in an equivalent form below.

Theorem 1.1A. *Let $R > 1$ be a real number and*

$$L' = \frac{1}{4}\Big[\sqrt{a_n a_{n+1}}\Big(1 + \frac{1}{R}\Big) + \sqrt{a_n a_{n+1}\Big(1 + \frac{1}{R}\Big)^2 + \frac{4}{R}}\ \Big]^2.$$

Then

(i) $D_{n-2} < R$ *and* $D_n < R$ *imply* $D_{n-1} > L'$;

(ii) $D_{n-2} > R$ *and* $D_n > R$ *imply* $D_{n-1} < L'$.

The proof of the equivalence of Theorem 1.1 and Theorem 1.1A is very easy since $R = L$ is equivalent to $r = L'$ as follows:

$$\begin{aligned}
R &= \Big(\sqrt{a_n a_{n+1}} + \frac{1}{\sqrt{r}}\Big)\Big/(\sqrt{r} - \sqrt{a_n a_{n+1}}),\\
rR - 1 &= \sqrt{a_n a_{n+1}}(R+1)\sqrt{r},\\
r - \sqrt{a_n a_{n+1}}&\Big(1 + \frac{1}{R}\Big)\sqrt{r} - \frac{1}{R} = 0,\\
r &= \frac{1}{4}\Big[\sqrt{a_n a_{n+1}}\Big(1 + \frac{1}{R}\Big) + \sqrt{a_n a_{n+1}\Big(1 + \frac{1}{R}\Big)^2 + \frac{4}{R}}\ \Big].
\end{aligned}$$

Theorem 1.1A suggests a very natural generalization of Theorem 1.1: if r, R are two real numbers greater than 1, and $D_{n-2} < r, D_n < R$, what expression can we have to estimate the magnitude of D_{n-1}? It is easily seen that Theorem 1.1 is the special case for $r = R$.

In this paper, using the idea developed in [2–4], we find this explicit expression in two parameters r, R.

§ 2. Preliminaries

Let $\xi = [a_0; a_1, \cdots, a_i, \cdots]$ and $\frac{p_i}{q_i} = [a_0; a_1, \cdots, a_i]$.

Let $D_n = [a_{n+1}; a_n, \cdots, a_1][a_{n+2}; a_{n+3}, \cdots]$ and $C_n = 1 + \frac{1}{D_n}$.

As pointed out in [1], we have

$$\xi - \frac{p_i}{q_i} = (-1)^n/(C_n q_n q_{n+1}). \tag{2.1}$$

Let $P = [a_{n+1}; a_{n+2}, \cdots]$ and $Q = [a_n; a_{n-1}, \cdots, a_1]$. Then it is easily seen that

$$D_{n-2} = (a_n + P^{-1})/(Q - a_n), \tag{2.2}$$

$$D_{n-1} = PQ, \tag{2.3}$$

$$D_n = (a_{n+1} + Q^{-1})/(P - a_{n+1}). \tag{2.4}$$

By (2.3) we have $Q = P^{-1}D_{n-1}$. Replacing Q in (2.2) by $P^{-1}D_{n-1}$ yields

$$D_{n-2} = (a_n + P^{-1})/(P^{-1}D_{n-1} - a_n). \tag{2.5}$$

From (2.5) we have

$$P^{-1} = a_n(D_{n-2} + 1)/(D_{n-2}D_{n-1} - 1). \tag{2.6}$$

By (2.3) we have $P = Q^{-1}D_{n-1}$. Replacing P in (2.4) by $Q^{-1}D_{n-1}$ yields

$$D_n = (a_{n+1} + Q^{-1})/(Q^{-1}D_{n-1} - a_{n+1}). \tag{2.7}$$

From (2.7) we have

$$Q^{-1} = a_{n+1}(D_n + 1)/(D_n D_{n-1} - 1). \tag{2.8}$$

From (2.3), (2.6) and (2.8), we have

$$D_{n-1} = \frac{(D_{n-2}D_{n-1} - 1)(D_n D_{n-1} - 1)}{a_n(1 + D_{n-2})a_{n+1}(1 + D_n)}, \tag{2.9}$$

which yields

$$D_{n-1}^2 - \Big[\frac{1}{D_{n-2}} + \frac{1}{D_n} + a_n a_{n+1}\Big(1 + \frac{1}{D_{n-2}}\Big)\Big(1 + \frac{1}{D_n}\Big)\Big]D_{n-1} + \frac{1}{D_{n-2}D_n} = 0. \tag{2.10}$$

The formula (2.10) plays a key role in the proof of our main result.

§ 3. Main Result

Theorem 3.1. *Let* $\xi = [a_0; a_1, \cdots, a_n, \cdots]$ *be an irrational number and*

$$D_n = [a_{n+1}; a_n, \cdots, a_1][a_{n+2}; a_{n+3}, \cdots].$$

If $r > 1, R > 1$ *are two real numbers and*

$$M = \frac{1}{2}\Big\{\frac{1}{r} + \frac{1}{R} + a_n a_{n+1}\Big(1 + \frac{1}{r}\Big)\Big(1 + \frac{1}{R}\Big) + \sqrt{\Big[\frac{1}{r} + \frac{1}{R} + a_n a_{n+1}\Big(1 + \frac{1}{r}\Big)\Big(1 + \frac{1}{R}\Big)\Big]^2 - \frac{4}{rR}}\ \Big\},$$

then

(i) $D_{n-2} < r$ *and* $D_n < R$ *imply* $D_{n-1} > M$;
(ii) $D_{n-2} > r$ *and* $D_n > R$ *imply* $D_{n-1} < M$.

Proof. Consider formula (2.10). Let

$$f(D_{n-2}, D_n) = D_{n-2}^{-1} + D_n^{-1} + a_n a_{n+1}(1 + D_{n-2}^{-1})(1 + D_n^{-1}).$$

Then formula (2.10) becomes

$$D_{n-1}^2 - f(D_{n-2}, D_n)D_{n-1} + (D_{n-2}D_n)^{-1} = 0. \tag{3.1}$$

Because $a_n \geq 1, a_{n+1} \geq 1$, it is easily seen that

$$f(D_{n-2}, D_n) \geq D_{n-2}^{-1} + D_n^{-1} + (1 + D_{n-2}^{-1})(1 + D_n^{-1}) > 2D_{n-2}^{-1} + 2D_n^{-1}.$$

From the quadratic equation (3.1) in D_{n-1}, we have $f^2(D_{n-2}, D_n) - 4(D_{n-2}D_n)^{-1} > 0$. Because $D_{n-2} > a_{n-1}a_n \geq 1, D_n > a_{n+1}a_{n+2} \geq 1$, we have

$$\begin{aligned} f^2(D_{n-2}, D_n) &\leq f^2(D_{n-2}, D_n) + 4 - 4(D_{n-2}D_n)^{-1} \\ &< f^2(D_{n-2}, D_n) - 4(D_{n-2}, D_n) + 4\sqrt{f^2(D_{n-2}, D_n) - 4(D_{n-2}D_n)^{-1}} + 4. \end{aligned}$$

Hence $f(D_{n-2}, D_n) - \sqrt{f^2(D_{n-2}, D_n) - 4(D_{n-2}D_n)^{-1}} < 2$. Therefore the quadratic equation (3.1) in D_{n-1} has only one solution as follows:

$$D_{n-1} = \frac{1}{2}[f(D_{n-2}, D_n) + \sqrt{f^2(D_{n-2}, D_n) - 4(D_{n-2}D_n)^{-1}}\ ]. \tag{3.2}$$

Now we show that D_{n-1} is a decreasing function of both D_{n-2} and D_n. It is obvious that $f(D_{n-2}, D_n)$ is a decreasing function of both D_{n-2} and D_n. We need only to show that the function $F(D_{n-2}, D_n) = f^2(D_{n-2}, D_n) - 4(D_{n-2}D_n)^{-1}$ is also a decreasing function of both D_{n-2} and D_n. This can be done easily by showing that $\partial F/\partial D_{n-2} < 0$ and $\partial F/\partial D_n < 0$. Because $F(D_{n-2}, D_n)$ is symmetric in D_{n-2} and D_n, we need only checked one of the partial derivatives.

$$\begin{aligned} \partial F/\partial D_n &= 2f(D_{n-2}, D_n)[-D_n^{-2} - a_n a_{n+1}(1 + D_{n-2}^{-1})D_n^{-2}] + 4D_{n-2}^{-1}D_n^{-2} \\ &< -4D_n^{-2} + 4D_{n-2}^{-1}D_n^{-2} < 0. \end{aligned}$$

The conclusion of Theorem 3.1 is immediate because

$$D_{n-1} = \frac{1}{2}[f(D_{n-2}, D_n) + \sqrt{F(D_{n-2}, D_n)}\ ]$$

is a decreasing function of both D_{n-2} and D_n.

For the convenience of application, Theorem 3.1 can be restated in terms of $C_n = 1 + D_n^{-1}$ instead of D_n.

Theorem 3.1A. *Let* $\xi = [a_0; a_1, \cdots, a_n, \cdots]$ *be an irrational number and* $\frac{p_n}{q_n} = [a_0; a_1, \cdots, a_n]$. *Let* C_n *be defined by* $\xi - \frac{p_n}{q_n} = (-1)^n/(C_n q_n q_{n+1})$. *If* $r > 1, R > 1$ *are two real numbers and*

$$K = \frac{1}{2}\Big[\frac{1}{r-1} + \frac{1}{R-1} + a_n a_{n+1} rR + \sqrt{\Big(\frac{1}{r-1} + \frac{1}{R-1} + a_n a_{n+1} rR\Big)^2 - \frac{4}{(r-1)(R-1)}}\ \Big],$$

then

(i) $C_{n-2} < r$ *and* $C_n < R$ *imply* $C_{n-1} > K$;

(ii) $C_{n-2} > r$ *and* $C_n > R$ *imply* $C_{n-1} < K$.

Remark 3.1. Letting $r = R$ in Theorem 3.1A, we have the result in [2]; letting $r = R = \sqrt{a_n^2 + 4}$, we have the result in [4].

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