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HOMOGENIZATION OF SEMILINEAR PARABOLIC EQUATIONS IN PERFORATED DOMAINS***

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Abstract

This paper is devoted to the homogenization of a semilinear parabolic equation with rapidly oscillating coefficients in a domain periodically perforated by ε-periodic holes of size ε. A Neumann condition is prescribed on the boundary of the holes.

The presence of the holes does not allow to prove a compactness of the solutions in L^2. To overcome this difficulty, the authors introduce a suitable auxiliary linear problem to which a corrector result is applied. Then, the asymptotic behaviour of the semilinear problem as $\varepsilon \to 0$ is described, and the limit equation is given.

Keywords Periodic homogenization, Perforated domains, Semilinear parabolic equations
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§ 1 . Introduction

In this paper we study the asymptotic behaviour of the following semilinear parabolic problem

$$\begin{cases} u_\varepsilon' - \operatorname{div}(A^\varepsilon \nabla u_\varepsilon) = f(u_\varepsilon) + g_\varepsilon & \text{in} \quad \Omega_\varepsilon \times (0,T), \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ A^\varepsilon \nabla u_\varepsilon \cdot \nu = 0 & \text{on} \quad \partial S_\varepsilon \times (0,T), \\ u_\varepsilon(x,0) = u_\varepsilon^0 & \text{in} \quad \Omega_\varepsilon, \end{cases} \tag{1.1}$$

where Ω is a bounded open subset of $\mathbb{R}^n$, $\Omega_\varepsilon = \Omega \backslash S_\varepsilon$ is a domain perforated by a closed subset S_ε of ε-periodic holes of the same size as the period, $g_\varepsilon \in L^2(0,T,L^2(\Omega_\varepsilon))$, $u_\varepsilon^0 \in L^2(\Omega_\varepsilon)$ and f is a continuous function with a linear growth. The matrix A^ε is of the form $A^\varepsilon(x) = A(\frac{x}{\varepsilon})$ and A is a periodic bounded matrix field uniformly positive definite.

The homogenization of the corresponding linear problem has been originally studied by S. Spagnolo [17, 16] in the (symmetric) general framework of the G-convergence. The homogenization and the correctors for the non symmetric periodic case have been studied

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by A. Bensoussan, J. L. Lions and G. Papanicolaou in [1] and by S. Brahim-Otsman, G. A. Francfort and F. Murat [4] in the framework of the H-convergence. The homogenization and the correctors in the case of a periodically perforated domain have been studied by the authors in [13, 14].

When there are no holes (i.e. $S_\varepsilon = \emptyset$), the homogenization of Problem (1.1) follows straightforward from the linear case. Indeed, under usual weak convergence assumptions on the data, the boundedness of u_ε in $L^2(0,T;\, H_0^1(\Omega))$ and of u'_ε in $L^2(0,T;\, H^{-1}(\Omega))$ implies that (up to a subsequence) u_ε strongly converges in $L^2(\Omega \times (0,T))$ to some function u. Hence, the continuity of the Nemyskii operator associated to f implies the convergence of $f(u_\varepsilon)$ to $f(u)$. Consequently, the convergence results of the linear case apply and give the limit problem solved by u.

In presence of holes, the problem is more complicated since, as already observed by the authors in the study of the linear case [14], the boundedness of $||u_\varepsilon||_{L^2(0,T;\, H^1(\Omega_\varepsilon))}$ and $||u'_\varepsilon||_{L^2(0,T;\, (H^1(\Omega_\varepsilon))')}$, does not provide the compactness in $L^2(\Omega \times (0,T))$ of any extension of u_ε to the whole of Ω. The best strong convergence we could derive in [14] is in $C([0,T];\, H^{-1}(\Omega))$. This convergence does not allow by itself to pass to the limit in the nonlinear term $f(u_\varepsilon)$.

Then, we look for reasonable assumptions on f and on the data g^ε and u^0_ε, allowing to overcome this difficulty. We prove here that if f is Lipschitz continuous and the data verify the following convergencies:

$$\begin{cases} \lim\limits_{\varepsilon \to 0} \|u^0_\varepsilon - u^0\|_{L^2(\Omega_\varepsilon)} = 0, \\ \lim\limits_{\varepsilon \to 0} \|g_\varepsilon - g\|_{L^2(0,T;L^2(\Omega_\varepsilon))} = 0, \end{cases} \tag{1.2}$$

then

$$\widetilde{f(u_\varepsilon)} \rightharpoonup \theta f(u) \quad \text{weakly in } \; L^2(0,T;\, L^2(\Omega)), \tag{1.3}$$

where $\sim$ denotes the zero extension to Ω, θ is the proportion of material and where we denoted by θu the limit of $\widetilde{u_\varepsilon}$ in $C([0,T];\, H^{-1}(\Omega))$.

This allows to show (Theorem 3.1) that u is the unique solution of the homogenized problem

$$\begin{cases} \theta u' - \operatorname{div}\left(A^0 \nabla u\right) = \theta f(u) + \theta g & \text{in} \quad \Omega \times (0,T), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u^0 & \text{in} \quad \Omega, \end{cases}$$

A^0 being the homogenized matrix of A^ε.

To prove convergence (1.3), we introduce the solution v_ε of an auxiliary linear problem (Problem (4.7)), whose right-hand side is the homogenized operator of the linear part of Equation (1.1). This approach is, in spirit, that used by A. Bensoussan, L. Boccardo and F. Murat in [2] for the homogenization of some nonlinear problems with quadratic growth in the gradient (see also P. Donato, A. Gaudiello and L. Sgambati [12] for an extension to the case of perforated domains). Here, the main step (Proposition 4.2) consists in proving that

$$\lim_{\varepsilon\to 0} ||u_\varepsilon - v_\varepsilon||_{L^2(\Omega_\varepsilon\times(0,T))} = 0.$$

As a consequence, since f is Lipschitz continuous, the limit of the nonlinear term $f(u_\varepsilon)$ is the same as that of $f(v_\varepsilon)$. Hence, the last step consists in computing this limit, by applying to v_ε the corrector result for the linear case.

We recall that the main difference between the elliptic and the parabolic case is that in the last one the corrector result needs a strong convergence of the data, even in the case without holes (see [4]). This is why the assumption (1.2) has been necessary.

In Section 2, we recall the results for the linear case, together with some preliminaries. In Section 3, we present the problem and we state the main convergence result. It is proved in Section 4, where we introduce the auxiliary problem and its properties.

§ 2 . Preliminaries

In this section, we introduce the perforated domain and recall the homogenization and corrector results for the linear parabolic problem.

Let Ω be a bounded connected open set of $\mathbb{R}^n,\ n\geq 2$, with boundary $\partial\Omega$. Let $Y =]0,l_1[\times\cdots\times]0,l_n[$ be the reference cell and $S\subset\subset Y$ an open subset (the reference hole) with a Lipschitz boundary ∂S. We denote by ε a positive parameter taking its values in a positive sequence which tends to zero. Introduce the set of holes in $\mathbb{R}^n$ defined by

$$\tau(\varepsilon\bar{S}) = \{\varepsilon(k(l)+\bar{S}), k\in\mathbb{Z}^n,\ k(l)=(k_1l_1,\cdots,k_nl_n)\}.$$

For simplicity, we assume that for every ε, the holes do not meet the boundary of Ω, i.e. that

$$\partial\Omega\cap\tau(\varepsilon\bar{S}) = \emptyset. \tag{2.1}$$

This means that there exists a subset $\mathcal{K}_\varepsilon$ of $\mathbb{Z}^n$ such that

$$\Omega\cap\tau(\varepsilon\bar{S}) = \bigcup_{k\in\mathcal{K}_\varepsilon}(\varepsilon(k(l)+\bar{S})).$$

Set

$$S_\varepsilon = \bigcup_{k\in\mathcal{K}_\varepsilon}(\varepsilon(k(l)+\bar{S})).$$

Then, the perforated domain Ω_ε is defined by

$$\Omega_\varepsilon = \Omega\backslash S_\varepsilon, \tag{2.2}$$

and from (2.1) we have

$$\partial\Omega\cap\partial S_\varepsilon = \emptyset,\quad \partial\Omega_\varepsilon = \partial\Omega\cup\partial S_\varepsilon.$$

In the following we use the notations:

- $Y^* = Y \backslash \bar{S}$;
- $|\omega|$ = the Lebesgue measure of any measurable set ω of $\mathbb{R}^n$;
- $\theta = |Y^*|/|Y|$ (the proportion of material);
- χ_ω = the characteristic function of the set ω, $\chi_\omega(x) = \begin{cases} 1, & \text{if } x \in \omega, \\ 0, & \text{elsewhere;} \end{cases}$
- $\widetilde{v}$ = the extension by zero on Ω of any function v defined on Ω_ε;
- $\nu = (\nu_i)_{i=1,\cdots,n}$ the unit external normal vector with respect to Y^* or Ω_ε;
- C = any constant independent of ε.

Recall that, as $\varepsilon \to 0$,

$$\chi_{\Omega_\varepsilon} \rightharpoonup \theta = |Y^*|/|Y| \qquad L^\infty(\Omega) \text{ weak } * \tag{2.3}$$

(see for instance [9, Chapter 2] for a proof). This is due to the fact that, from the assumption (2.1), one has

$$\chi_{\Omega_\varepsilon}(x) = (\chi_{Y^*})^\# \Big(\frac{x}{\varepsilon}\Big),$$

where $(\chi_{Y^*})^\#$ is defined by

$$(\chi_{Y^*})^\#(y + k\, l_i\, e_i) = \chi_{Y^*}(y) \qquad \text{a.e. on } Y, \quad \forall\, k \in \mathbb{Z},\ \forall\, i \in \{1, \cdots, n\},$$

and $\{e_1, \cdots, e_n\}$ is the canonical basis of $\mathbb{R}^n$.

Let $A(y) = (a_{ij}(y))_{1 \le i,j \le n}$ be a $n \times n$ matrix-valued function defined on $\mathbb{R}^n$ such that

$$\begin{cases} A \in (L^\infty(Y))^{n^2}, \\ A \text{ is } Y\text{-periodic}, \\ \text{there exists } \ \alpha > 0 \ \text{ such that for any } \ \lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n, \\ \displaystyle\sum_{i,j=1}^n a_{ij}(y)\lambda_i\lambda_j \ge \alpha\, \|\lambda\|^2 \qquad \text{a.e. on } Y, \end{cases} \tag{2.4}$$

and set for any ε,

$$A^\varepsilon(x) = A\Big(\frac{x}{\varepsilon}\Big) \qquad \text{a.e. on } \Omega.$$

Let V_ε be the Hilbert space

$$V_\varepsilon = \{v \in H^1(\Omega_\varepsilon) \ : \ v_{|\partial\Omega} = 0\}, \tag{2.5}$$

equipped with the $H^1(\Omega_\varepsilon)$-norm and denote by V_ε' its dual.

We will need in the sequel the following compactness result, which is a particular case of Lemma 3.3 of [14].

Lemma 2.1. (cf. [14]) *Let $\{\Omega_\varepsilon\} \subset \Omega$ be given by (2.2), $\{v_\varepsilon\}_\varepsilon$ be a sequence in $L^\infty(0,T;$ $L^2(\Omega_\varepsilon))$ such that $\{v_\varepsilon'\}_\varepsilon$ is in $L^2(0,T;V_\varepsilon')$ and*

$$\begin{aligned} &\text{(i) } \widetilde{v_\varepsilon} \rightharpoonup \theta v \quad \textit{weakly in } \ L^\infty(0,T;L^2(\Omega)), \\ &\text{(ii) } \|v_\varepsilon'\|_{L^2(0,T;V_\varepsilon')} \le c, \end{aligned} \tag{2.6}$$

where θ is the constant given by (2.2). *Then*

$$\widetilde{v_\varepsilon} \longrightarrow \theta v \quad \textit{strongly in} \;\; C^0([0,T];H^{-1}(\Omega)).$$

For every $T > 0$, we introduce a family $\{P_\varepsilon\}$ of extension operators for time-dependant functions, recalled in the following lemma (see [6, 8]).

Lemma 2.2. (cf. [6, 8]) *For all $\varepsilon > 0$, there exists an extension operator*

$$P_\varepsilon \in \mathcal{L}(L^2(0,T;H^k(\Omega_\varepsilon)),L^2(0,T;H^k(\Omega))), \qquad k = 0,1,$$

such that, for all $\varphi \in L^2(0,T;H^k(\Omega_\varepsilon))$ and $\varphi' \in L^2(0,T;L^2(\Omega_\varepsilon))$, one has

(i) $P_\varepsilon\varphi = \varphi \quad in \;\; \Omega_\varepsilon \times (0,T)$,
(ii) $P_\varepsilon\varphi' = (P_\varepsilon\varphi)' \quad in \;\; \Omega \times (0,T)$,
(iii) $\|P_\varepsilon\varphi\|_{L^2(0,T;L^2(\Omega))} \leq c_0\|\varphi\|_{L^2(0,T;L^2(\Omega_\varepsilon))}$,
(iv) $\|P_\varepsilon\varphi'\|_{L^2(0,T;L^2(\Omega))} \leq c_0\|\varphi'\|_{L^2(0,T;L^2(\Omega_\varepsilon))}$,
(v) $\|\nabla(P_\varepsilon\varphi)\|_{L^2(0,T;[L^2(\Omega)]^n)} \leq c_0\|\nabla\varphi\|_{L^2(0,T;[L^2(\Omega_\varepsilon)]^n)}$,

where c_0 is a constant independent of ε.

Remark 2.1. This lemma provides a Poincaré inequality in V_ε with a constant independent of ε. Indeed,

$$\forall v \in V_\varepsilon, \quad \|v\|_{L^2(\Omega_\varepsilon)} \leq C_\Omega\|\nabla v\|_{[L^2(\Omega_\varepsilon)]^n},$$

where $C_\Omega = c_0 C(\Omega)$, $C(\Omega)$ being the constant in the Poincaré inequality for $H^1_0(\Omega)$.

Let us consider the following linear parabolic problem

$$\begin{cases} u'_\varepsilon - \operatorname{div}(A^\varepsilon\nabla u_\varepsilon) = h_\varepsilon + P^*_\varepsilon(\sigma) & \text{in} \quad \Omega_\varepsilon \times (0,T), \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ A^\varepsilon\nabla u_\varepsilon \cdot \nu = 0 & \text{on} \quad \partial S_\varepsilon \times (0,T), \\ u_\varepsilon(x,0) = u^0_\varepsilon & \text{in} \quad \Omega_\varepsilon, \end{cases} \tag{2.7}$$

where we denote by $P^*_\varepsilon \in \mathcal{L}\left(L^2(0,T;H^{-1}(\Omega)),L^2(0,T;V_\varepsilon)\right)$ the adjoint of the operator P_ε introduced in Lemma 2.2.

It is well known (see [11, Chapter XVIII, §3]) that if $\sigma \in L^2(0,T,H^{-1}(\Omega))$, $h_\varepsilon \in L^2(\Omega_\varepsilon \times (0,T))$ and $u^0_\varepsilon \in L^2(\Omega_\varepsilon)$, Problem (2.7) has a unique solution u_ε such that

$$u_\varepsilon \in L^2(0,T;\, V_\varepsilon) \cap C^0([0,T];\, L^2(\Omega_\varepsilon)). \tag{2.8}$$

In the following, we will need some homogenization and corrector results for Problem (2.7), proved in [14]. We recall them here for the reader's convenience.

To do that, let us introduce the homogenized matrix A^0, which is the same as in the elliptic case studied in [10]. For any $\lambda \in \mathbb{R}^n$, let $\widehat{\chi}_\lambda$ be the solution of the following problem

$$\begin{cases} -\operatorname{div}(A\nabla(y\cdot\lambda - \widehat{\chi}_\lambda)) = 0 & \text{in} \quad Y^*, \\ (A\nabla(y\cdot\lambda - \widehat{\chi}_\lambda))\cdot\nu = 0 & \text{on} \quad \partial S, \\ \widehat{\chi}_\lambda \quad Y\text{-periodic}, \\ \displaystyle\int_{Y^*} \widehat{\chi}_\lambda(y)\, dy = 0, \end{cases} \tag{2.9}$$

where A is the matrix given by (2.4). Set

$$\widehat{w}^\lambda(y) = -\widehat{\chi}_\lambda(y) + \lambda \cdot y \qquad \text{a.e. on } Y^*.$$

Then the $n \times n$ homogenized matrix $A^0 = \{a^0_{ij}\}_{1\le i,j\le n}$ is defined by

$$A^0\lambda = \frac{1}{|Y|}\int_{Y^*} A\nabla\widehat{w}^\lambda\, dy \qquad \text{for any } \lambda \in \mathbb{R}^n. \tag{2.10}$$

We also introduce the $(n \times n)$ Y-periodic corrector matrix $C(y) = \{C_{ij}(y)\}_{1\le i,j\le n}$, defined by

$$C_{ij}(y) = \delta_{ij}(y) - \frac{\partial\widehat{\chi}^j}{\partial y_i}(y) = \frac{\partial\widehat{w}^j}{\partial y_i}(y) \qquad \text{a.e. on } Y^*,$$

with

$$\widehat{w}^j = x_j - \widehat{\chi}^j,$$

where $\widehat{\chi}^j$ is the solution of (2.9) for $\lambda = e_j$ and δ_{ij} is the Kronecker symbol.

We define

$$C^\varepsilon(x) = C\Big(\frac{x}{\varepsilon}\Big), \qquad \text{a.e. on } \Omega_\varepsilon, \tag{2.11}$$

which, by construction, is εY-periodic.

The asymptotic behaviour of Problem (2.7) is given by the following theorem.

Theorem 2.1. (cf. [13, 14]) *Let $\sigma \in L^2(0,T,H^{-1}(\Omega))$ and $(\{u^0_\varepsilon\},\{h_\varepsilon\}) \subset L^2(\Omega_\varepsilon) \times L^2(\Omega_\varepsilon \times (0,T))$ be two sequences such that*

$$\begin{aligned} &(\mathrm{i})\quad \widetilde{u^0_\varepsilon} \rightharpoonup \theta u^0 \quad \textit{weakly in } L^2(\Omega),\\ &(\mathrm{ii})\quad \widetilde{h_\varepsilon} \rightharpoonup \theta h \quad \textit{weakly in } L^2(\Omega\times(0,T)). \end{aligned} \tag{2.12}$$

Under the assumption (2.4)*, let u_ε be the solution of Problem* (2.7) *and $\{P_\varepsilon\}$ given by Lemma* 2.2*. Then, as $\varepsilon \to 0$, the following convergencies hold:*

$$\begin{array}{lll} (\mathrm{i}) & P^\varepsilon u_\varepsilon \rightharpoonup u & \textit{weakly in } L^2(0,T;\, H^1_0(\Omega)),\\ (\mathrm{ii}) & \widetilde{A^\varepsilon\nabla u_\varepsilon} \rightharpoonup A^0\nabla u & \textit{weakly in } [L^2(\Omega\times(0,T))]^n,\\ (\mathrm{iii}) & \widetilde{u_\varepsilon} \rightharpoonup \theta u & \textit{weakly in } L^\infty(0,T;\, L^2(\Omega)),\\ (\mathrm{iv}) & \widetilde{u_\varepsilon} \longrightarrow \theta u & \textit{strongly in } C^0([0,T];\, H^{-1}(\Omega)), \end{array} \tag{2.13}$$

where u is the solution of the homogenized equation

$$\begin{cases} \theta u' - \operatorname{div}(A^0\nabla u) = \theta h + \sigma & \textit{in } \Omega\times(0,T),\\ u = 0 & \textit{on } \partial\Omega\times(0,T),\\ u(x,0) = u^0 & \textit{in } \Omega \end{cases} \tag{2.14}$$

with A^0 given by (2.10).

Moreover, if the data satisfy

$$\begin{aligned} &(\mathrm{i})\quad \lim_{\varepsilon\to 0}\|u^0_\varepsilon - u^0\|_{L^2(\Omega_\varepsilon)} = 0,\\ &(\mathrm{ii})\quad \lim_{\varepsilon\to 0}\|h_\varepsilon - h\|_{L^2(0,T;L^2(\Omega_\varepsilon))} = 0, \end{aligned} \tag{2.15}$$

then we have the following corrector result

$$\begin{aligned} &(\mathrm{i})\ \lim_{\varepsilon\to 0}\|u_\varepsilon - u\|_{C^0([0,T];\,L^2(\Omega_\varepsilon))} = 0,\\ &(\mathrm{ii})\ \lim_{\varepsilon\to 0}\|\nabla u_\varepsilon - C^\varepsilon\nabla u\|_{L^2((0,T);\,[L^1(\Omega_\varepsilon)]^n)} = 0. \end{aligned} \tag{2.16}$$

Remark 2.2. Theorem 2.1 has been proved in [14] in the case $\sigma = 0$. The same proof with the obvious modifications is still valid when $\sigma \neq 0$. The result in this case can also be derived from a more general one, proved, in a forthcoming paper, in the framework of the H^0-convergence. This convergence, introduced by M. Briane, A. Damlamian and P. Donato in [5], extends the H-convergence to (not necessarily periodic) perforated domains.

§ 3. Position of the Problem and Main Result

In this section we state the main result of this paper. We will prove it in the next section.

Let us consider the following nonlinear parabolic problem

$$\begin{cases} u'_\varepsilon - \mathrm{div}(A^\varepsilon\nabla u_\varepsilon) = f(u_\varepsilon) + g_\varepsilon & \text{in}\quad \Omega_\varepsilon\times(0,T),\\ u_\varepsilon = 0 & \text{on}\quad \partial\Omega\times(0,T),\\ A^\varepsilon\nabla u_\varepsilon\cdot\nu = 0 & \text{on}\quad \partial S_\varepsilon\times(0,T),\\ u_\varepsilon(x,0) = u^0_\varepsilon & \text{in}\quad \Omega_\varepsilon, \end{cases} \tag{3.1}$$

under the following assumptions:

$$\begin{aligned} &(\mathrm{i})\ (u^0_\varepsilon, u^0)\in L^2(\Omega_\varepsilon)\times L^2(\Omega)\ \text{ and }\ \lim_{\varepsilon\to 0}\|u^0_\varepsilon - u^0\|_{L^2(\Omega_\varepsilon)} = 0,\\ &(\mathrm{ii})\ (g_\varepsilon, g)\in L^2(0,T,L^2(\Omega_\varepsilon))\times L^2(0,T,L^2(\Omega)),\quad \lim_{\varepsilon\to 0}\|g^0_\varepsilon - g\|_{L^2(0,T,L^2(\Omega_\varepsilon))} = 0,\\ &(\mathrm{iii})\ f:\ \mathbb{R}\longrightarrow\mathbb{R}\ \text{ is globally Lipschitz continuous.} \end{aligned} \tag{3.2}$$

It is well known (see for instance [7]), that if $g_\varepsilon \in L^2(0,T,L^2(\Omega_\varepsilon))$ and $u^0_\varepsilon \in L^2(\Omega_\varepsilon)$, Problem (3.1) has a unique solution u_ε such that

$$u_\varepsilon \in L^2(0,T;\,V_\varepsilon)\cap C^0([0,T];\,L^2(\Omega_\varepsilon)).$$

Remark 3.1. In [14, Lemma 5.1] it is proved that the assumption (3.2)(i) is equivalent to the following one:

$$\begin{aligned} &(\mathrm{i})\ \widetilde{u^0_\varepsilon} \rightharpoonup \theta u^0\quad \text{weakly in}\ \ L^2(\Omega),\\ &(\mathrm{ii})\ \|u^0_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} \longrightarrow \theta\|u^0\|^2_{L^2(\Omega)}. \end{aligned} \tag{3.3}$$

In particular, if $u^0_\varepsilon = h_{|\Omega_\varepsilon}$ for some h in $L^2(\Omega)$, then $u^0 = h$ and (i) and (ii) are satisfied.

Similarly, the assumption (3.2)(ii) is equivalent to the following

$$\begin{aligned} &(\mathrm{i})\ \widetilde{g_\varepsilon} \rightharpoonup \theta g\quad \text{weakly in}\ \ L^2(0,T;L^2(\Omega)),\\ &(\mathrm{ii})\ \|g_\varepsilon\|^2_{L^2(0,T;L^2(\Omega_\varepsilon))} \longrightarrow \theta\|g\|^2_{L^2(0,T;L^2(\Omega))}. \end{aligned} \tag{3.4}$$

Also, observe that the assumption (3.2)(iii) implies that there exists $m \in \mathbb{R}_+$ such that

$$\forall\, s \in \mathbb{R}, \quad |f(s)| \leq m(1+|s|). \tag{3.5}$$

The asymptotic behavior of the semilinear problem (3.1) is given by the following theorem.

Theorem 3.1. *Under the assumption* (3.2)*, let* u_ε *be the solution of Problem* (3.1)*. Then*

$$\begin{aligned}
&(\mathrm{i})\ \ P^\varepsilon u_\varepsilon \rightharpoonup u \quad \textit{weakly in} \ \ L^2(0,T;\, H_0^1(\Omega)),\\
&(\mathrm{ii})\ \ \widetilde{A^\varepsilon \nabla u_\varepsilon} \rightharpoonup A^0 \nabla u \quad \textit{weakly in} \ \ [L^2(\Omega \times (0,T))]^n,\\
&(\mathrm{iii})\ \ \lim_{\varepsilon \to 0} \|f(u_\varepsilon) - f(u)\|_{L^2(\Omega_\varepsilon \times (0,T))} = 0,\\
&(\mathrm{iv})\ \ \lim_{\varepsilon \to 0} \|u_\varepsilon - u\|_{C^0([0,T];\, L^2(\Omega_\varepsilon))} = 0,\\
&(\mathrm{v})\ \ \lim_{\varepsilon \to 0} \|\nabla u_\varepsilon - C^\varepsilon \nabla u\|_{L^2((0,T);\, [L^1(\Omega_\varepsilon)]^n)} = 0,
\end{aligned} \tag{3.6}$$

where u *is the solution of the homogenized equation*

$$\begin{cases}
\theta u' - \operatorname{div}(A^0 \nabla u) = \theta f(u) + \theta g & \textit{in} \quad \Omega \times (0,T),\\
u = 0 & \textit{on} \quad \partial\Omega \times (0,T),\\
u(x,0) = u^0 & \textit{in} \quad \Omega
\end{cases} \tag{3.7}$$

with A^0 *given by* (2.10)*.*

Remark 3.2. As seen in Remark 3.1, convergence (3.6)(iii) is equivalent to the following ones:

$$\begin{aligned}
&(\mathrm{i})\ \ \widetilde{f(u_\varepsilon)} \rightharpoonup \theta f(u) \quad \text{weakly in} \ \ L^2(\Omega \times (0,T)),\\
&(\mathrm{ii})\ \ \lim_{\varepsilon \to 0} \|f(u_\varepsilon)\|^2_{L^2(\Omega_\varepsilon \times (0,T))} = \theta \|f(u)\|^2_{L^2(\Omega \times (0,T))}.
\end{aligned}$$

This describes the limit behaviour, as $\varepsilon \to 0$, of the nonlinear term $f(u_\varepsilon)$ on Ω_ε.

§ 4. Proof of the Main Result

The proof of Theorem 3.1 is given here in several steps. First, in Subsection 4.1, we prove some suitable a priori estimates on u_ε. In Subsection 4.2 we introduce the solution v_ε of the auxiliary linear problem and we show that the L^2-norm of $u_\varepsilon - v_\varepsilon$ goes to zero. There we also prove convergence (3.6)(iii). The conclusion of the proof is given in the last subsection.

4.1. A Priori Estimates

The variational formulation of Problem (3.1) is

$$\begin{cases}
\text{Find } u_\varepsilon \in L^2(0,T;V_\varepsilon) \text{ with } u_\varepsilon' \in L^2(0,T;V_\varepsilon'), \text{ such that for every } v \in V_\varepsilon,\\
\langle u_\varepsilon'(t), v\rangle_{V_\varepsilon', V_\varepsilon} + \displaystyle\int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla v\, dx = \int_{\Omega_\varepsilon} f(u_\varepsilon) v\, dx + \int_{\Omega_\varepsilon} g_\varepsilon v\, dx \quad \text{in} \ \ D'(0,T),\\
u_\varepsilon(0) = u_\varepsilon^0 \quad \text{in} \ \ \Omega_\varepsilon.
\end{cases} \tag{4.1}$$

Proposition 4.1. *Under the assumption* (3.2)*, let* u_ε *be the solution of Problem* (3.1)*. Then*

$$\begin{aligned} &(\mathrm{i}) \quad \|u_\varepsilon\|_{L^\infty(0,T,L^2(\Omega_\varepsilon))} \leq C,\\ &(\mathrm{ii}) \quad \|u_\varepsilon\|_{L^2(0,T,V_\varepsilon)} \leq C,\\ &(\mathrm{iii}) \quad \|u'_\varepsilon\|_{L^2(0,T,V'_\varepsilon)} \leq C, \end{aligned} \tag{4.2}$$

where C *is a constant independent of* ε*.*

Proof. Let us choose $v = u_\varepsilon$ in (4.1) and integrate in time between 0 and t. We have

$$\begin{aligned} &\frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + \int_0^t \int_{\Omega_\varepsilon} A^\varepsilon \nabla u_\varepsilon \nabla u_\varepsilon dxds\\ &= \frac{1}{2}\|u_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + \int_0^t \int_{\Omega_\varepsilon} f(u_\varepsilon)u_\varepsilon dxds + \int_0^t \int_{\Omega_\varepsilon} g_\varepsilon u_\varepsilon dxds. \end{aligned} \tag{4.3}$$

On the other hand, the assumption (3.2) implies that

$$|f(u_\varepsilon)u_\varepsilon| \leq m(1 + |u_\varepsilon|)|u_\varepsilon| \leq m(1 + |u_\varepsilon|^2) + m|u_\varepsilon|^2 = m + 2m|u_\varepsilon|^2,$$

where m is given by (3.5).

Hence, taking into account the coerciveness of A^ε, one has

$$\begin{aligned} &\frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + \alpha \int_0^t \|\nabla u_\varepsilon\|^2_{[L^2(\Omega_\varepsilon)]^n} ds\\ &\leq \frac{1}{2}\|u_\varepsilon(0)\|^2_{L^2(\Omega_\varepsilon)} + m|\Omega|t + 2m\int_0^t \|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} ds + \int_0^t \|g_\varepsilon\|_{L^2(\Omega_\varepsilon)}\|u_\varepsilon\|_{L^2(\Omega_\varepsilon)} ds. \end{aligned}$$

Consequently

$$\begin{aligned} &\frac{1}{2}\|u_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + \alpha \int_0^t \|\nabla u_\varepsilon\|^2_{[L^2(\Omega_\varepsilon)]^n} ds\\ &\leq \frac{1}{2}\|\widetilde{u^0_\varepsilon}\|^2_{L^2(\Omega)} + m|\Omega|t + 2m\int_0^t \|u_\varepsilon\|^2_{L^2(\Omega_\varepsilon)} ds + \frac{1}{2}\int_0^t \|\widetilde{g_\varepsilon}\|^2_{L^2(\Omega)}, \end{aligned}$$

and from the classical Gronwall Lemma we deduce estimates (4.2)(i) and (4.2)(ii).

To prove (4.2)(iii), let v be in $L^2(0,T;V_\varepsilon)$. From the variational formulation (4.1), it is easy to see that

$$\begin{aligned} \Big| \int_0^T \langle u'_\varepsilon(t), v\rangle_{V'_\varepsilon, V_\varepsilon} ds \Big| &\leq \|A^\varepsilon \nabla u_\varepsilon\|_{L^2(0,T;[L^2(\Omega_\varepsilon)]^n)} \|\nabla v\|_{L^2(0,T;[L^2(\Omega_\varepsilon)]^n)}\\ &+ |g_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))}\|v\|_{L^2(0,T;L^2(\Omega_\varepsilon))}\\ &+ \|f(u_\varepsilon)\|_{L^2(0,T;L^2(\Omega_\varepsilon))}\|v\|_{L^2(0,T;L^2(\Omega_\varepsilon))}. \end{aligned}$$

Hence, from (4.2)(i), (4.2)(ii), the assumption (3.2) and the Poincaré inequality on V_ε, we deduce

$$\Big| \int_0^T \langle u'_\varepsilon(t), v\rangle_{V'_\varepsilon, V_\varepsilon} \Big| ds \leq C\|v\|_{L^2(0,T;V_\varepsilon)}, \qquad \forall\, v \in L^2(0,T;V_\varepsilon),$$

where C is a constant independent of ε, which gives (4.2)(iii).

Corollary 4.1. *Under the assumptions of Proposition* 4.1*, there exist a subsequence of* $\{u_\varepsilon\}_\varepsilon$*, still denoted by* ε*, and* $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T,L^2(\Omega))$ *such that the following convergencies hold:*

$$\begin{aligned}
&(\mathrm{i})\ \ P^\varepsilon u_\varepsilon \rightharpoonup u \quad \textit{weakly} \quad \textit{in}\ \ L^2(0,T;H_0^1(\Omega)),\\
&(\mathrm{ii})\ \ \widetilde{u_\varepsilon} \rightharpoonup \theta u \quad \textit{weakly} * \ \textit{in}\ \ L^\infty(0,T;L^2(\Omega)),\\
&(\mathrm{iii})\ \ \widetilde{u_\varepsilon} \to \theta u \quad \textit{strongly} \quad \textit{in}\ \ C([0,T];H^{-1}(\Omega)).
\end{aligned} \tag{4.4}$$

Moreover

$$u(x,0) = u^0. \tag{4.5}$$

Proof. From Proposition 4.1 and Lemma 2.2, one has

$$\begin{aligned}
&(\mathrm{i})\ \ \|P^\varepsilon u_\varepsilon\|_{L^\infty(0,T,L^2(\Omega))} \le C,\\
&(\mathrm{ii})\ \ \|\widetilde{u_\varepsilon}\|_{L^\infty(0,T,L^2(\Omega))} \le C,\\
&(\mathrm{iii})\ \ \|P^\varepsilon u_\varepsilon\|_{L^2(0,T,H_0^1(\Omega))} \le C.
\end{aligned} \tag{4.6}$$

Then, there exists a subsequence (still denoted by ε) and $u \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(0,T,L^2(\Omega))$ such that (4.4)(i) holds. Convergence (4.4)(ii) follows from (4.4)(i) and convergence (2.3) (see Lemma 3.2 of [14] for a detailed proof). Finally, to prove (4.4)(iii), it suffices to observe that, thanks to (4.4)(ii) and (4.2)(iii), Lemma 2.1 can be applied to $\{u_\varepsilon\}$.

To prove (4.5), we observe that from (4.4)(iii) one has

$$\widetilde{u_\varepsilon}(0) \to \theta u(0) \qquad \text{strongly in}\ \ H^{-1}(\Omega).$$

On the other hand, from (3.3) and the initial condition in Problem (3.1), we can deduce that

$$\widetilde{u_\varepsilon}(0) = \widetilde{u_\varepsilon^0} \rightharpoonup \theta u^0 \qquad \text{weakly in}\ \ L^2(\Omega).$$

The uniqueness of the weak limit gives the claimed result.

4.2. The Auxiliary Problem and Some Convergence Results

We introduce here the solution $v_\varepsilon \in L^2(0,T;\,V_\varepsilon) \cap C^0([0,T];\,L^2(\Omega_\varepsilon))$ of the following problem

$$\begin{cases}
v_\varepsilon' - \mathrm{div}(A^\varepsilon \nabla v_\varepsilon) = P_\varepsilon^*(\theta u' - \mathrm{div}(A^0\nabla u)) + (g_\varepsilon - g_{/\Omega_\varepsilon}) & \text{in} \quad \Omega_\varepsilon \times (0,T),\\
v_\varepsilon = 0 & \text{on} \quad \partial\Omega \times (0,T),\\
A^\varepsilon \nabla v_\varepsilon \cdot \nu = 0 & \text{on} \quad \partial S_\varepsilon \times (0,T),\\
v_\varepsilon(x,0) = u_\varepsilon^0 & \text{in} \quad \Omega_\varepsilon,
\end{cases} \tag{4.7}$$

where P_ε^* denotes the adjoint of P_ε, $g_{/\Omega_\varepsilon}$ is the restriction of the function g to the domain Ω_ε and u is given by Corollary 4.1.

Thanks to the assumption (3.2) and convergence (2.3), Theorem 2.1, written for $h_\varepsilon = g_\varepsilon - g_{/\Omega_\varepsilon}$ and $\sigma = \theta u' - \mathrm{div}(A^0\nabla u)$, applies to Problem (4.7).

Then, $P^\varepsilon v_\varepsilon$ weakly converges to the solution v of

$$\begin{cases} \theta v' - \operatorname{div}(A^0\nabla v) = \theta u' - \operatorname{div}(A^0\nabla u) & \text{in } \ \Omega \times (0,T), \\ v = 0 & \text{on } \ \partial\Omega \times (0,T), \\ v(x,0) = u^0 & \text{in } \ \Omega. \end{cases} \tag{4.8}$$

The uniqueness of this problem, together with (4.5), implies that $v = u$. Hence

$$\begin{aligned} &(\mathrm{i})\ \ P^\varepsilon v_\varepsilon \rightharpoonup u \quad \text{weakly in } \ L^2(0,T;\, H_0^1(\Omega)), \\ &(\mathrm{ii})\ \ \widetilde{v_\varepsilon} \rightharpoonup \theta u \quad \text{weakly in } \ L^\infty(0,T;\, L^2(\Omega)), \\ &(\mathrm{iii})\ \ \widetilde{v_\varepsilon} \to \theta u \quad \text{strongly in } \ C([0,T], H^{-1}(\Omega)), \\ &(\mathrm{iv})\ \ \lim_{\varepsilon\to 0} \|v_\varepsilon - u\|_{C^0([0,T];\, L^2(\Omega_\varepsilon))} = 0, \\ &(\mathrm{v})\ \ \lim_{\varepsilon\to 0} \|\nabla v_\varepsilon - C^\varepsilon \nabla u\|_{L^2((0,T);\, [L^1(\Omega_\varepsilon)]^n)} = 0. \end{aligned} \tag{4.9}$$

The following proposition is the main tool for proving Theorem 3.1.

Proposition 4.2. *Under the assumption* (3.2)*, we have*

$$\lim_{\varepsilon\to 0} \|u_\varepsilon - v_\varepsilon\|_{L^2(0,T;\, L^2(\Omega_\varepsilon))} = 0, \tag{4.10}$$

where u_ε and v_ε are respectively solutions of (3.1) *and* (4.7)*.*

Proof. Set $w_\varepsilon = u_\varepsilon - v_\varepsilon$. One has

$$\begin{cases} w_\varepsilon' - \operatorname{div}(A^\varepsilon \nabla w_\varepsilon) = f(u_\varepsilon) - P_\varepsilon^*(\theta u' - \operatorname{div}(A^0\nabla u)) + g_{/\Omega_\varepsilon} & \text{in } \ \Omega_\varepsilon \times (0,T), \\ w_\varepsilon = 0 & \text{on } \ \partial\Omega \times (0,T), \\ A^\varepsilon \nabla w_\varepsilon \cdot \nu = 0 & \text{on } \ \partial S_\varepsilon \times (0,T), \\ w_\varepsilon(x,0) = 0 & \text{in } \ \Omega_\varepsilon. \end{cases}$$

Choosing w_ε as test function in the variational formulation of this problem and integrating in time between 0 and t, we obtain

$$\begin{aligned} &\frac{1}{2}\|w_\varepsilon(t)\|^2_{L^2(\Omega_\varepsilon)} + \int_0^t \int_{\Omega_\varepsilon} A^\varepsilon \nabla w_\varepsilon \nabla w_\varepsilon dxds \\ &= \int_0^t \int_{\Omega_\varepsilon} f(u_\varepsilon) w_\varepsilon dxds - \int_0^t \langle \theta u', P^\varepsilon w_\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\, ds \\ &\quad + \int_0^t \left\langle \operatorname{div}(A^0\nabla u), P^\varepsilon w_\varepsilon \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} ds + \int_0^t \int_\Omega \chi_{\Omega_\varepsilon} g \widetilde{w_\varepsilon} dxds. \end{aligned} \tag{4.11}$$

We want now to pass to the limit, as $\varepsilon \to 0$, in this equation. From convergencies (4.4) and (4.9), it is easy to see that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \int_0^t \int_\Omega \chi_{\Omega_\varepsilon} g \widetilde{w_\varepsilon} dxds = 0, \\ &\lim_{\varepsilon\to 0} \int_0^t \langle \theta u', P^\varepsilon w_\varepsilon \rangle_{H^{-1}(\Omega), H_0^1(\Omega)}\, ds = 0, \\ &\lim_{\varepsilon\to 0} \int_0^t \left\langle \operatorname{div}(A^0\nabla u), P^\varepsilon w_\varepsilon \right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} ds = 0. \end{aligned} \tag{4.12}$$

The main difficulty is to prove that

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\Omega_\varepsilon} f(u_\varepsilon)w_\varepsilon dxds = 0. \tag{4.13}$$

Indeed, due to the lack of compactness of $P^\varepsilon u_\varepsilon$ in $L^2(0,T,L^2(\Omega))$, both terms $\widetilde{f(u_\varepsilon)}$ and $\widetilde{w_\varepsilon}$ only weakly converge in $L^2(0,T,L^2(\Omega))$. Hence, we use the strong convergence to zero of $\widetilde{w_\varepsilon}$ in $C^0([0,T];\, H^{-1}(\Omega))$, as follows.

From convergencies (4.4)(ii) and (4.9)(ii), one has

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\Omega_\varepsilon} f(0)w_\varepsilon dxds = 0.$$

Hence, to prove (4.13) it suffices to show that

$$\lim_{\varepsilon\to 0}\int_0^t\int_{\Omega_\varepsilon} f_1(u_\varepsilon)w_\varepsilon dxds = 0, \tag{4.14}$$

where $f_1(s) = f(s) - f(0)$.

Obviously, $f_1(0) = 0$ and in view of (3.2)(iii), also f_1 is globally Lipschitz continuous. Then, f_1 is derivable a.e. and f_1' is in $L^\infty(\mathbb{R})$.

Then, from known results (see [15, p.54] as well as the more general proof of A. Ancona given in [3]), one has that $f_1(P^\varepsilon u_\varepsilon)$ is in $L^2(0,T,H_0^1(\Omega))$ and

$$\nabla(f_1(P^\varepsilon u_\varepsilon)) = f_1'(P^\varepsilon u_\varepsilon)\nabla(P^\varepsilon u_\varepsilon).$$

Since

$$\|f(P^\varepsilon u_\varepsilon)\|^2_{L^2(0,T,H_0^1(\Omega))} \leq \int_0^T (\|f(P^\varepsilon u_\varepsilon)\|^2_{L^2(\Omega)} + \|f'(P^\varepsilon u_\varepsilon)\|^2_{L^\infty(\Omega)}\|\nabla(f(P^\varepsilon u_\varepsilon))\|^2_{L^2(\Omega)}),$$

this, together with the assumption (3.2)(iii) and the estimate (4.6)(iii) implies that

$$f_1(P^\varepsilon u_\varepsilon) \quad \text{is bounded in} \quad L^2(0,T,H_0^1(\Omega)). \tag{4.15}$$

On the other hand, from (4.4)(iii) and (4.9)(iii), we have (up to a subsequence, still denoted by ε)

$$\widetilde{w_\varepsilon} \to 0 \quad \text{strongly in} \quad C([0,T],H^{-1}(\Omega)). \tag{4.16}$$

From (4.16) and (4.15), one has

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_0^t\int_{\Omega_\varepsilon} f(u_\varepsilon)w_\varepsilon dxds &= \lim_{\varepsilon\to 0}\int_0^t\int_{\Omega} f(P^\varepsilon u_\varepsilon)\widetilde{w_\varepsilon} dxds\\ &= \lim_{\varepsilon\to 0}\int_0^t \langle\widetilde{w_\varepsilon}, f(P^\varepsilon u_\varepsilon)\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\\ &= 0,\end{aligned}$$

which proves (4.14) and then (4.13). We can now pass to the limit in (4.11) and we deduce that

$$\limsup_{\varepsilon\to 0}\|\widetilde{w_\varepsilon}(t)\|^2_{L^2(\Omega)} = 0, \qquad \forall\, t\in(0,T).$$

Finally, by using the Lebesgue dominated convergence theorem, we conclude the proof.

Corollary 4.2. *Under the assumptions of Proposition* 4.2*, we have*

$$\lim_{\varepsilon\to 0}\|f(u_\varepsilon)-f(u)\|_{L^2(\Omega_\varepsilon\times(0,T))}=0, \tag{4.17}$$

where u is given by Corollary 4.1*.*

Proof. Observe that, since f is Lipschitz continuous,

$$\begin{aligned}&\|f(u_\varepsilon)-f(u)\|_{L^2(\Omega_\varepsilon\times(0,T))}\\ &\le C\|u_\varepsilon-u\|_{L^2(\Omega_\varepsilon\times(0,T))}\\ &\le \|u_\varepsilon-v_\varepsilon\|_{L^2(\Omega_\varepsilon\times(0,T))}+\|v_\varepsilon-u\|_{L^2(\Omega_\varepsilon\times(0,T))}.\end{aligned}$$

Hence, from convergencies (4.9)(iv) and (4.10), we deduce the desired convergence.

4.3. End of the Proof

To conclude the proof, it only remains to prove that the function u given in (4.4) is the unique solution of Problem (2.14). To do that, since (3.2)(i) holds, it suffices to check that $h_\varepsilon=f(u_\varepsilon)+g_\varepsilon$ satisfies (2.15)(ii) of Theorem 2.1. This follows straightforward from Remark 3.1, the assumption (3.2)(ii) and Corollary 4.2.

Hence, by applying Theorem 2.1 with $\sigma=0$ and $h_\varepsilon=g_\varepsilon+f(u_\varepsilon)$, we deduce that u_ε verifies

$$\begin{array}{lll}\text{(i)} & P^\varepsilon u_\varepsilon \rightharpoonup u & \text{weakly in } L^2(0,T;\, H_0^1(\Omega)),\\ \text{(ii)} & \widetilde{A^\varepsilon\nabla u_\varepsilon} \rightharpoonup A^0\nabla u & \text{weakly in } [L^2(\Omega\times(0,T))]^n,\\ \text{(iii)} & \lim\limits_{\varepsilon\to 0}\|u_\varepsilon-u\|_{C^0([0,T];\,L^2(\Omega_\varepsilon))}=0, & \\ \text{(iv)} & \lim\limits_{\varepsilon\to 0}\|\nabla u_\varepsilon-C^\varepsilon\nabla u\|_{L^2((0,T);\,[L^1(\Omega_\varepsilon)]^n)}=0, & \end{array}$$

where u is the solution of the homogenized equation

$$\begin{cases}\theta u'-\operatorname{div}(A^0\nabla u)=\theta f(u)+\theta g & \text{in}\quad \Omega\times(0,T),\\ u=0 & \text{on}\quad \partial\Omega\times(0,T),\\ u(x,0)=u^0 & \text{in}\quad \Omega.\end{cases}$$

This concludes the proof of Theorem 3.1, since this problem has a unique solution.

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