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REGULAR RELATIONS AND MONOTONE NORMAL ORDERED SPACES***

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Abstract

In this paper the classical theorem of Zareckiǐ about regular relations is generalized and an intrinsic characterization of regularity is obtained. Based on the generalized Zareckiǐ theorem and the intrinsic characterization of regularity, the authors give a characterization of monotone normality of ordered spaces. A new proof of the Urysohn-Nachbin lemma is presented which is quite different from the classical one.

Keywords Regular relation, Complete distributivity, Monotone normality
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From the point of view of lattice-ordered structures, binary relations have attracted a considerable deal of attention since the work of Raney and Zareckiǐ. In [8] Raney proved that if a binary relation ρ on a set X is idempotent, then the complete lattice $(\Phi_\rho(X), \subseteq)$ is completely distributive, where

$$\Phi_\rho(X) = \{\rho(A) : A \subseteq X\}, \quad \rho(A) = \{y \in X : \exists a \in A \text{ with } (a, y) \in \rho\}.$$

In [13] (see also [12]) Zareckiǐ generalized the work of Raney and proved the following remarkable result: a binary relation ρ on a set X is regular if and only if $(\Phi_\rho(X), \subseteq)$ is completely distributive. Further criteria for regularity were given by Markowsky [5] and Schein [10] (see also [1]).

In this paper we generalize the Zareckii theorem and give an intrinsic characterization of regularity. It is showed that the generalized Zareckiǐ theorem can be naturally deduced from the Raney theorem. Based on the generalized Zareckiǐ theorem and the intrinsic characterization of regularity we give a characterization of monotone normality of ordered spaces. A new proof of the Urysohn-Nachbin lemma is presented, which is quite different from the classical one.

§ 1 . Preliminaries

In this paper, **Set** denotes the class of sets. The class of complete lattices is denoted by **Com**. $\forall L \in$**Com**, $x \in L$, and $A \subseteq L$, let

$$\uparrow x = \{y \in L : x \le y\}, \qquad \uparrow A = \bigcup_{a \in A} \uparrow a;$$

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dually define $\downarrow x$ and $\downarrow A$. For two complete lattices L_1 and L_2, the symbol $L_1 \cong L_2$ stands for the statement that L_1 is isomorphic to L_2. $\forall X \in \mathbf{Set}$, let

$$X^{(<\omega)} = \{F \subseteq X : F \text{ is finite and nonempty}\}.$$

For a topology η on a set X, the set of all closed subsets of X is denoted by η^c, i.e.,

$$\eta^c = \{X \setminus U : U \in \eta\}.$$

If X is a poset, then define

$$\eta^{\uparrow} = \{U \in \eta : U = \uparrow U\}, \qquad \eta^{c\uparrow} = \{U \in \eta^c : U = \uparrow U\}.$$

The families $\eta^{\downarrow}$ and $\eta^{c\downarrow}$ are defined in the same way.

For a poset P, the topology generated by the collection of sets $P \setminus \downarrow x$ (as subbasic open subsets) is called the upper topology on P and denoted by $\upsilon(P)$; dually define the lower topology $\omega(P)$ on P. The topology $\theta(P) = \omega(P) \bigvee \upsilon(P)$ is called the interval topology on P. In this paper the unit closed interval $[0,1]$ is always endowed with the interval topology.

$\forall X, Y \in \mathbf{Set}$, we call ϱ a binary relation between X and Y, written $\varrho : X \rightharpoonup Y$, if $\varrho \subseteq X \times Y$. When $X = Y$, ϱ is usually called a binary relation on X. Let $\mathbf{Rel}(X,Y)$ denote the set of all binary relations between X and Y. For $\varrho : X \rightharpoonup Y$ and $\tau : Y \rightharpoonup Z$, define $\tau \circ \varrho = \{(x,z) \in X \times Z : \exists y \in Y \text{ with } (x,y) \in \varrho \text{ and } (y,z) \in \tau\}$. The relation $\tau \circ \varrho : X \rightharpoonup Z$ is called the composition of ϱ and τ.

For a binary relation $\rho : X \rightharpoonup Y$ and $A \subseteq X$, define

$$\rho(A) = \{y \in Y : \exists a \in A, (a,y) \in \rho\},$$

called the image of A under ρ. Let

$$\Phi_\rho(X,Y) = \{\rho(A) : A \subseteq X\}.$$

When $Y = X$, $\Phi_\rho(X,Y)$ is shortly written as $\Phi_\rho(X)$. Clearly, $(\Phi_\rho(X,Y), \subseteq)$ is a complete lattice in which the join operation $\vee$ is the set union operator $\cup$. But the meet operation $\wedge$ in $(\Phi_\rho(X,Y), \subseteq)$ is not the set intersection operator $\cap$ in general.

Definition 1.1. *A complete lattice L is called completely distributive if the following condition holds for all $\{x_{ij} : j \in J_i, i \in I\} \subseteq L$:*

$$\bigwedge_{i\in I} \bigvee_{j \in J_i} x_{ij} = \bigvee_{\varphi \in \prod J_i} \bigwedge_{i \in I} x_{i\varphi(i)}.$$

The following result is well known (see [3, 11] or [8, 9]).

Theorem 1.1. *Let L be a completely distributive lattice. Then L can be embedded into a cube $[0,1]^X$ via a complete lattice homomorphism.*

Corollary 1.1. *Let L be a completely distributive lattice. Then $\forall x, y \in L$ with $x \not\leq y$, there is a complete lattice homomorphism $f : L \to [0,1]$ such that $f(x) = 1$ and $f(y) = 0$.*

For a poset P and $A \subseteq P$, define

$$A^{\uparrow} = \{x \in P : a \leq x \text{ for all } a \in A\},$$
$$A^{\downarrow} = \{x \in P : x \leq a \text{ for all } a \in A\}.$$

Let

$$\delta(P) = \{A^{\uparrow\downarrow} : A \subseteq P\} = \{B^{\downarrow} : B \subseteq P\}.$$

The complete lattice $(\delta(P), \subseteq)$ is called the normal completion (or completion by cuts) of P (see [2]). It is readily verified that P is complete if and only if $P \cong (\delta(P), \subseteq)$.

§ 2. Regularity and Complete Distributivity

$\forall L \in \mathbf{Com}$, define a binary relation $\lhd$ (called the completely below relation) on L by $x \lhd y \Leftrightarrow \forall A \subseteq L,\ y \leq \vee A \Rightarrow \exists a \in A$ with $x \leq a$. Let

$$\Downarrow x = \{u \in L : u \lhd x\}, \qquad \Uparrow x = \{v \in L : x \lhd v\}.$$

By the definition of $\lhd$ we get the following

Lemma 2.1. *Let $L \in$* **Com**, *$u \in L$ and $x \in L \setminus \{0\}$. Then*

$$u \lhd x \Leftrightarrow x \not\leq \vee(L \setminus \uparrow u).$$

Therefore

$$\Uparrow x = L \setminus \downarrow \vee(L \setminus \uparrow x) \in \upsilon(L).$$

Lemma 2.2. (cf. [8]) *A complete lattice L is completely distributive $\Leftrightarrow \forall x \in L$, $x = \vee \Downarrow x$.*

$\forall X \in \mathbf{Set}$, it is easy to see that $(\mathbf{Rel}(X,X), \circ)$ is a semigroup with the unit $\triangle(X) = \{(x,x) : x \in X\}$. For $\rho \in \mathbf{Rel}(X,X)$, if $\rho^2 = \rho \circ \rho = \rho$, then ρ is called idempotent; ρ is called regular if ρ is a regular element in the semigroup $(\mathbf{Rel}(X,X), \circ)$, i.e., $\exists \sigma \in \mathbf{Rel}(X,X)$ with $\rho \circ \sigma \circ \rho = \rho$. This motivates the following

Definition 2.1. *A binary relation $\rho : X \rightharpoonup Y$ is called regular if $\exists \sigma : Y \rightharpoonup X$ such that $\rho \circ \sigma \circ \rho = \rho$.*

Example 2.1. (1) The relation $\in$ between a set X and its powerset $\mathcal{P}(X)$ is regular. In fact, define a binary relation $\sigma : \mathcal{P}(X) \rightharpoonup X$ by $(A,a) \in \sigma \Leftrightarrow A = \{a\}$. Then $\in \circ\, \sigma \circ \in = \in$.

(2) Every mapping $f : X \to Y$ is regular when it is regarded as a binary relation between X and Y. In fact, define a binary relation $\tau : Y \rightharpoonup X$ by $(y,x) \in \tau \Leftrightarrow y = f(x)$. Then $f \circ \tau \circ f = f$.

In [13] Zareckiĭ proved the following remarkable result, which improves greatly the Raney theorem.

Theorem 2.1. (cf. [13]) *For a binary relation ρ on a set X, the following two conditions are equivalent:*

(1) *ρ is regular;*

(2) *$(\Phi_\rho(X), \subseteq)$ is a completely distributive lattice.*

For a binary relation $\rho : X \rightharpoonup Y$, let $X^* = X \times \{1\}$, $Y^* = Y \times \{2\}$. Then $X^* \cap Y^* = \emptyset$. Define a relation $\rho^* : X^* \rightharpoonup Y^*$ by $((x,1),(y,2)) \in \rho^* \Leftrightarrow (x,y) \in \rho$. Then the following lemma is readily verified.

Lemma 2.3. (1) *ρ is regular $\Leftrightarrow \rho^*$ is regular.*

(2) $(\Phi_\rho(X,Y), \subseteq) \cong (\Phi_{\rho^*}(X^*,Y^*), \subseteq)$.

For relations between regularity and idempotence we have the following

Lemma 2.4. *For a binary relation $\rho : X \rightharpoonup Y$ with $X \cap Y = \emptyset$, let $Z = X \cup Y$. Then*

(1) *$\rho : X \rightharpoonup Y$ is regular $\Leftrightarrow \exists \tau : Y \rightharpoonup X$ such that $\sigma = \rho \cup \rho \circ \tau : Z \rightharpoonup Z$ is idempotent.*

(2) *For $\tau : Y \rightharpoonup X$, if $\sigma = \rho \cup \rho \circ \tau : Z \rightharpoonup Z$ is idempotent, then $(\Phi_\rho(X,Y), \subseteq)$ is a complete sublattice of $(\Phi_\sigma(Z), \subseteq)$.*

Proof. (1) If ρ is regular, then there is $\tau : Y \rightharpoonup X$ with $\rho \circ \tau \circ \rho = \rho$. We verify that $\sigma = \rho \cup \rho \circ \tau : Z \rightharpoonup Z$ is idempotent. Since $X \cap Y = \emptyset$,

$$\sigma \circ \sigma = (\rho \cup \rho \circ \tau) \circ (\rho \cup \rho \circ \tau) = \rho \circ \tau \circ \rho \cup \rho \circ \tau \circ \rho \circ \tau = \rho \cup \rho \circ \tau = \sigma,$$

i.e., σ is idempotent. Conversely, suppose that there is $\tau : Y \rightharpoonup X$ such that $\sigma = \rho \cup \rho \circ \tau : Z \rightharpoonup Z$ is idempotent. By $X \cap Y = \emptyset$,

$$(\rho \cup \rho \circ \tau) \circ (\rho \cup \rho \circ \tau) = \rho \circ \tau \circ \rho \cup \rho \circ \tau \circ \rho \circ \tau.$$

Hence

$$\rho \circ \tau \circ \rho \cup \rho \circ \tau \circ \rho \circ \tau = \rho \cup \rho \circ \tau.$$

Since $\{\rho, \rho \circ \tau \circ \rho\} \subseteq \mathbf{Rel}(X, Y)$, $\{\rho \circ \tau, \rho \circ \tau \circ \rho \circ \tau\} \subseteq \mathbf{Rel}(Y, Y)$ and $X \cap Y = \emptyset$, from $\rho \circ \tau \circ \rho \cup \rho \circ \tau \circ \rho \circ \tau = \rho \cup \rho \circ \tau$, we have

$$\rho \circ \tau \circ \rho = \rho \quad \text{and} \quad \rho \circ \tau \circ \rho \circ \tau = \rho \circ \tau.$$

Whence ρ is regular.

(2) For $\tau : Y \rightharpoonup X$, if $\sigma = \rho \cup \rho \circ \tau : Z \rightharpoonup Z$ is idempotent, then $\forall A \subseteq X, \tau(A) = \emptyset$ (since $X \cap Y = \emptyset$); and hence

$$\sigma(A) = \rho(A) \cup \rho(\tau(A)) = \rho(A) \cup \rho(\emptyset) = \rho(A).$$

Whence $\rho(A) \in \Phi_\sigma(Z)$. Let $L = \Phi_\sigma(Z)$ and $M = \Phi_\rho(X, Y)$. Then $M \subseteq L$. $\forall \{A_i : i \in I\} \subseteq \mathcal{P}(X)$, we have

$$\vee_L \{\rho(A_i) : i \in I\} = \vee_L \{\sigma(A_i) : i \in I\} = \sigma\Big(\bigcup_{i \in I} A_i\Big) = \rho\Big(\bigcup_{i \in I} A_i\Big) \in M.$$

Now we verify that

$$\wedge_L \{\rho(A_i) : i \in I\} = \wedge_M \{\rho(A_i) : i \in I\}.$$

Clearly

$$\wedge_M \{\rho(A_i) : i \in I\} \subseteq \wedge_L \{\rho(A_i) : i \in I\}.$$

On the other hand, since $\wedge_L \{\rho(A_i) : i \in I\} \in L$, there is $U \subseteq Z = X \cup Y$ with $\sigma(U) = \wedge_L \{\rho(A_i) : i \in I\} \in M$. In fact, by $\sigma = \rho \cup \rho \circ \tau, \rho \in \mathbf{Rel}(X, Y), \tau \in \mathbf{Rel}(Y, X)$ and $X \cap Y = \emptyset$, we have

$$\sigma(U) = \rho(U \cap X) \cup \rho \circ \tau(U \cap Y) = \rho((U \cap X) \cup \tau(U \cap Y)) \in M.$$

Whence $\wedge_L \{\rho(A_i) : i \in I\} \in M$; and hence

$$\wedge_L \{\rho(A_i) : i \in I\} \subseteq \wedge_M \{\rho(A_i) : i \in I\} \in M.$$

Therefore

$$\wedge_L \{\rho(A_i) : i \in I\} = \wedge_M \{\rho(A_i) : i \in I\} \in M.$$

Thus $(\Phi_\rho(X, Y), \subseteq)$ is a complete sublattice of $(\Phi_\sigma(Z), \subseteq)$.

Now we generalize the Zareckiǐ theorem and show that this generalized Zareckiǐ theorem can be naturally deduced from the Raney theorem. Meanwhile, we give an intrinsic characterization (the condition (3) in Theorem 2.2) of regularity.

Theorem 2.2. *For a binary relation $\rho : X \rightharpoonup Y$, the following two conditions are equivalent:*
(1) *ρ is regular;*
(2) *$(\Phi_\rho(X), \subseteq)$ is a completely distributive lattice;*
(3) *$\forall (x, y) \in X \times Y, (x, y) \in \rho \Rightarrow \exists (u, v) \in X \times Y$ such that*
 (a) *$(x, v) \in \rho, (u, y) \in \rho$, and*
 (b) *$\forall (s, t) \in X \times Y, (s, v) \in \rho$ and $(u, t) \in \rho \Rightarrow (s, t) \in \rho$.*

Proof. (1)$\Rightarrow$(2). Let $\rho : X \rightharpoonup Y$ be regular. Then by Lemma 2.3, $\rho^* : X^* \rightharpoonup Y^*$ is regular. Therefore, by Lemma 2.4 and the Raney theorem, $(\Phi_{\rho^*}(X^*, Y^*), \subseteq)$ is completely distributive. Hence by Lemma 2.3, $(\Phi_\rho(X, Y), \subseteq)$ is completely distributive.

(2)$\Rightarrow$(3). Suppose that $(x, y) \in \rho$. $\forall z \in \rho(x)$, let

$$\mathcal{N}(z) = \{N \in \Phi_\rho(X, Y) : z \in N\}.$$

Since $(\Phi_\rho(X, Y), \subseteq)$ is completely distributive and the complete distributivity is self-dual (see [7]), we have

$$\rho(x) \supseteq \bigcup_{z \in \rho(x)} \bigwedge \mathcal{N}(z) = \bigwedge_{\phi \in \prod \mathcal{N}(z)} \bigcup_{z \in \rho(x)} \phi(z) \supseteq \rho(x).$$

It follows that $y \in \rho(x) = \bigcup_{z \in \rho(x)} \bigwedge \mathcal{N}(z)$. Therefore, there is $v \in \rho(x)$ with $y \in \bigwedge \mathcal{N}(v)$. Since $\bigwedge \mathcal{N}(v) \in \Phi_\rho(X, Y)$, there is $A \subseteq X$ with $\rho(A) = \bigwedge \mathcal{N}(v)$. Whence $\exists u \in A$ with $y \in \rho(u) \subseteq \bigwedge \mathcal{N}(v)$. Thus we get the condition (a). Now we check the condition (b). $\forall (s, t) \in X \times Y$, if $(s, v) \in \rho$ and $(u, t) \in \rho$, then $t \in \rho(u) \subseteq \bigwedge \mathcal{N}(v) \subseteq \rho(s)$. Hence $(s, t) \in \rho$.

(3)$\Rightarrow$(1). Define a binary relation $\rho^\dagger : Y \rightharpoonup X$ by

$$(p, q) \in \rho^\dagger \Leftrightarrow \forall (c, d) \in X \times Y, \quad (c, q) \in \rho \quad \text{and} \quad (p, d) \in \rho \Rightarrow (c, d) \in \rho.$$

We show that $\rho \circ \rho^\dagger \circ \rho = \rho$. $\forall (x, y) \in X \times Y$, if $(x, y) \in \rho$, then by (2), there exists $(u, v) \in X \times Y$ which satisfies the conditions (a) and (b) in (2). By the definition of $\rho^\dagger$, $(u, v) \in \rho^\dagger$; and hence by $(x, v) \in \rho$ and $(u, y) \in \rho$, $(x, y) \in \rho \circ \rho^\dagger \circ \rho$. On the other hand, if $(x, y) \in \rho \circ \rho^\dagger \circ \rho$, then there exists $(p, q) \in X \times Y$ such that $(x, q) \in \rho, (q, p) \in \rho^\dagger$ and $(p, y) \in \rho$. By the definition of $\rho^\dagger$, $(x, y) \in \rho$. Hence $\rho = \rho \circ \rho^\dagger \circ \rho$. Thus ρ is regular.

For a poset $(P, \leq)$, let $\rho = \not\leq$. Then $\forall A \in P$, $\rho(A) = P \setminus A^\uparrow$; and hence

$$\Phi_\rho(P) = \{P \setminus A^\uparrow : A \subseteq P\}.$$

Define a mapping $\varphi : (\Phi_\rho(P), \subseteq) \to (\delta(P), \subseteq)$ by $\varphi(P \setminus A^\uparrow) = A^{\uparrow\downarrow}$. It is readily verified that φ is a lattice isomorphism. Hence by Theorem 2.2 we get the following

Corollary 2.1. *For a poset $(P, \leq)$, the following three conditions are equivalent:*
(1) *the relation $\not\leq$ on P is regular;*
(2) *$\forall x, y \in P, x \not\leq y \Rightarrow \exists u, v \in P$ such that*
 (a) *$x \not\leq v, u \not\leq y$, and*
 (b) *$\forall z \in P,\ u \leq z$ or $z \leq v$;*
(3) *the normal completion $(\delta(P), \subseteq)$ of P is completely distributive.*

The condition (2) in Corollary 2.1 is the intrinsic characterization of complete distributivity. It was first obtained by Raney [9] by using the tight Galois connections. Corollary 2.1 shows that the intrinsic characterization of complete distributivity is another expression of the regularity of $\not\leq$.

§ 3 . Regular Relations and Monotone Normal Ordered Spaces

For a poset $(P, \leq)$ and a topology δ on P, the triple $(P, \leq, \delta)$ is called a partially ordered topological space. We customarily denote $(P, \leq, \delta)$ simply by (P, δ) and call it an ordered space for short. The partial order $\leq$ on P is called semiclosed if $\downarrow x$ and $\uparrow x$ are closed in (P, δ) for each $x \in P$.

Definition 3.1. *Let (P, δ) be ordered space. $\forall A \in P$, define*

$$\mathrm{cl}^{\uparrow}(A) = \cap\{G \in \delta^c : G = \uparrow G \text{ and } A \subseteq G\}.$$

Definition 3.2. *An ordered space (P, δ) is said to be monotone normal if $\forall (A, B) \in \delta^{c\uparrow} \times \delta^{c\downarrow}$ with $A \bigcap B = \emptyset$, there exists $(U, V) \in \delta^{c\uparrow} \times \delta^{c\downarrow}$ such that $A \subseteq U, B \subseteq V$, and $U \bigcap V = \emptyset$.*

It is easy to get the following

Lemma 3.1. *An ordered space (P, δ) is monotone normal if and only if $\forall (A, U) \in \delta^{c\uparrow} \times \delta^{\uparrow}$ with $A \subseteq U$, there exists $V \in \delta^{\uparrow}$ such that $A \subseteq V \subseteq \mathrm{cl}^{\uparrow}(V) \subseteq U$.*

For an ordered space (P, δ), define a binary relation $\rho : \delta^c \rightharpoonup \delta$ by

$$(A, U) \in \rho \Leftrightarrow A = \uparrow A, \quad U = \uparrow U \quad \text{and} \quad A \subseteq U. \tag{$*$}$$

Theorem 3.1. *For an ordered space (P, δ), let $\rho : \delta^c \rightharpoonup \delta$ be the relation defined by $(*)$ and $L = (\Phi_\rho(\delta^c), \subseteq)$. Consider the following three conditions:*
(1) (P, δ) is monotone normal.
(2) ρ is regular and the function $g : (P, \delta) \to (L, \theta(L))$, $g(x) = \rho(\mathrm{cl}^{\uparrow}(\{x\}))$, is monotone and continuous.
(3) ρ is regular.
Then (1)⇔(2)⇒(3). If the partial order $\leq$ on P is semiclosed, then (3)⇒(1); and hence all three conditions are equivalent.

Proof. (1)⇒(2). Suppose that (P, δ) is a monotone normal ordered space. Define a binary relation $\sigma : \delta \rightharpoonup \delta^c$ by

$$(V, B) \in \sigma \Leftrightarrow V = \uparrow V, \quad B = \uparrow B \quad \text{and} \quad V \subseteq B.$$

We show that $\rho \circ \sigma \circ \rho = \rho$. $\forall (A, U) \in \delta^c \times \delta$, if $(A, U) \in \rho$, then by the monotone normality of (P, δ), there is a $(G, H) \in \delta^{\uparrow} \times \delta^{\downarrow}$ with $A \subseteq G, P \setminus U \subseteq H$ and $G \cap H = \emptyset$. Hence $(A, G) \in \rho, (G, P \setminus H) \in \sigma$ and $(P \setminus H, U) \in \rho$. It follows that $(A, U) \in \rho \circ \sigma \circ \rho$. Conversely, if $(A, U) \in \rho \circ \sigma \circ \rho$, then there exists a $(C, W) \in \delta^c \times \delta$ such that $(A, W) \in \rho, (W, B) \in \sigma$ and $(B, U) \in \rho$, i.e.,

$$A = \uparrow A \subseteq W = \uparrow W \subseteq B = \uparrow B \subseteq U = \uparrow U.$$

Whence $(A, U) \in \rho$. Therefore, $\rho \circ \tau \circ \rho = \rho$. Thus ρ is regular.

Clearly, the function $g : (P, \delta) \to (L, \theta(L))$, $g(x) = \rho(\mathrm{cl}^{\uparrow}(\{x\}))$, is monotone. Now we show that g is continuous, i.e.,

$$g^{-1}(L \setminus \downarrow_L \rho(\mathcal{A})) \in \delta \quad \text{and} \quad g^{-1}(L \setminus \uparrow_L \rho(\mathcal{A})) \in \delta \qquad \text{for all } \mathcal{A} \subseteq \delta^c.$$

(i) $g^{-1}(L \setminus \downarrow_L \rho(\mathcal{A})) \in \delta$.

$\forall x \in P$, if

$$x \in g^{-1}(L \setminus \downarrow_L \rho(\mathcal{A})) = \Big\{u \in P : g(u) = \rho(\mathrm{cl}^{\uparrow}(\{u\})) \not\subseteq \bigcup_{A \in \mathcal{A}} \rho(A)\Big\},$$

then

$$\rho(\mathrm{cl}^{\uparrow}(\{x\})) \not\subseteq \bigcup_{A \in \mathcal{A}} \rho(A).$$

Hence there is a $U \in \delta^{\uparrow}$ such that $U \in \rho(\mathrm{cl}^{\uparrow}(\{x\}))$ and $U \notin \bigcup_{A \in \mathcal{A}} \rho(A)$. By the monotone normality of (P, δ), there is a $(V, W) \in \delta^{\uparrow} \times \delta^{\downarrow}$ with $\mathrm{cl}^{\uparrow}(\{x\}) \subseteq V, P \setminus U \subseteq W$ and $V \cap W = \emptyset$. $\forall y \in V$, we have $y \in P \setminus W \subseteq U$. Whence $U \in \rho(\mathrm{cl}^{\uparrow}(\{y\}))$; and hence

$$x \in V \subseteq g^{-1}(L \setminus \downarrow_L \rho(\mathcal{A})).$$

Therefore, $g^{-1}(L \setminus \downarrow_L \rho(\mathcal{A})) \in \delta^{\uparrow}$.

(ii) $g^{-1}(L \setminus \uparrow_L \rho(\mathcal{A})) \in \delta$.

$\forall x \in P$, if

$$x \in g^{-1}(L \setminus \uparrow_L \rho(\mathcal{A})) = \Big\{v \in P : \bigcup_{A \in \mathcal{A}} \rho(A) \not\subseteq g(v) = \rho(\mathrm{cl}^{\uparrow}(\{v\}))\Big\},$$

then $\exists A \in \mathcal{A}$ with $\rho(A) \not\subseteq \rho(\mathrm{cl}^{\uparrow}(\{x\}))$. Whence $\exists H \in \delta^{\uparrow}$ such that $H \in \rho(A)$ and $H \notin \rho(\mathrm{cl}^{\uparrow}(\{x\}))$. By the monotone normality of (P, δ), there is a $(U, V) \in \delta^{\uparrow} \times \delta^{\downarrow}$ with $A \subseteq U, P \setminus H \subseteq V$ and $U \cap V = \emptyset$, $\forall y \in V$, since $U \in \rho(A), U \notin \rho(\mathrm{cl}^{\uparrow}(\{y\})), H \notin \rho(\mathrm{cl}^{\uparrow}(\{x\}))$ and $P \setminus V \subseteq H$, we have $\rho(A) \not\subseteq \rho(\mathrm{cl}^{\uparrow}(\{y\}))$ and $x \in V$. Whence $x \in V \subseteq g^{-1}(L \setminus \uparrow_L \rho(\mathcal{A}))$. Therefore, $g^{-1}(L \setminus \downarrow_L \rho(\mathcal{A})) \in \delta^{\downarrow}$.

(2)$\Rightarrow$(1). $\forall (A, B) \in \delta^{c\uparrow} \times \delta^{c\downarrow}$ with $A \cap B = \emptyset$, let

$$p = \rho(A) \quad \text{and} \quad q = \bigcup_{b \in B} g(b) = \bigcup_{b \in B} \rho(\mathrm{cl}^{\uparrow}(\{b\})).$$

Then

$$g(A) = \{g(a) : a \in A\} \subseteq \uparrow_L p, \quad g(B) = \{g(b) : b \in B\} \subseteq \downarrow_L q.$$

Since $P \setminus B \in \rho(A)$ and $P \setminus B \notin \bigcup_{b \in B} \rho(\mathrm{cl}^{\uparrow}(\{b\}))$, we have that $p \not\leq_L q$ (i.e., $p \not\subseteq q$). By the complete distributivity of L and Lemma 2.2, there is a $u \in L$ with $u \lhd x$ and $u \not\subseteq q$. Let $U = g^{-1}(\Uparrow_L u)$ and $V = g^{-1}(L \setminus \uparrow_L u)$. Since g is monotone and continuous, by Lemma 2.1 we have that $(U, V) \in \delta^{\uparrow} \times \delta^{\downarrow}$. Clearly, $A \subseteq U,\ B \subseteq V$ and $U \cap V = \emptyset$. Thus (P, δ) is monotone normal.

(2)$\Rightarrow$(3). Trivial.

(3)$\Rightarrow$(1). Suppose that the partial order $\leq$ on P is semiclosed and the relation ρ is regular. $\forall (A, B) \in \delta^{c\uparrow} \times \delta^{c\downarrow}$, if $A \cap B = \emptyset$, then $(A, P \setminus B) \in \rho$. By the intrinsic characterization of regular relations (i.e, the condition (2) in Theorem 2.2), there is $(C, V) \in \delta^c \times \delta$ such that

(a) $(A, V) \in \rho, (C, P \setminus B) \in \rho$, and

(b) $\forall (D, W) \in \delta^c \times \delta, (D, V) \in \rho$ and $(C, W) \in \rho \Rightarrow (D, W) \in \rho$.

We claim that $V \subseteq C$. Suppose that there is a $v \in V \setminus C$. Let $D_0 = \uparrow v$ and $W_0 = X \setminus \downarrow v$. Since the partial order $\leq$ on P is semiclosed, $(D_0, W_0) \in \delta^{c\uparrow} \times \delta^{\uparrow}$. Clearly, $(D_0, V) \in \rho$ and $(C, W_0) \in \rho$. Whence by (b), $(D_0, W_0) \in \rho$ which contradicts $v \notin W_0$. Hence $V \subseteq C$. Therefore, by $(A, V) \in \rho$ and $(B, U) \in \rho$, we have $A \subseteq V \in \delta^{\uparrow}, B \subseteq P \setminus C \in \delta^{\downarrow}$ and $V \cap (P \setminus C) = \emptyset$. Whence (P, δ) is monotone normal.

For a monotone normal ordered space (P, δ), by Theorem 3.1 we know that the relation ρ defined by $(*)$ is regular. Using the regularity of those relations we can present a proof of the Urysohn-Nachbin lemma which is quite different from the classical one (see [6] or [3]).

Corollary 3.1. (Urysohn-Nachbin Lemma) *Let (P, δ) be a monotone normal ordered space. Then $\forall (A, B) \in \delta^{c\uparrow} \times \delta^{c\downarrow}$ with $A \cap B = \emptyset$, there exists a monotone continuous function $f : (P, \delta) \to [0, 1]$ such that $f(A) = \{1\}$ and $f(B) = \{0\}$.*

Proof. By Theorem 2.2 and Theorem 3.1, the complete lattice $L = (\Phi_\rho(\delta^c), \subseteq)$ is completely distributive and the function $g : (P, \delta) \to (L, \theta(L))$, $g(x) = \rho(\mathrm{cl}^\uparrow(\{x\}))$, is monotone and continuous. Let $p = \rho(A)$ and $q = \bigcup_{b \in B} g(b)$. Then

$$g(A) = \{g(a) : a \in A\} \subseteq \uparrow_L p, g(B) = \{g(b) : b \in B\} \subseteq \downarrow_L q,$$

and $p \not\leq_L q$ (i.e., $p \not\subseteq q$). By Corollary 1.1, there is a complete lattice homomorphism $h : L \to [0, 1]$ such that $f(p) = 1$ and $f(q) = 0$. Obviously, $h : (L, \theta(L)) \to [0, 1]$ is monotone and continuous. Let $f = h \circ g$. Then $f : (P, \delta) \to [0, 1]$ is monotone continuous and $f(A) = \{1\}$ and $f(B) = \{0\}$.

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