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POLYNOMIAL RECURRENCE FOR LÉVY PROCESSES***

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Abstract

In this paper, the authors study the w-transience and w-recurrence for Lévy processes with any weight function w, give a relation between w-recurrence and the last exit times. As a special case, the polynomial recurrence and polynomial transience are also studied.

Keywords Polynomial recurrence, Polynomial transience, Lévy process
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§ 1 . Introduction

Let $X = \{\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P\}$ be a Lévy process on $\mathbb{R}^n$ starting at 0 with convolution semigroup $\{\pi_t\}$ and Lévy exponent ϕ, that is, $\hat{\pi}_t = e^{-t\phi}$. Without loss of generality, we assume that X is genuinely n-dimensional, that is, the linear space generated by $\operatorname{supp}\pi_1$ is $\mathbb{R}^n$ itself. For any Borel set A, let L_A be the last exit time from A, that is,

$$L_A = \sup\{t > 0 : X_t \in A\},$$

where $\sup\emptyset = 0$. Then L_A is $\mathcal{F}$-measurable. Let $\aleph$ be the set of all bounded open sets N with $0 \in N$. Lévy processes are divided into two classes: recurrence and transience. The Lévy process X is transient if and only if $P(L_N < \infty) = 1$ for all $N \in \aleph$, and recurrent if and only if $P(L_N = \infty) = 1$ for all $N \in \aleph$ (see e.g., [4]). If $n \geq 3$, then X is transient. The Lévy process X is recurrent if and only if for some $N \in \aleph$ (and also for all $N \in \aleph$),

$$\int_N \mathrm{Re}\Big(\frac{1}{\phi(x)}\Big)dx = \infty.$$

Port and Stone [3] contributed a lot to the study of the recurrence and transience for Lévy processes. They also discuss the concepts of weak transience and strong transience. A transient Lévy process is called weakly transient if $E(L_N) = \infty$ for all $N \in \aleph$; strongly transient if $E(L_N) < \infty$ for all $N \in \aleph$. A transient Lévy process is either weakly transient or strongly transient. If $n \geq 5$, then X is strongly transient.

We have discussed α-transience and α-recurrence for Lévy processes with $\alpha \leq 0$ in [6]. The 0-transient and 0-recurrent are the usual transient and recurrent respectively. When $\alpha < 0$, X is α-recurrent if and only if $E(e^{-\alpha L_N}) = \infty$ for all $N \in \aleph$; while X is α-transient

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if and only if $E(e^{-\alpha L_N}) < \infty$ for all $N \in \aleph$. Thus the classification of α-transient and α-recurrent is determined by the exponential moments of the last exit times.

We call X quasi-symmetric if it satisfies

Condition 1. *There exists a compact subset K such that* $\limsup_{t\to\infty} \pi_t(K)^{\frac{1}{t}} = 1$.

Condition 1 was introduced by S. C. Port and C. J. Stone [3] to give a ratio limit theorem. In [6], we prove that X is quasi-symmetric if and only if X is α-recurrent for all $\alpha < 0$, that is to say, the speed at which X escapes from any bounded open set N can not be any exponential. This gives a probabilistic explanation of quasi-symmetry. It actually says that X is quasi-symmetric if and only if it is (0-)-recurrent. Particularly, any recurrent Lévy process is quasi-symmetric. Since any symmetric Lévy process is quasi-symmetric while X is transient when $n \geq 3$, the class of quasi-symmetric Lévy processes is much bigger than that of recurrent Lévy processes.

To classify the quasi-symmetric Lévy processes more finely, we introduce the concept of polynomial recurrence and polynomial transience. They are studied in §2. In general, given a weight function w, we may define w-recurrence and w-transience. In §3, we shall discuss w-transience and w-recurrence.

§ 2 . Polynomial Recurrence

Given any $\alpha \in \mathbb{R}$, for any Borel subset A on $\mathbb{R}^n$, define

$$V^\alpha(A) = \int_0^\infty e^{-\alpha t}\pi_t(A)\, dt.$$

The support of V^α is independent of α and we denote it by Σ. It is a closed semigroup containing 0. The closed group generated by Σ is denoted by G which is $\overline{\Sigma - \Sigma}$. If $\alpha > 0$, then $V^\alpha(\mathbb{R}^n) = \frac{1}{\alpha} < \infty$. Considering $\alpha \leq 0$, if V^α is a Radon measure, then X is said to be α-transient; if $V^\alpha(x+N) = \infty$ for all $x \in \Sigma$ and all $N \in \aleph$, then X is said to be α-recurrent.

We say X is degenerate if there is $u \neq 0$ and $a \neq 0$, such that $\operatorname{supp}\pi_1 \subseteq \{x : (u, x) = a\}$. We say X is not one-sided if for all $u \neq 0$, $\pi_1\{x : (u, x) > 0\} > 0$. The semigroup Σ is a group if and only if X is not one-sided. If X is quasi-symmetric, then X is not one-sided. If X is not one-sided, then X is non-degenerate.

Given any $x, y \in \mathbb{R}^n$, we say that y can be reached from x, and write $x \curvearrowright y$, if for any $N \in \aleph$, $\pi_t(y + N - x) > 0$ for some $t > 0$. We say that x and y communicate, and write $x \leftrightarrow y$, if x and y can be reached from each other. If $x \curvearrowright y$ and $y \curvearrowright z$, then $x \curvearrowright z$. The relation $\leftrightarrow$ is an equivalence relation on $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, the set that can be reached by x is $x + \Sigma$ while the set that can be communicated by x is $x + \Sigma \cap (-\Sigma)$.

Given $\beta > -1$, for any Borel set A on $\mathbb{R}^n$, we define

$$H^\beta(A) := \int_0^\infty t^\beta \pi_t(A)\, dt.$$

Then H^β is a measure with support Σ. Clearly, $H^\beta(A) = \infty$ if and only if

$$\int_s^\infty t^\beta \pi_t(A)\, dt = \infty$$

for some finite $s > 0$ (and hence for all finite $s > 0$).

Definition 2.1. *A state $x \in G$ is β-polynomially recurrent if $H^\beta(x+N) = \infty$ for all $N \in \aleph$, and β-polynomially transient if $H^\beta(x+N) < \infty$ for some $N \in \aleph$.*

For any $x \in G$, x is either β-polynomially recurrent or β-polynomially transient. If $x \notin \Sigma$, then x is β-polynomially transient.

Proposition 2.1. *Suppose that $x \curvearrowright y$. If x is β-polynomially recurrent, then y is β-polynomially recurrent. If y is β-polynomially transient, then so is x.*

Proof. For any $N \in \aleph$, there is $N_1 \in \aleph$ such that $N_1 + N_1 \subset N$. Since $x \curvearrowright y$, there is $s > 0$ such that $\pi_s(y + N_1 - x) > 0$. Then

$$\pi_{t+s}(y+N) \geq \int_{z\in N_1} \pi_s(y+N-x-dz)\pi_t(x+dz) \geq \pi_s(y-x+N_1)\pi_t(x+N_1)$$

for all $t > 0$. If $\beta \geq 0$, then $(t+s)^\beta \geq t^\beta$ for all $t > 0$. If $\beta < 0$, then $(t+s)^\beta \geq (2t)^\beta = 2^\beta t^\beta$ for all $t > s$. Thus the result holds.

Definition 2.2. *The Lévy process $\{X_t\}$ is said to be β-polynomially transient if H^β is a Radon measure. It is said to be β-polynomially recurrent if $H^\beta(N+x) = \infty$ for all $x \in \Sigma$ and $N \in \aleph$.*

For simplicity, we write β-p-transient for β-polynomially transient, and β-p-recurrent for β-polynomially recurrent. Then 0-p-recurrent and 0-p-transient are the usual recurrent and transient respectively. As β increases, the set of β-p-transient Lévy processes decreases, while the set of β-p-recurrent ones increases.

Proposition 2.2. (1) *The Lévy process X is β-p-recurrent* (*resp. β-p-transient*) *if and only if all states $x \in G$ are β-p-recurrent* (*resp. β-p-transient*).

(2) *The Lévy process X is either β-p-recurrent or β-p-transient.*

(3) *The Lévy process X is β-p-transient if and only if $H^\beta(N) < \infty$ for some $N \in \aleph$.*

(4) *The Lévy process X is β-p-recurrent if and only if $H^\beta(N) = \infty$ for some $N \in \aleph$.*

Proof. If X is α-transient for some $\alpha < 0$, then for any $\beta > -1$, X is β-p-transient and all states $x \in G$ are β-p-transient. Therefore we need only consider the case that X is quasi-symmetric. Then $G = \Sigma$ that is a communicating class. By Proposition 2.1, either all states $x \in G$ are β-p-transient or all states are β-p-recurrent. If all $x \in G$ are β-p-transient, then for any compact set K, by the finite covering theorem, there exist finite points $x_1, \cdots, x_m$ and open sets $N_1, \cdots, N_m \in \aleph$ such that $K \subset \bigcup_1^m (x_i + N_i)$ and $H^\beta(x_i + N_i) < \infty$ for $1 \leq i \leq m$. So

$$H^\beta(K) \leq \sum_1^m H^\beta(x_i + N_i) < \infty.$$

Therefore H^β is a Radon measure or X is β-p-transient. Thus X is β-p-recurrent (resp. β-p-transient) if and only if x is β-p-recurrent (resp. β-p-transient) for some $x \in G$. Particularly, it is equivalent to that 0 is β-p-recurrent (resp. β-p-transient). Therefore our result holds.

For any $a > 0$ and $x \in \mathbb{R}$, let

$$f_a(x) = \frac{\sin^2 ax}{(ax)^2},$$
$$g_a(x) = \frac{1}{2a}\Big(1 - \frac{|x|}{2a}\Big)1_{[-2a,2a]}(x).$$

Then

$$\hat{f}_a(z) = 2\pi g_a(z), \quad \hat{g}_a(z) = f_a(z) \qquad \text{for any} \quad z \in \mathbb{R}.$$

Theorem 2.1. *Fix a set $N \in \aleph$. Then (1) implies (2). If X is symmetric, then they are equivalent to each other. Here we let $\frac{1}{0} = \infty$.*

(1) *The Lévy process X is β-p-recurrent.*

(2) *That*

$$\int_N \frac{1}{[\operatorname{Re}\phi(x)]^{\beta+1}}\, dx = \infty.$$

Proof. At first, we shall prove that (1) implies (2). Suppose that X is β-p-recurrent. There is $b > 0$ such that

$$N \supseteq \{(x_1, x_2, \cdots, x_n) : |x_i| \le 2b, \; i = 1, \cdots, n\}.$$

Let

$$f((x_1, \cdots, x_n)) = f_b(x_1) \cdots f_b(x_n).$$

Then f is bounded, continuous, nonnegative, integrable and

$$\hat{f}((y_1, \cdots, y_n)) = (2\pi)^n g_b(y_1) \cdots g_b(y_n).$$

Thus $\hat{f}$ is nonnegative and vanishes outside N. When $x \in N$,

$$\hat{f}(x) \le \frac{\pi^n}{b^n}.$$

We have

$$\begin{aligned}(H^\beta, f) = \int_0^\infty t^\beta (\pi_t, f)\, dt &= (2\pi)^{-n} \int_0^\infty t^\beta (\hat{\pi}_t, \hat{f})\, dt \\ &= (2\pi)^{-n} \int_0^\infty t^\beta\, dt \int_N \hat{f}(x) e^{-t\phi(x)}\, dx \\ &= (2\pi)^{-n} \int_N \hat{f}(x)\, dx \int_0^\infty e^{-t\phi(x)} t^\beta\, dt.\end{aligned}$$

Since

$$|e^{-t\phi(x)}| = e^{-t\operatorname{Re}\phi(x)} \quad \text{and} \quad \operatorname{Re}\phi(x) \ge 0,$$

we have

$$(H^\beta, f) \le (2\pi)^{-n} \int_N \frac{\pi^n}{b^n}\, dx \int_0^\infty e^{-t\operatorname{Re}\phi(x)} t^\beta\, dt = (2b)^{-n} \int_N \frac{\Gamma(\beta+1)}{[\operatorname{Re}\phi(x)]^{\beta+1}}\, dx.$$

Since $f \ge 0$ and $f(0) > 0$,

$$(H^\beta, f) = \infty.$$

Hence (2) holds.

Next, suppose that X is symmetric. Then $\phi = \operatorname{Re}\phi$ that is nonnegative. We shall show that (2) implies (1). Let

$$g((x_1, \cdots, x_n)) = g_a(x_1) \cdots g_a(x_n).$$

Then

$$\hat{g}((y_1, \cdots, y_n)) = f_a(y_1) \cdots f_a(y_n).$$

Choose $a > 0$ such that

$$c := \inf_{x \in N} \hat{g}(x) > 0.$$

Since g is nonnegative, bounded and vanishes outside a compact set and $\hat{g} \geq 0$,

$$(H^\beta, g) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{g}(x)\,dx \int_0^\infty e^{-t\phi(x)} t^\beta\,dt \geq (2\pi)^{-n} \int_N \frac{c\Gamma(\beta+1)}{[\operatorname{Re}\phi(x)]^{\beta+1}}\,dx.$$

By (2), $(H^\beta, g) = \infty$ which implies that X is β-p-recurrent.

Use $X^\sharp$ to denote the Lévy process with Lévy exponent $\operatorname{Re}\phi$, which is the symmetric part of X. Suppose its convolution semigroup is $\{b_t\}$. Then

$$b_t = \pi_{t/2} * \tilde{\pi}_{t/2},$$

where $\tilde{\pi}_{t/2}$ is the dual of $\pi_{t/2}$ defined by

$$\tilde{\pi}_{t/2}(A) = \pi_{t/2}(-A) \qquad \text{for any Borel set } A.$$

Corollary 2.1. *If X is β-p-recurrent, then $X^\sharp$ is β-p-recurrent.*

Proposition 2.3. *If $n > 2(1+\beta)$, then X is β-p-transient.*

Proof. We need only consider that X is quasi-symmetric. Then X is non-degenerate. By the proof of Theorem 37.8 of [4], there are constants $c > 0$ and $a > 0$ such that $\operatorname{Re}\phi(x) \geq c\|x\|^2$ on B_a, where $B_a = \{x : \|x\| \leq a\}$. Let c_n be the surface measure of the unit sphere. We get

$$\int_{B_a} [\operatorname{Re}\phi(x)]^{-(\beta+1)}\,dx \leq \frac{1}{c^{\beta+1}} \int_{B_a} \|x\|^{-2(\beta+1)}\,dx = \frac{c_n}{c^{\beta+1}} \int_0^a r^{n-1-2(\beta+1)}\,dr.$$

Hence if $n - 2\beta - 3 > -1$, that is if $n > 2(1+\beta)$, then

$$\int_{B_a} [\operatorname{Re}\phi(x)]^{-(\beta+1)} < \infty$$

and hence X is β-p-transient.

Corollary 2.2. *If $n \geq 1$, then X is β-p-transient for any $\beta < -\frac{1}{2}$. If $n \geq 2$, then X is β-p-transient for any $\beta < 0$.*

When $n = 0$, $\pi_t = \delta_0$, $X_t = 0$ and the process X is trivial. The corollary above tell us that we need only to consider $\beta \geq -\frac{1}{2}$ when $\pi_1 \neq \delta_0$. If X is β-p-recurrent for some $\beta < 0$, then the genuinely dimension $n = 1$ or $\pi_1 = \delta_0$.

Theorem 2.2. *For any bounded open set A with $A \cap G \neq \emptyset$, let*

$$\beta_0 := \limsup_{t\to\infty} \frac{\ln \pi_t(A)}{\ln t} \quad \text{and} \quad \beta_1 := \liminf_{t\to\infty} \frac{\ln \pi_t(A)}{\ln t}.$$

Then X is β-p-transient provided $\beta < -(\beta_0 + 1)$, and is β-p-recurrent provided that $\beta > -(\beta_1 + 1)$.

Proof. If $\beta < -(\beta_0+1)$, then there is $\gamma > \beta_0$ such that $\beta+\gamma < -1$. Thus there exists $t_0 > 0$ such that whenever $t > t_0$, $\frac{\ln \pi_t(A)}{\ln t} < \gamma$, that is, $\pi_t(A) < t^\gamma$. Then

$$\int_{t_0}^{\infty} t^\beta \pi_t(A)\, dt < \int_{t_0}^{\infty} t^{\beta+\gamma} dt < \infty.$$

Hence X is β-p-transient.

If $\beta > -(\beta_1+1)$, then there is $\gamma < \beta_1$ such that $\beta+\gamma > -1$. Thus there exists $t_0 > 0$ such that whenever $t > t_0$, $\frac{\ln \pi_t(A)}{\ln t} > \gamma$, that is, $\pi_t(A) > t^\gamma$. Then

$$\int_{t_0}^{\infty} t^\beta \pi_t(A)\, dt > \int_{t_0}^{\infty} t^{\beta+\gamma} dt = \infty.$$

It follows that X is β-p-recurrent.

By this theorem, if there is a $N \in \aleph$ such that

$$\beta_0 := \lim_{t\to\infty} \frac{\ln \pi_t(N)}{\ln t}$$

exists, then X is β-p-transient provided $\beta < -(\beta_0+1)$, and is β-p-recurrent provided that $\beta > -(\beta_0+1)$. In other words, $-(\beta_0+1)$ is the critical point.

Example 2.1. Let X be an α-semi-stable Lévy process with $0 < \alpha \le 2$. Then there is a constant $c > 0$ such that $\operatorname{Re}\phi(x) \ge c\|x\|^\alpha$ for all $x \in \mathbb{R}^n$ (see [4, Proposition 24.20]). For any $\beta > -1$,

$$\int_{B_1} \frac{dx}{\|x\|^{\alpha(\beta+1)}} = c_n \int_0^1 r^{n-1-\alpha(\beta+1)}\, dr,$$

where B_1 is the open unit ball and c_n is the surface measure of the unit sphere. Thus X is β-p-transient if $n-1-\alpha(\beta+1) > -1$, that is, if $\beta < \frac{n}{\alpha} - 1$.

Example 2.2. Let $0 < \alpha \le 2$. Let X be a symmetric α-stable Lévy process. If $\alpha = 2$, then X is a Gaussian process with mean 0 and hence $\phi(x) = (x, Ax)$ for some symmetric positive-definite $n \times n$ matrix. Thus there are $k_1, k_2 > 0$ such that

$$k_1\|x\|^2 \le \phi(x) \le k_2\|x\|^2 \qquad \text{for all } \ x \in \mathbb{R}^n.$$

If $\alpha < 2$, then

$$\phi(x) = \int_S |(x,y)|^\alpha \lambda(\,dy)$$

with a symmetric finite measure λ on S, where S is the unit sphere (see [4, Theorem 14.13]). Thus $\lambda(S) < \infty$ and

$$\phi(x) = \int_S |(x,y)|^\alpha \lambda(\,dy) \le \int_S \|x\|^\alpha \|y\|^\alpha \lambda(\,dy) = \lambda(S)\|x\|^\alpha.$$

Consequently, X is β-p-recurrent if and only if $\beta \ge \frac{n}{\alpha} - 1$.

Example 2.3. Let X be the cauchy process on $\mathbb{R}$ with a drift $x_0 \in \mathbb{R}$. Let

$$f_t(x) = \frac{t}{\pi(t^2 + (x - tx_0)^2)}.$$

Then

$$\pi_t(dx) = f_t(x)dx,$$

where dx is the Lebesgue measure on $\mathbb{R}$. Clearly, for all $x \in \mathbb{R}$,

$$f_t(x) \le (\pi t)^{-1}.$$

For any compact set K, there is t_0 and $c > 0$ such that for any $x \in K$ and any $t > t_0$,

$$f_t(x) \ge ct^{-1}.$$

Thus X is η-recurrent if $\eta \ge 0$ and is η-transient if $\eta < 0$. When $x_0 = 0$, X is a symmetric 1-stable Lévy process and our result is consistent with that in Example 2.2.

§ 3. Recurrent and Transient with a Weight Function

In the last section, we have discussed polynomial recurrence and polynomial transience. In this section, we shall discuss w-recurrent and w-transient with any function w, which serves as a weight for recurrence and transience to describe roughly the asymptotic behavior of π_t as $t \longrightarrow \infty$. We have used exponential function and polynomial function as weights to discuss recurrence and transience. If necessary, we may also use logarithmic weight $\ln t$ to classify the 1-p-recurrent Lévy processes more finely.

Suppose that w is a real-valued monotone function on $[0, \infty)$, absolutely continuous on every closed interval $[0, a]$ and $w(t) > 0$ for all $t > 0$. When w is increasing, we suppose that $w(0) = 0$. When w is decreasing, we suppose that for any $s > 0$, there is $t_0 \ge 0$ and $c > 0$ such that when $t > t_0$, $w(t + s) \ge cw(t)$. Clearly

$$\int_0^s w(t)\, dt \le s[w(s) \vee w(0)] < \infty$$

for all finite $s > 0$.

For any Borel set A on $\mathbb{R}^n$, define

$$G^w(A) = \int_0^\infty w(t)\pi_t(A)\, dt.$$

Then G^w is a measure with support Σ. Clearly, $G^w(A) = \infty$ if and only if

$$\int_s^\infty w(t)\pi_t(A)\, dt = \infty$$

for some finite $s > 0$ (and hence for all finite $s > 0$).

Definition 3.1. *A state $x \in G$ is w-recurrent if $G^w(x + N) = \infty$ for all $N \in \aleph$, and w-transient if $G^w(x + N) < \infty$ for some $N \in \aleph$.*

For any $x \in G$, x is either w-recurrent or w-transient. If $x \notin \Sigma$, then x is w-transient.

Proposition 3.1. *Suppose that $x \curvearrowright y$. If x is w-recurrent, then y is w-recurrent. If y is w-transient, then so is x.*

Proof. For any $N \in \aleph$, there is $N_1 \in \aleph$ such that $N_1 + N_1 \subset N$. Since $x \curvearrowright y$, there is $s > 0$ such that $\pi_s(y + N_1 - x) > 0$. Then

$$\pi_{t+s}(y + N) \ge \pi_s(y - x + N_1)\pi_t(x + N_1) \qquad \text{for all} \quad t > 0.$$

If w is increasing, then

$$w(t + s) \ge w(t) \qquad \text{for all} \quad t > 0.$$

If w is decreasing, there is $c > 0$ and $t_0 \ge 0$ such that $w(t + s) \ge cw(t)$ provided $t > t_0$. Thus our result holds.

Definition 3.2. *The Lévy process $\{X_t\}$ is said to be w-transient if G^w is a Radon measure. It is said to be w-recurrent if $G^w(N+x)=\infty$ for all $x \in \Sigma$ and $N \in \aleph$.*

If $w_1 \leq w_2$, then any w_2-transient Lévy process is w_1-transient while any w_1-recurrent Lévy process is w_2-recurrent. In fact it is also true when there exist $t_0 > 0$ and $k > 0$ such that $w_2(t) \geq k w_1(t)$ for all $t > t_0$. We shall say $w_2 \succcurlyeq w_1$ if such condition holds. If $w_1 \succcurlyeq w_2$ and $w_2 \succcurlyeq w_1$, then we say $w_1 \sim w_2$. That is to say, $w_1 \sim w_2$ if and only if there exist t_0 and $k_1, k_2 > 0$ such that $k_1 w_1(t) \leq w_2(t) \leq k_2 w_1(t)$ for all $t > t_0$. When $w_1 \sim w_2$, the class of w_1-recurrent (resp. w_1-transient) Lévy processes coincides to that of w_2-recurrent (resp. w_2-transient) Lévy processes. If w is increasing, then $w \succcurlyeq 1$ and any w-transient Lévy process is transient. If w is decreasing, then $1 \succcurlyeq w$ and any w-recurrent Lévy process is recurrent. Thus the w-recurrence and w-transience is a finer classification for transient Lévy processes when w is increasing, and is for recurrent Lévy processes when w is decreasing.

Clearly, X is w-recurrent (resp. w-transient) if and only if all states $x \in \Sigma$ are w-recurrent (resp. w-transient). By Proposition 3.1, X is w-recurrent if and only if the state 0 is w-recurrent. If X is not one-sided, then X is w-transient if and only if 0 is w-transient. Thus we get the following properties.

Proposition 3.2. *Suppose that X is not one-sided.*
(1) *The Lévy process X is either w-recurrent or w-transient.*
(2) *The Lévy process X is w-transient if and only if $G^w(N) < \infty$ for some $N \in \aleph$.*
(3) *The Lévy process X is w-recurrent if and only if $G^w(N) = \infty$ for some $N \in \aleph$.*

Theorem 3.1. *Fix $N \in \aleph$. Then (1) implies (2). If X is symmetric, then (1) is equivalent to (2).*

(1) *That X is w-recurrent.*

(2) *That*

$$\int_N dx \int_0^\infty e^{-t \operatorname{Re} \phi(x)} w(t)\, dt = \infty.$$

Proof. The proof is similar as that of Theorem 2.1. Choose f and g as in the proof of Theorem 2.1. Suppose that X is w-recurrent. We have

$$\begin{aligned}
(G^w, f) &= \int_0^\infty w(t)(\pi_t, f)\, dt = (2\pi)^{-n} \int_0^\infty w(t)(\hat{\pi}_t, \hat{f})\, dt \\
&= (2\pi)^{-n} \int_0^\infty w(t)\, dt \int_N \hat{f}(x) e^{-t\phi(x)}\, dx \\
&= (2\pi)^{-n} \int_N \hat{f}(x)\, dx \int_0^\infty e^{-t\phi(x)} w(t)\, dt \\
&\leq (2b)^{-n} \int_N dx \int_0^\infty e^{-t \operatorname{Re} \phi(x)} w(t)\, dt.
\end{aligned}$$

Since f is nonnegative continuous and $f(0) > 0$,

$$(G^w, f) = \infty.$$

It follows (2).

Next, suppose that X is symmetric. Then $\phi = \operatorname{Re} \phi$ that is nonnegative. Suppose that

(2) holds. Since $\hat{g} \geq c1_N$,

$$(G^w, g) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{g}(x)\, dx \int_0^\infty e^{-t\phi(x)} w(t)\, dt$$
$$\geq c(2\pi)^{-n} \int_N dx \int_0^\infty e^{-t \operatorname{Re}\phi(x)} w(t)\, dt = \infty.$$

Consequently, $(G^w, g) = \infty$. Since X is symmetric, X is not one-sided. That g is continuous, $g(0) > 0$ and g vanishes outside a bounded open set M implies that $G^w(M) = \infty$. By Proposition 3.2, X is w-recurrent.

Corollary 3.1. (1) *If X is w-recurrent, then $X^\sharp$ is w-recurrent.*

(2) *If X is degenerate, then X is w-transient.*

(3) *If $\int_N dx \int_0^\infty e^{-t\|x\|^2} w(t)\, dt < \infty$ for some $N \in \aleph$, then $\{X_{bt}\}$ is w-transient for some $b > 0$.*

(4) *If all Gaussian processes with mean* 0 *are w-transient, then X is w-transient.*

Proof. By Theorem 3.1, (1) holds. If X is degenerate, since X is genuinely n-dimensional, there is $u \neq 0$ such that

$$\operatorname{supp} \pi_t \subseteq \{x : (u, x) = t\}, \qquad t > 0.$$

For any compact set K, let

$$\tau = \sup_{x \in K} |(u, x)|.$$

Then $\tau < \infty$. Thus $\pi_t(K) = 0$ whenever $t > \tau$. It follows that $G^w(K) < \infty$. Therefore X is w-transient.

To prove (3) and (4), by (2), we need only consider the case that X is non-degenerate. Then there are constants $c > 0$ and $a > 0$ such that

$$\operatorname{Re}\phi(x) \geq c\|x\|^2 \qquad \text{on} \quad B_a,$$

where $B_a = \{x : \|x\| < a\}$. If

$$\int_N dx \int_0^\infty e^{-t\|x\|^2} w(t)\, dt < \infty \qquad \text{for some} \quad N \in \aleph,$$

then

$$\int_{N \cap B_a} dx \int_0^\infty e^{-t \frac{\operatorname{Re}\phi(x)}{c}} w(t)\, dt < \infty.$$

Thus $\{X_{t/c}\}$ is w-transient. Hence (3) holds. If all Gaussian processes with mean 0 are w-transient, then the Gaussian process with Lévy exponent $c\|x\|^2$ is w-transient and hence X is w-transient. Thus (4) holds.

Now we consider the case that w is increasing. Let

$$w(\infty) := \lim_{t \to \infty} w(t).$$

If $w(\infty) < \infty$, then $w \sim 1$, w-recurrent is equivalent to recurrent and w-transient is equivalent to transient.

Theorem 3.2. *Suppose $w(\infty) = \infty$. Then*

(1) X is w-transient if and only if $E(w(L_N)) < \infty$ for all $N \in \aleph$.

(2) X is w-recurrent if and only if $E(w(L_{x+N})) = \infty$ for all $x \in \Sigma$ and $N \in \aleph$.

Proof. If X is recurrent, then $P(L_{x+N} = \infty) = 1$ and hence $E(w(L_{x+N})) = \infty$ for all $x \in \Sigma$ and $N \in \aleph$. Thus we need only to prove the case that X is transient. Since w is absolutely continuous on every interval $[0, a]$ and w is increasing, w' exists a.e. and $w' \geq 0$ a.e. with respect to the Lebesgue measure on $[0, \infty)$. By Fubini Theorem, for any Borel set A,

$$G^w(A) = \int_0^\infty w(t)\pi_t(A)\,dt = \int_0^\infty \pi_t(A)\,dt \int_0^t w'(s)\,ds = \int_0^\infty w'(s)\,ds \int_s^\infty \pi_t(A)\,dt,$$

$$E(w(L_A)) = E\int_0^\infty w'(s)1_{\{L_A>s\}}\,ds = \int_0^\infty w'(s)P(L_A > s)\,ds.$$

For any $N \in \aleph$, there is $N_1 \in \aleph$ such that $\overline{N}_1 + N_1 \subseteq N$. By Corollary 3.3 of [6], for any $x \in \mathbb{R}^n$ and any $s > 0$, we have

$$V^0(N_1)P(L_{x+N_1} > s) \leq \int_s^\infty \pi_t(x+N)\,dt \leq V^0(N - \overline{N})P(L_{x+N} > s).$$

Thus

$$V^0(N_1)E(w(L_{x+N_1})) \leq G^w(x+N) \leq V^0(N - \overline{N})E(w(L_{x+N})).$$

Since X is transient, $0 < V^0(N_1) < \infty$ and $0 < V^0(N - \overline{N}) < \infty$. It implies our result.

Applying this theorem to $w(t) = t^\eta$ with $\eta > 0$, we get

Corollary 3.2. *Suppose $\eta > 0$. Then X is η-p-transient if and only if $E(L_N^\eta) < \infty$ for all $N \in \aleph$; is η-p-recurrent if and only if $E(L_{x+N}^\eta) = \infty$ for all $x \in \Sigma$ and $N \in \aleph$.*

Thus X is strongly transient if and only if X is 1-p-transient.

Remark 3.1. This paper is based on a part of the first author's Ph.D. dissertation. Around the period of her final defense (December 2002), she received a manuscript from Professor K. Sato, which shows that he (jointly with T. Watanabe) did a very similar work as in §2.

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