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PROPERTIES OF THE BOUNDARY FLUX OF A SINGULAR DIFFUSION PROCESS***

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Abstract

The authors study the singular diffusion equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(\rho^\alpha |\nabla u|^{p-2} \nabla u), \qquad (x,t) \in Q_T = \Omega \times (0,T),$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with appropriately smooth boundary $\partial\Omega$, $\rho(x) = \operatorname{dist}(x, \partial\Omega)$, and prove that if $\alpha \geq p-1$, the equation admits a unique solution subject only to a given initial datum without any boundary value condition, while if $0 < \alpha < p-1$, for a given initial datum, the equation admits different solutions for different boundary value conditions.

Keywords Boundary flux, Singular diffusion, Boundary degeneracy
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§ 1 . Introduction

In this paper, we are concerned with a special model of heat transfer process governed by the following singular diffusion equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(\rho^\alpha |\nabla u|^{p-2} \nabla u), \qquad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with appropriately smooth boundary $\partial\Omega$, occupied by some object, $p > 1$, $\alpha > 0$, and $\rho(x) = \operatorname{dist}(x, \partial\Omega)$. If $\alpha = 0$, then the equation (1.1) becomes the evolutionary p-Laplacian equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2} \nabla u).$$

In particular, if $p = 2$, then the equation is just the classical heat conduction equation. For $p \neq 2$, it is more natural to use the equation to describe the heat conduction, since it reflects even more exactly the physical reality, for example, if $p > 2$, the solutions of such equation may possess the properties of finite speed of propagation of perturbations (see for example [1, 2]). There are a tremendous amount of related works for such equations, see for example [1–8].

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For the equation (1.1), the diffusion coefficient depends on the distance to the boundary and vanishes on the boundary. Thus the equation degenerates on the boundary. The memoir by F. Tricomi [9], as well as subsequent investigations of equations of mixed type, elicited interest in the general study of elliptic equations degenerating on the boundary of the domain. The 1951 paper of M. V. Keldyš [10] played a significant role in the development of the theory. It was this paper that first brought to light the fact that in the case of elliptic equations degenerating on the boundary, under definite assumptions a portion of the boundary may be free from the prescription of boundary conditions. Later, G. Fichera [11, 12] and O. A. Oleǐnik [13, 14] developed and perfected the general theory of second order equations with nonnegative characteristic form, which, in particular, contains those degenerating on the boundary.

The equation considered by Fichera and Oleǐnik is linear, and the second order derivatives of coefficients of principal part are bounded. They obtained the existence and uniqueness of solution for the Dirichlet problem, and investigated the properties of solutions too. Their results can be applied to the equation (1.1) with $p = 2$ and $\alpha \geq 2$, revealing that there is no flux on the boundary no matter how the outer temperature varies.

In this paper, we study the singular diffusion equation (1.1) with $p > 1$ and $\alpha > 0$. We are more interested in the behavior of the heat transfer process governed by (1.1) near the boundary. Since the diffusion coefficient vanishes on the boundary, it seems that there is no heat flux across the boundary. However, the fact might not coincide with what we image. The purpose of this paper will exhibit the fact, which is different from the usual imagination, that α, the exponent characterizing the vanishing ratio of the diffusion coefficient near the boundary, does determine the behavior of the heat transfer near the boundary. We will show that if the ratio is relatively small, the outer temperature may affect the diffusion process of the inner temperature of the object, while if the ratio is relatively large, there is no flux on the boundary no matter how the outer temperature varies. Exactly, we will prove that $p-1$ is the critical value for the exponent α. If $0 < \alpha < p-1$, then for a given initial datum, the equation (1.1) admits different solutions for different boundary values. While if $\alpha \geq p-1$, then only the initial value will completely determine the unique solution.

Since the equation we consider is degenerate on the boundary and may be degenerate or singular at points where $|\nabla u| = 0$, we should consider weak solutions instead of classical solutions. Now we present the following

Definition 1.1. *A function $u(x,t)$ is said to be a weak solution of the equation* (1.1)*, if $u \in C(0,T; L^2(\Omega)) \cap L^\infty(Q_T)$, $\dfrac{\partial u}{\partial t} \in L^2(Q_T)$, $\rho^\alpha |\nabla u|^p \in L^1(Q_T)$, and for any test function $\varphi \in C_0^\infty(Q_T)$, the following integral equality holds*

$$\iint_{Q_T} \Big(\frac{\partial u}{\partial t}\varphi + \rho^\alpha |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi\Big) dxdt = 0.$$

Although we consider the equation in a bounded domain, as mentioned above, we could not supplement the initial and boundary value conditions as usual. In fact, only in the case that $0 < \alpha < p-1$, can we impose the Dirichlet boundary value condition

$$u(x,t) = g(x,t), \qquad (x,t) \in \partial\Omega \times (0,T). \tag{1.2}$$

We always assume that $g(x,t)$ is a function which can be extended to be defined on $\overline{Q}_T$ and is appropriately smooth. However, the initial value condition is always required

$$u(x,0) = u_0(x), \qquad x \in \Omega. \tag{1.3}$$

In any cases, the solution u is said to satisfy the initial boundary value conditions (1.2) and (1.3), if (1.2) and (1.3) hold in the trace sense. The main results of this paper are as follows.

Theorem 1.1. *Let $0 < \alpha < p-1$. Then for any u_0 satisfying $u_0 \in L^\infty(\Omega)$ and $\rho^\alpha|\nabla u_0|^p \in L^1(\Omega)$, there exists at least one weak solution of the first initial-boundary problem* (1.1)–(1.3)*. Moreover, the solution of the problem is unique.*

Theorem 1.2. *Let $\alpha \geq p-1$. Then the equation* (1.1) *admits at most one weak solution with initial value u_0, no matter what the boundary value is. Moreover, for any u_0 as in Theorem* 1.1*, there exists at least one weak solution of the equation* (1.1) *with initial value* (1.3)*.*

§ 2 . Proof of the Main Results

To study the equation (1.1), we first consider the regularizing problem

$$\frac{\partial u_\varepsilon}{\partial t} - \operatorname{div}(\rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon) = 0, \qquad (x,t) \in Q_T, \tag{2.1}$$

$$u_\varepsilon(x,t) = g(x,t), \qquad (x,t) \in \partial\Omega \times (0,T), \tag{2.2}$$

$$u_\varepsilon(x,0) = u_{\varepsilon,0}(x), \qquad x \in \Omega, \tag{2.3}$$

where $\rho_\varepsilon = \rho+\varepsilon$, $\varepsilon > 0$. Similar to the theory for evolutionary p-Laplacian equation, for any $u_{\varepsilon,0}$ satisfying $u_{\varepsilon,0} \in L^\infty(\Omega)$ and $\rho_\varepsilon^\alpha|\nabla u_{\varepsilon,0}|^p \in L^1(\Omega)$, the above problem admits a unique weak solution $u_\varepsilon \in C(0,T;L^2(\Omega)) \cap L^p(0,T;W^{1,p}(\Omega))$ with $\frac{\partial u_\varepsilon}{\partial t} \in L^2(Q_T)$, in the sense that, for any test function $\varphi \in C_0^\infty(Q_T)$, u_ε satisfies the following integral equality

$$\iint_{Q_T} \Big(\frac{\partial u_\varepsilon}{\partial t}\varphi + \rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla\varphi\Big) dxdt = 0,$$

and (2.2), (2.3) hold in the trace sense.

Lemma 2.1. *Assume*

$$u_0 \in L^\infty(\Omega), \qquad \rho^\alpha|\nabla u_0|^p \in L^1(\Omega),$$

$\|u_{\varepsilon,0}\|_{L^\infty(\Omega)}$ and $\|\rho_\varepsilon^\alpha|\nabla u_{\varepsilon,0}|^p\|_{L^1(\Omega)}$ are uniformly bounded, and $u_{\varepsilon,0}$ converges to u_0 in $W^{1,p}_{\text{loc}}(\Omega)$. Then the weak solution of the first initial-boundary problem (2.1)–(2.3) *u_ε is convergent in $L^2(Q_T)$ and the limit function is the weak solution of the equation* (1.1) *with initial value condition* (1.3)*.*

Proof. Using the maximum principle and a rather standard technique, we may easily show that there exists a constant C depending on $\|u_{\varepsilon,0}\|_{L^\infty(\Omega)}$, $\|\rho_\varepsilon^\alpha|\nabla u_{\varepsilon,0}|^p\|_{L^1(\Omega)}$, g and independent of ε such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leq C, \quad \iint_{Q_T} \rho_\varepsilon^\alpha|\nabla u_\varepsilon|^p dxdt \leq C, \quad \iint_{Q_T} \Big(\frac{\partial u_\varepsilon}{\partial t}\Big)^2 dxdt \leq C.$$

So there exist a function u and an n-dimensional vector function $\vec{\zeta} = (\zeta_1, \cdots, \zeta_n)$ such that

$$u \in C(0,T;L^2(\Omega)) \cap L^\infty(Q_T), \quad \frac{\partial u}{\partial t} \in L^2(Q_T), \quad |\vec{\zeta}| \in L^{p/(p-1)}(Q_T)$$

and

$$u_\varepsilon \to u \quad \text{in } L^2(Q_T), \qquad \nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^p_{\rm loc}(Q_T),$$
$$\frac{\partial u_\varepsilon}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } L^2(Q_T), \qquad \rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \rightharpoonup \vec{\zeta} \quad \text{in } L^{p/(p-1)}(Q_T;\mathbb{R}^n).$$

In addition, u satisfies (1.3) in the trace sense. To prove that u satisfies the equation (1.1), we note that for any test function $\varphi \in C_0^\infty(Q_T)$, the integral equality

$$\iint_{Q_T} \Big(\frac{\partial u_\varepsilon}{\partial t}\varphi + \rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla\varphi\Big) dxdt = 0 \tag{2.4}$$

holds, which implies, by letting $\varepsilon \to 0$, that

$$\iint_{Q_T} \Big(\frac{\partial u}{\partial t}\varphi + \vec{\zeta} \cdot \nabla\varphi\Big) dxdt = 0. \tag{2.5}$$

It remains to show that for any $\varphi \in C_0^\infty(Q_T)$,

$$\iint_{Q_T} \rho^\alpha |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi dxdt = \iint_{Q_T} \vec{\zeta} \cdot \nabla\varphi dxdt. \tag{2.6}$$

Let $0 \le \psi \in C_0^\infty(Q_T)$ and $\psi = 1$ on suppφ. Choosing $\varphi = \psi u_\varepsilon$ in (2.4), we see that

$$\iint_{Q_T} \psi\rho_\varepsilon^\alpha |\nabla u_\varepsilon|^p dxdt = \frac{1}{2}\iint_{Q_T} u_\varepsilon^2 \frac{\partial \psi}{\partial t} dxdt - \iint_{Q_T} \rho_\varepsilon^\alpha u_\varepsilon |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla\psi dxdt.$$

Let $v \in C(0,T;L^2(\Omega)) \cap L^\infty(Q_T)$ and $\rho^\alpha|\nabla v|^p \in L^1(Q_T)$. It is obvious that

$$\iint_{Q_T} \psi\rho_\varepsilon^\alpha (|\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon - |\nabla v|^{p-2}\nabla v)\cdot(\nabla u_\varepsilon - \nabla v) dxdt \ge 0.$$

Therefore

$$\begin{aligned}
&\frac{1}{2}\iint_{Q_T} u_\varepsilon^2 \frac{\partial \psi}{\partial t} dxdt - \iint_{Q_T} \rho_\varepsilon^\alpha u_\varepsilon |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla\psi dxdt \\
&- \iint_{Q_T} \psi\rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{p-2}\nabla u_\varepsilon \cdot \nabla v dxdt \\
&- \iint_{Q_T} \psi\rho^\alpha |\nabla v|^{p-2}\nabla v \cdot (\nabla u_\varepsilon - \nabla v) dxdt \\
&+ \iint_{Q_T} \psi(\rho^\alpha - \rho_\varepsilon^\alpha)|\nabla v|^{p-2}\nabla v \cdot (\nabla u_\varepsilon - \nabla v) dxdt \ge 0.
\end{aligned}$$

Letting $\varepsilon \to 0$ and noticing that

$$\begin{aligned}
&\Big|\iint_{Q_T} \psi(\rho^\alpha - \rho_\varepsilon^\alpha)|\nabla v|^{p-2}\nabla v \cdot (\nabla u_\varepsilon - \nabla v) dxdt\Big| \\
\le& \sup_{(x,t)\in Q_T} \frac{\psi|\rho^\alpha - \rho_\varepsilon^\alpha|}{\rho^\alpha} \iint_{Q_T} \rho^\alpha |\nabla v|^{p-1}|\nabla u_\varepsilon - \nabla v| dxdt,
\end{aligned}$$

we obtain

$$\begin{aligned}
&\frac{1}{2}\iint_{Q_T} u^2 \frac{\partial \psi}{\partial t} dxdt - \iint_{Q_T} u\, \vec{\zeta} \cdot \nabla\psi dxdt \\
&- \iint_{Q_T} \psi\, \vec{\zeta} \cdot \nabla v dxdt - \iint_{Q_T} \psi\rho^\alpha |\nabla v|^{p-2}\nabla v \cdot (\nabla u - \nabla v) dxdt \ge 0.
\end{aligned}$$

By the choice that $\varphi = \psi u$ in (2.5), we see that

$$\iint_{Q_T} \psi \vec{\zeta} \cdot \nabla u dxdt = \frac{1}{2}\iint_{Q_T} u^2 \frac{\partial \psi}{\partial t} dxdt - \iint_{Q_T} u \vec{\zeta} \cdot \nabla \psi dxdt.$$

Therefore

$$\iint_{Q_T} \psi(\vec{\zeta} - \rho^\alpha |\nabla v|^{p-2}\nabla v) \cdot (\nabla u - \nabla v) dxdt \geq 0. \tag{2.7}$$

Choosing $v = u - \lambda\varphi$ with $\lambda > 0$ in (2.7), we get

$$\iint_{Q_T} \psi(\vec{\zeta} - \rho^\alpha |\nabla(u - \lambda\varphi)|^{p-2}\nabla(u - \lambda\varphi)) \cdot \nabla\varphi dxdt \geq 0,$$

which implies, by letting $\lambda \to 0$, that

$$\iint_{Q_T} \psi(\vec{\zeta} - \rho^\alpha |\nabla u|^{p-2}\nabla u) \cdot \nabla\varphi dxdt \geq 0.$$

If we choose $\lambda < 0$, we get the inequality with opposite sign. Thus

$$\iint_{Q_T} \psi(\vec{\zeta} - \rho^\alpha |\nabla u|^{p-2}\nabla u) \cdot \nabla\varphi dxdt = 0.$$

Noticing that $\psi = 1$ on suppφ, we see that (2.6) holds. The proof is complete.

Proof of Theorem 1.1. We first prove the existence. For all $\varepsilon > 0$, choose $u_{\varepsilon,0}$ such that $\|u_{\varepsilon,0}\|_{L^\infty(\Omega)}$ and $\|\rho_\varepsilon^\alpha |\nabla u_{\varepsilon,0}|^p\|_{L^1(\Omega)}$ are uniformly bounded, and $u_{\varepsilon,0}$ converges to u_0 in $W^{1,p}_{\text{loc}}(\Omega)$. Let u_ε be the weak solution of the first initial-boundary problem (2.1)–(2.3). From Lemma 2.1, we see that u_ε is convergent in $L^2(Q_T)$ and the limit function u satisfies the equation (1.1) with the initial condition (1.3). Now we prove u also satisfies the boundary condition (1.2), thus u is the weak solution of the first initial-boundary problem (1.1)–(1.3).

Since $\frac{\alpha}{p-1} < 1$ and $p - \alpha > 1$, there exists a constant $\beta \in (\frac{\alpha}{p-1}, 1)$ such that $p - \frac{\alpha}{\beta} > 1$. Since $\beta < 1$ and $p - \frac{\alpha}{\beta} > 1$, there exists a constant $\gamma \in (1, p - \frac{\alpha}{\beta})$ such that $\beta\gamma < 1$. Therefore

$$\begin{aligned}
\iint_{Q_T} |\nabla u_\varepsilon|^\gamma dxdt &= \iint_{\{(x,t)\in Q_T; \rho_\varepsilon^\beta |\nabla u_\varepsilon| \leq 1\}} |\nabla u_\varepsilon|^\gamma dxdt \\
&\quad + \iint_{\{(x,t)\in Q_T; \rho_\varepsilon^\beta |\nabla u_\varepsilon| > 1\}} |\nabla u_\varepsilon|^\gamma dxdt \\
&\leq \iint_{Q_T} \rho_\varepsilon^{-\beta\gamma} dxdt + \iint_{Q_T} \rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{\alpha/\beta+\gamma} dxdt \\
&\leq \iint_{Q_T} \rho_\varepsilon^{-\beta\gamma} dxdt + \iint_{Q_T} \rho_\varepsilon^\alpha (1 + |\nabla u_\varepsilon|^p) dxdt \\
&\leq C,
\end{aligned}$$

where C is a positive constant independent of ε. Thus ∇u_ε is uniformly bounded in $L^\gamma(Q_T)$. So u satisfies the boundary condition (1.2).

Now, we prove the uniqueness. Let u and v be two weak solutions of the first initial-boundary problem (1.1)–(1.3). From the definition of solutions, we see that

$$\iint_{Q_T} \varphi \frac{\partial(u - v)}{\partial t} dxdt = -\iint_{Q_T} \rho^\alpha (|\nabla u|^{p-2}\nabla u - |\nabla v|^{p-2}\nabla v) \cdot \nabla\varphi dxdt$$

holds for any $\varphi \in C_0^\infty(Q_T)$. For any fixed $s \in [0,T]$, after an approximate procedure, we may choose $\chi_{[0,s]}(u-v)$ as a test function in the above equality, where $\chi_{[0,s]}$ is the characteristic function on $[0,s]$. Thus

$$\begin{aligned}&\iint_{Q_s}(u-v)\frac{\partial(u-v)}{\partial t}dxdt\\=&-\iint_{Q_s}\rho^\alpha(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)dxdt\le 0,\end{aligned}$$

and hence

$$\begin{aligned}\int_\Omega(u(x,s)-v(x,s))^2dx&=\int_\Omega(u(x,0)-v(x,0))^2dx+\iint_{Q_s}\frac{\partial}{\partial t}(u-v)^2dxdt\\&=2\iint_{Q_s}(u-v)\frac{\partial(u-v)}{\partial t}dxdt\\&\le 0,\end{aligned}$$

which implies that

$$u(x,s)=v(x,s),\qquad \text{a.e. } (x,s)\in Q_T.$$

The proof is complete.

Proof of Theorem 1.2. The existence has already been done in Lemma 2.1. We need only to show the uniqueness. Denote

$$\Omega_\varepsilon=\{x\in\Omega \mid \operatorname{dist}(x,\partial\Omega)>\varepsilon\}.$$

Let $\xi_\varepsilon\in C_0^\infty(\Omega_\varepsilon)$ such that $\xi_\varepsilon=1$ on $\Omega_{2\varepsilon}$, $0\le\xi_\varepsilon\le 1$ and

$$|\nabla\xi_\varepsilon|\le\frac{C}{\varepsilon},$$

where C is a constant independent of ε. Let u and v be two weak solutions of the equation (1.1) with initial value (1.3). From the definition of solutions, we get

$$\iint_{Q_T}\varphi\frac{\partial(u-v)}{\partial t}dxdt=-\iint_{Q_T}\rho^\alpha(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla\varphi dxdt$$

for any $\varphi\in C_0^\infty(Q_T)$. For any fixed $s\in[0,T]$, after an approximate procedure, we may choose $\chi_{[0,s]}(u-v)\xi_\varepsilon$ as a test function in the above equality, where $\chi_{[0,s]}$ is the characteristic function on $[0,s]$. Thus

$$\begin{aligned}&\iint_{Q_s}(u-v)\xi_\varepsilon\frac{\partial(u-v)}{\partial t}dxdt\\=&-\iint_{Q_s}\rho^\alpha(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla((u-v)\xi_\varepsilon)dxdt.\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_\Omega (u(x,s)-v(x,s))^2\xi_\varepsilon dx\\
=&\int_\Omega (u(x,0)-v(x,0))^2\xi_\varepsilon dxdt\\
&-2\iint_{Q_s}\xi_\varepsilon\rho^\alpha(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla(u-v)dxdt\\
&-2\iint_{Q_s}(u-v)\rho^\alpha(|\nabla u|^{p-2}\nabla u-|\nabla v|^{p-2}\nabla v)\cdot\nabla\xi_\varepsilon dxdt\\
\le& 2\iint_{Q_s}|u-v|\rho^\alpha(|\nabla u|^{p-1}+|\nabla v|^{p-1})|\nabla\xi_\varepsilon|dxdt\\
\le& C\Big(\int_0^T\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}}\rho^\alpha(|\nabla u|^p+|\nabla v|^p)dxdt\Big)^{(p-1)/p}\\
&\cdot\Big(\int_0^T\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}}\rho^\alpha|\nabla\xi_\varepsilon|^p dxdt\Big)^{1/p}\\
\le& C\Big(\int_0^T\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}}\rho^\alpha(|\nabla u|^p+|\nabla v|^p)dxdt\Big)^{(p-1)/p}\\
&\cdot\Big(\int_0^T\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}}\varepsilon^{\alpha-p}dxdt\Big)^{1/p}\\
\le& C\varepsilon^{(\alpha+1-p)/p}\Big(\int_0^T\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}}\rho^\alpha(|\nabla u|^p+|\nabla v|^p)dxdt\Big)^{(p-1)/p}\\
\le& C\Big(\int_0^T\int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}}\rho^\alpha(|\nabla u|^p+|\nabla v|^p)dxdt\Big)^{(p-1)/p},
\end{aligned}$$

where C is a constant independent of ε. Since $\rho^\alpha|\nabla u|^p,\rho^\alpha|\nabla v|^p\in L^1(Q_T)$, letting $\varepsilon\to 0$, we see that

$$\int_\Omega (u(x,s)-v(x,s))^2dx\le 0,$$

which implies that

$$u(x,s)=v(x,s),\qquad \text{a.e. } (x,s)\in Q_T.$$

The proof is complete.

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