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ESTIMATE OF THE UPPER CRITICAL FIELD AND CONCENTRATION FOR SUPERCONDUCTOR***

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Abstract

The effect of an applied magnetic field on an inhomogeneous superconductor is studied and the value of the upper critical magnetic field H_{c_3} at which superconductivity can nucleate is estimated. In addition, the authors locate the concentration of the order parameter, which depends on the inhomogeneous term $a(x)$. Unlikely to the homogeneous case, the order parameter may concentrate in the interior of the superconducting material, due to the influence of the inhomogeneous term $a(x)$.

Keywords Superconductor, Critical field, Nucleation, Ginzburg-Landau
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§ 1 . Introduction

Consider the following functional

$$J(\psi, A) = \frac{1}{2}\int_\Omega |\nabla_{\kappa A}\psi|^2 + \frac{1}{2}\kappa^2(a(x) - |\psi|^2)^2 + \kappa^2|h - h_{ex}|^2, \tag{1.1}$$

which corresponds to the free energy of a superconductor in a prescribed constant magnetic field h_{ex}. Here, $\Omega \subset \mathbb{R}^2$ is the smooth, bounded, simply connected section of the superconductor and $a(x) : \Omega \to \mathbb{R}^2$ is a given function satisfying $0 < \min\limits_{\overline{\Omega}} a(x) \leq a(x)$ in Ω. The unknowns are the complex-valued order parameter $\psi \in H^1(\Omega, \mathbb{C})$ and the $U(1)$ connection $A \in H^1(\Omega, \mathbb{R}^2)$. $h = \mathrm{curl}A$ is the induced magnetic field, $\nabla_{\kappa A}\psi = \nabla\psi - i\kappa\psi A$. We denote

$$\partial_j = \frac{\partial}{\partial x_j}, \quad \mathrm{curl}A = \partial_1 A_2 - \partial_2 A_1, \quad \mathrm{curl}^2 A = (\partial_2(\mathrm{curl}A), -\partial_1(\mathrm{curl}A)),$$
$$\nabla_A^2\psi = (\nabla - iA)^2\psi = \triangle\psi - i(A\cdot\nabla\psi + \psi\mathrm{div}A) - |A|^2\psi.$$

The order parameter ψ indicates the local state of the material, viz., $|\psi|$ is the density of superconducting electron pairs so that, when $|\psi| \simeq 1$, the material is in its superconducting state, whereas when $|\psi| \simeq 0$, it is in its normal state. $\kappa > 0$ is the Ginzburg-Landau

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parameter depending on the material. The modified Ginzburg-Landau functional (1.1) was first written down by Likharev [14]. Then, this model has been used and developed by Aftalion, Sandier, Serfaty, Chapman and Richardson [2, 8]. The minima of $a(x)$ corresponds to the impurities in the material.

It is well known that a superconductor placed in an applied magnetic field may change its phase when the field varies. If the field is sufficiently strong, it penetrates through the entire sample and the superconductor is in a normal state. As the field is gradually reduced to a certain value H_{c_3}, called upper critical field, the nucleation of superconductivity occurs. It is important both in theory and in applications to estimate the values of the critical fields for superconductors, especially for type II superconductors with large value of κ.

Recently, there have been extensive works on the Ginzburg-Landau system with or without the applied magnetic field (see [1–23]). We do not attempt to give an exhaustive list of references, but briefly summarize the advances concerning this problem. In the case $a(x) \equiv 1$ in Ω, the physicists Saint-James and De Gennes [19] were the first to study the nucleation phenomenon for semi-infinite superconductor occupying the half space. They found that the nucleation of superconductivity occurs first on the surface. Chapman [6] made a study of the half-plane problem on H_{c_3} by using formal mathematical analysis. Bauman, Phillips and Tang [3] rigorously estimated H_{c_3} and found the location of nucleation for a disk occupying a 2-dimensional cross section of a cylinder. Bernoff and Sternberg [5] considered an arbitrary simply connected smooth bounded region in $\mathbb{R}^2$ occupying an infinite cylinder with 2-dimensional cross section. They estimated H_{c_3} and found the location of nucleation by using formal asymptotic expansions. In [11, 17], Helffer, Pan and Lu, Pan rigorously obtained the estimates for H_{c_3} and the locations of nucleation for a cylindrical sample which is placed in an applied magnetic field.

In this paper, assuming $a(x) \neq 1$, we want to address the question how the term $a(x)$ will modify the properties of the superconductor in the presence of an applied magnetic field. We obtain rigorously the estimates for H_{c_3} and the location of nucleation for a cylinder which is placed in an applied magnetic field. We found a new phenomenon that if the applied field reduced to H_{c_3}, the nucleation of superconductivity may occur in the interior of the domain since the term $a(x)$ happens. Before starting our main results, we recall that the minimizers of $J(\psi, A)$ in $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ satisfy the following Euler equations

$$-(\nabla - i\kappa A)^2\psi = \kappa^2(a - |\psi|^2)\psi \qquad \text{in} \quad \Omega, \tag{1.2}$$

$$\text{curl}^2 A = -\frac{i}{2\kappa}(\overline{\psi}\,\nabla\psi - \psi\nabla\overline{\psi}) - |\psi|^2 A \qquad \text{in} \quad \Omega, \tag{1.3}$$

$$(\nabla_{\kappa A}\psi)\cdot\nu = 0 \text{ and } \text{curl}A = h_{ex} \qquad \text{on} \quad \partial\Omega, \tag{1.4}$$

where ν is the unit out-normal vector on the boundary of Ω.

In order to make our discussion clear, we replace h_{ex} by σ and A by σA respectively, then the Ginzburg-Landau functional (1.1) can be rewritten as

$$G(\psi, A) = \frac{1}{2}\int_\Omega |\nabla_{\sigma\kappa A}\psi|^2 + (\sigma\kappa)^2|\text{curl}A - 1|^2 + \frac{\kappa^2}{2}(a(x) - |\psi|^2)^2. \tag{1.5}$$

It is well known that there exists a unique smooth vector field F on $\overline{\Omega}$ such that

$$\text{curl}F = 1, \quad \text{div}F = 0 \quad \text{in} \quad \Omega \qquad \text{and} \qquad F\cdot\nu = 0 \quad \text{on} \quad \partial\Omega. \tag{1.6}$$

Note that $(0, F)$ is a trivial critical point of the functional G. Moreover, $(0, F)$ is the only minimizer if σ is large enough, which means that a sufficiently strong magnetic field penetrates the entire superconductor and completely destroys superconductivity. Now, we define

$\sigma^*(\kappa)$ as

$$\sigma^*(\kappa) = \inf\{\sigma > 0 : (0, F) \text{ is the only minimizer of G in } H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)\}. \tag{1.7}$$

Then, it naturally leads to the definition of the upper critical applied magnetic field $H_{c_3}(\kappa) = \sigma^*(\kappa)$. Denote

$$\gamma_0 = \min\Big\{\min_{x\in\Omega} \frac{1}{a(x)},\ \beta_0 \min_{x\in\partial\Omega} \frac{1}{a(x)}\Big\}, \tag{1.8}$$

where β_0 is defined as $\beta_0 = \inf\limits_{\psi\in H^1(\mathbb{R}^2_+,\mathbb{C})} \frac{\int_{\mathbb{R}^2_+} |\nabla_\omega \psi|^2}{\int_{\mathbb{R}^2_+} |\psi|^2}$. Then we have the following theorem which estimates the upper critical field $H_{c_3}(\kappa)$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, smooth and simply connected domain. Assume that $a(x) \in C^2(\overline{\Omega})$ and $0 < a_0 = \min\limits_{\overline{\Omega}} a(x) \le a(x)$ in $\overline{\Omega}$. Then, we have*

$$\lim_{\kappa\to\infty} \frac{\sigma^*}{\kappa} = \frac{1}{\gamma_0}. \tag{1.9}$$

In particular,

$$H_{c_3}(\kappa) = \sigma^*(\kappa) = \frac{\kappa}{\gamma_0}(1 + o(1)) \qquad \text{as} \ \ \kappa \to +\infty. \tag{1.10}$$

The nucleation phenomenon can be described by the concentration behavior of the order parameter ψ when σ is close to H_{c_3}. Denote

$$\Omega_m = \Big\{x \in \Omega : \frac{1}{a(x_0)} = \min_{x\in\Omega} \frac{1}{a(x)}\Big\}, \tag{1.11}$$

$$(\partial\Omega)_m = \Big\{x \in \partial\Omega : \frac{1}{a(x_0)} = \min_{x\in\partial\Omega} \frac{1}{a(x)}\Big\}. \tag{1.12}$$

Then we have the following theorem which locates the concentration of the order parameter ψ.

Theorem 1.2. *Under the assumptions of Theorem 1.1, let $\kappa_n \to +\infty$, $\sigma_n < \sigma^*(\kappa_n)$ and $\frac{\sigma_n}{\kappa_n} \to \frac{1}{\gamma_0}$. Let (ψ_n, A_n) be a non-trivial minimizer of the functional G with $\kappa = \kappa_n$ and $\sigma = \sigma_n$. Then $\mathrm{curl} A_n \to 1$ in $C^\alpha(\Omega)$, $\| \psi_n \|_{L^\infty(\Omega)} \to 0$ and $\frac{\psi_n(x)}{\|\psi_n\|_{L^\infty(\Omega)}} \to 0$ on $\overline{\Omega} \setminus (\Omega_m \cup (\partial\Omega)_m)$ as $n \to +\infty$.*

Corollary 1.1. *Assume that $a(x) \in C^2(\overline{\Omega})$, $0 < a_0 \le a(x) \le 1$ in Ω and $a(x)$ has a unique maximum point $x_0 \in \Omega$ and $\frac{1}{a(x_0)} < \beta_0 \min\limits_{x\in\partial\Omega} \frac{1}{a(x)}$. Then*

$$H_{c_3} = \kappa a(x_0)(1 + o(1)) \qquad \text{as} \ \ \kappa \to +\infty. \tag{1.13}$$

Moreover, let (ψ_n, A_n) be a sequence of minimizers as in Theorem 1.2. Then, as $n \to +\infty$,

$$\begin{aligned} \frac{|\psi_n(x)|}{\| \psi_n \|_{L^\infty(\Omega)}} &\to 0 \qquad \text{if} \ \ x \in \overline{\Omega} \setminus \{x_0\}, \\ \frac{|\psi_n(x)|}{\| \psi_n \|_{L^\infty(\Omega)}} &\to 1 \qquad \text{if} \ \ x = x_0. \end{aligned} \tag{1.14}$$

Corollary 1.1 indicates that the order parameter may concentrate in the interior of Ω and have a spike-layer.

We organize this paper as follows: In Section 2, we collect some basic results needed in this paper. In Section 3, we derive an asymptotic estimate for a variational problem. In Section 4, we give a proof of Theorem 1.1. In Section 5, Theorem 1.2 is proven.

§ 2 . Preliminary Lemmas and Notations

In this section, some preliminary results which will be used in this paper are given. Throughout this paper, we denote

$$\omega(x) = \frac{1}{2}(-x_2, x_1). \tag{2.1}$$

Consider the following functional

$$\alpha(h) = \inf_{\psi \in H^1(\mathbb{R}^2, \mathbb{C})} \frac{\int_{\mathbb{R}^2} |\nabla_{h\omega}\psi|^2}{\int_{\mathbb{R}^2} |\psi|^2}. \tag{2.2}$$

Then the minimizers of (2.2) satisfy

$$-\nabla^2_{h\omega}\psi = \alpha\psi \qquad \text{in} \ \ \mathbb{R}^2. \tag{2.3}$$

Let

$$\beta(h) = \inf_{\psi \in H^1(\mathbb{R}^2_+, \mathbb{C})} \frac{\int_{\mathbb{R}^2_+} |\nabla_{h\omega}\psi|^2}{\int_{\mathbb{R}^2_+} |\psi|^2}. \tag{2.4}$$

Then the associated eigenvalue problem can be defined as

$$-\nabla^2_{h\omega}\psi = \beta\psi \qquad \text{in} \quad \mathbb{R}^2_+, \tag{2.5}$$

$$(\nabla_{h\omega}\psi)\cdot\nu = 0 \qquad \text{on} \quad \partial\mathbb{R}^2_+, \tag{2.6}$$

where $\nu = (0, -1)$ is the outward normal vector to $\partial\mathbb{R}^2_+$.

Lemma 2.1. (cf. [18, Lemma 2.1])

(i) *For every $h \neq 0$, $\alpha(h) = |h|$. The associated eigenfunctions are given by*

$$\psi(x) = \begin{cases} f(x)\exp\Big(-\dfrac{|h|r^2}{4}\Big) & \text{if} \ \ h > 0, \\ \overline{f(x)}\exp\Big(-\dfrac{|h|r^2}{4}\Big) & \text{if} \ \ h < 0, \end{cases}$$

where $r = |x|$ and $f(x)$ is any function analytic in $\mathbb{R}^2$ such that

$$f(x)\exp\Big(-\frac{|h|r^2}{4}\Big) \in L^2(\mathbb{R}^2).$$

Moreover, let $\alpha < \alpha(h)$. Then (2.3) *has no nontrivial bounded solution.*

(ii) *There exists a constant β_0 with $0 < \beta_0 < 1 - \dfrac{1}{\sqrt{2e\pi}}$ such that $\beta(h) = \beta_0|h|$. Moreover,* (2.5)–(2.6) *has no nontrivial bounded solution for all $\beta < \beta(h)$.*

Lemma 2.2. (cf. [17, Proposition 2.5]) *Let ψ satisfy the following equations respectively:*

$$-\nabla^2_{h\omega}\psi = \lambda(1 - |\psi|^2)\psi \quad \text{in} \ \ \mathbb{R}^2, \tag{2.7}$$

$$-\nabla^2_{h\omega}\psi = \lambda(1 - |\psi|^2)\psi \quad \text{in} \ \ \mathbb{R}^2_+, \qquad (\nabla_{h\omega}\psi)\cdot\nu = 0 \quad \text{on} \ \ \partial\mathbb{R}^2_+. \tag{2.8}$$

Then the only bounded solution of (2.7) *is $\psi \equiv 0$ for $0 \le \lambda \le |h|$. Similarly, the only bounded solution of* (2.8) *is $\psi \equiv 0$ for $0 \le \lambda \le \beta_0|h|$.*

In later sections, the gauge transformation will be used frequently and thus a decomposition of vector field into gradient and curl parts is needed. Let $A(x) = (A^1(x), A^2(x)) \in C^2(\overline{B}_R)$. Here $\overline{B}_R$ is the closed ball with radius R. Denote $a^i_j = \frac{\partial A^i}{\partial x_j}(0)$, $a^i_{jk} = \frac{\partial^2 A^i}{\partial x_j \partial x_k}(0)$, $a^1 = A^1(0)$ and $a^2(0) = A^2(0)$ respectively. Let $H(x) = \text{curl}\, A(x)$. Then $\text{curl}\,^2 A(x) = (\partial_2 H, -\partial_1 H)$ and we have the following lemma.

Lemma 2.3. (cf. [18, Lemma 3.1]) *A vector field $A(x) = (A^1(x), A^2(x)) \in C^2(\overline{B}_R)$ can be decomposed as*

$$A(x) = A(0) + \nabla\xi(x) + \nabla\zeta(x) + \text{curl}\, A(0)\, \omega(x) - \frac{1}{2}|x|^2 \text{curl}^2 A(0) + D(x), \tag{2.9}$$

where

$$\begin{aligned} \xi(x) &= \frac{1}{2}[a^1_1 {x_1}^2 + (a^1_2 + a^2_1)x_1x_2 + a^2_2 {x_2}^2], \\ \zeta(x) &= \frac{1}{6}[c_1 {x_1}^3 + 3c_2 {x_1}^2 x_2 + 3c_3 x_1 {x_2}^2 + c_4 {x_2}^3] \end{aligned} \tag{2.10}$$

with $c_1 = a^2_{11} + \partial_2 H(0)$, $c_2 = a^1_{12}$, $c_3 = a^2_{12}$ and $c_4 = a^2_{22} - \partial_1 H(0)$ respectively. Moreover, $|D(x)| = o(|x|^2)$ as $x \to 0$ and if $A \in C^3(\overline{B}_R)$, then $|D(x)| \le C(R)|x|^3$ in B_R. Here $C(R)$ is a constant which depends on a given radius R.

In the following we assume that Ω is a smooth, bounded, simply connected domain in $\mathbb{R}^2$ and $0 \in \partial\Omega$. The boundary $\partial\Omega$ of Ω can be represented as $z = z(s)$, where s is the arclength of $\partial\Omega$. Let $\tau(s) = (\tau_1, \tau_2) = z'(s)$ be the unit tangent vector and $\nu(s) = (\nu_1, \nu_2)$ be the unit outer normal. We choose the positive direction of $\partial\Omega$ in such a way that the orientation of (ν, τ) is coincident with the orientation of the x_1x_2 coordinates. Then, $\tau_1 = -\nu_2$, $\tau_2 = \nu_1$. We denote the relative curvature of $\partial\Omega$ under the given orientation by k_r. The mapping

$$x = \mathcal{F}(s,t) = z(s) - t\nu(s) \tag{2.11}$$

determines a C^1-transformation of coordinates, and $g(s,t) = |\det D\mathcal{F}| = 1 - tk_r(s)$. In the following, we change the variables $y_1 = s$ and $y_2 = t$ with $y = (y_1, y_2)$. Denote $e_1 = \tau$, $e_2 = -\nu$. Then, for a given vector field $A(x)$, a new vector field $\overline{A}$ associated with A can be defined by

$$\overline{A}(y) = \overline{A}^1(y)e_1 + \overline{A}^2 e_2, \tag{2.12}$$

where

$$\overline{A}^1(y) = g(y)A(\mathcal{F}(y)) \cdot e_1(y) \quad \text{and} \quad \overline{A}^2(y) = A(\mathcal{F}(y)) \cdot e_2(y). \tag{2.13}$$

One can easily verify that

$$\begin{aligned} \text{curl} A(x) &= \frac{1}{g}(\partial_1 \overline{A}^2 - \partial_2 \overline{A}^1), \\ \text{div} A(x) &= \frac{1}{g}\Big[\partial_1\Big(\frac{\overline{A}^1}{g}\Big) + \partial_2(g\overline{A}^2)\Big], \\ \text{curl}^2 A(x) &= \partial_2 \widetilde{H}(y) \cdot e_1 - \frac{1}{g}\partial_1 \widetilde{H}(y) \cdot e_2, \end{aligned}$$

where $\widetilde{H}(y) = (\partial_1 \overline{A}^2 - \partial_2 \overline{A}^1)$. Now, define $D(g)W = \frac{1}{g}\partial_1 W e_1 + \partial_2 W e_2$ and $D(g)_{\overline{A}} W = [D(g)_{\overline{A}^1} W]e_1 + [D(g)_{\overline{A}^2} W]e_2$ with $D(g)_{\overline{A}^1} W = \frac{1}{g}(\partial_1 - i\overline{A}^1)W$ and $D(g)_{\overline{A}^2} W = (\partial_2 - i\overline{A}^2)W$ respectively. Then, we can define

$$D(g)^*_{\overline{A}^1} W = \frac{1}{g}(\partial_1 - i\overline{A}^1)W, \quad D(g)^*_{\overline{A}^2} W = \frac{1}{g}[\partial_2(gW) - i\overline{A}^2 gW]$$

and

$$\begin{aligned}\Delta(g)_{\overline{A}}W &= D(g)^*_{\overline{A}^1}D(g)_{\overline{A}^1}W + D(g)^*_{\overline{A}^2}D(g)_{\overline{A}^2}W]\\ &= \frac{1}{g}\Big\{\partial_1\Big[\frac{1}{g}(\partial_1 W - i\overline{A}^1 W)\Big] - \frac{i}{g}\overline{A}^1(\partial_1 W - i\overline{A}^1 W)\Big\}\\ &\quad + \frac{1}{g}\{\partial_2[g(\partial_2 W - i\overline{A}^2 W)] - i\overline{A}^2 g(\partial_2 W - i\overline{A}^2 W)\}.\end{aligned}$$

If $\widetilde{\psi}(y) = \psi(\mathcal{F}(y))$, then $\nabla_A\psi = D(g)_{\overline{A}}\widetilde{\psi},\ \nabla_A^2\psi = \Delta(g)_{\overline{A}}\widetilde{\psi}$ and $\Delta\psi = \Delta(g)\widetilde{\psi}$. Now, let

$$\nabla\chi = (\partial_1\chi)e_1 + (\partial_2\chi)e_2. \tag{2.14}$$

Then operators $D(g)_{\overline{A}}$ and $\Delta(g)_{\overline{A}}$ have the following gauge invariant properties:

$$D(g)_{\overline{A}+\nabla\chi}(e^{i\chi}\varphi) = e^{i\chi}D(g)_{\overline{A}}\varphi, \quad \Delta(g)_{\overline{A}+\nabla\chi}(e^{i\chi}\varphi) = e^{i\chi}\Delta(g)_{\overline{A}}\varphi. \tag{2.15}$$

Lemma 2.4. (cf. [18, Lemma 3.2]) *Let Ω be a smooth domain in $\mathbb{R}^2$ with $0 \in \partial\Omega$. Assume that $A \in C^2(\overline{\Omega \cap \mathcal{F}(B_R)})$. Then, in the new coordinates y straightening the boundary, the vector field $\overline{A}(y)$ associated with $A(x)$ has the following decomposition for $y \in B_R$:*

$$\begin{aligned}(\overline{A}^1, \overline{A}^2) &= A(0) + \nabla\widetilde{\xi}(y) + \nabla\widetilde{\zeta}(y) + \operatorname{curl} A(0)\,\omega(y)\\ &\quad - \frac{|y|^2}{2}[\operatorname{curl}^2 A(0) - k_r(0)\operatorname{curl} A(0)\tau(0)] + \widetilde{D}(y),\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}\widetilde{\xi}(y) &= \frac{1}{2}[(a_1^1 + k_r(0)a^2){y_1}^2 + (a_2^1 + a_1^2 - 2k_r(0)a^1)y_1y_2 + a_2^2{y_2}^2],\\ \widetilde{\zeta}(y) &= \frac{1}{6}[\tilde{c}_1{y_1}^3 + 3\tilde{c}_2{y_1}^2y_2 + 3\tilde{c}_3y_1{y_2}^2 + \tilde{c}_4{y_2}^3]\end{aligned}$$

with

$$\begin{aligned}\tilde{c}_1 &= a_{11}^1 - a_{22}^1 + a_{12}^2 + k_r(0)(a_1^2 + a_2^1) - {k_r}^2(0)a^1 + k_r'(0)a^2,\\ \tilde{c}_2 &= a_{12}^1 + k_r(0)(a_2^2 - 2a_1^1) - {k_r}^2(0)a^2 - k_r'(0)a^1,\\ \tilde{c}_3 &= a_{12}^2 - k_r(0)(a_1^2 + a_2^1),\\ \tilde{c}_4 &= a_{12}^1 - a_{11}^2 + a_{22}^2.\end{aligned}$$

Moreover, $|\widetilde{D}(y)| = o(|y|^2)$ as $y \to 0$ and if $A \in C^3(\overline{\Omega \cap \mathcal{F}(B_R)})$, then $|\widetilde{D}(y)| \le C(R)|y|^3$ in B_R^+. Here $C(R)$ is a constant which depends on the radius R.

Noting $\nabla\widetilde{\xi} = (\partial_{y_1}\widetilde{\xi}, \partial_{y_2}\widetilde{\xi})$ in (2.16), and using (2.12) and (2.14), we can rewrite (2.16) as

$$\begin{aligned}\overline{A}(y) &= A(0) + \nabla\widetilde{\xi}(y) + \nabla\widetilde{\zeta}(y) + \operatorname{curl} A(0)\,\widetilde{\omega}(y)\\ &\quad - \frac{|y|^2}{2}[\operatorname{curl}^2 A(0) - k_r(0)\operatorname{curl} A(0)\,\tau(0)] + \widetilde{D}(y),\end{aligned} \tag{2.17}$$

where $\widetilde{\omega}(y) = -\frac{y_2}{2}e_1 + \frac{y_1}{2}e_2$.

§ 3 . Variational Problem

In this section, the asymptotic behavior of the following variational problem

$$\mu(\sigma A) = \inf_{\psi \in H^1(\Omega,\mathbb{C})} \frac{\int_\Omega |(\nabla - i\sigma A)\psi|^2}{\int_\Omega a(x)|\psi|^2} \tag{3.1}$$

is examined in the region $\sigma \gg 1$. Here A is any vector field with $\operatorname{curl} A$ equal to 1 and $a(x) \in C^2(\overline{\Omega})$ with $0 < \min_{\overline{\Omega}} a(x) = a_0$. Then, the associated eigenvalue problem is

$$-\nabla^2_{\sigma A}\psi = \mu a(x)\psi \quad \text{in} \ \ \Omega, \qquad (\nabla_{\sigma A}\psi)\cdot\nu = 0 \quad \text{on} \ \ \partial\Omega. \tag{3.2}$$

$\mu(\sigma A)$ is the first eigenvalue of this problem.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open, and simply connected domain with $\partial\Omega \in C^2$. Assume that $a(x) \in C^2(\overline{\Omega})$ with $0 < a_0 \le a(x)$ in $\overline{\Omega}$, $A \in C^2(\overline{\Omega})$, and $\operatorname{curl} A = 1$ in Ω. Then, there exists a universal constant β_0, which is given in Lemma 2.1, with $0 < \beta_0 < 1$ such that*

$$\lim_{\sigma\to\infty} \frac{\mu(\sigma A)}{|\sigma|} = \min\Big\{ \min_{x\in\Omega} \frac{1}{a(x)}, \beta_0 \min_{x\in\partial\Omega} \frac{1}{a(x)} \Big\}. \tag{3.3}$$

To prove Theorem 3.1, several lemmas will be given first.

Lemma 3.1. *Under the assumptions of Theorem 3.1, we have*

$$\limsup_{\sigma\to\infty} \frac{\mu(\sigma A)}{|\sigma|} \le \min_{x\in\Omega} \frac{1}{a(x)}. \tag{3.4}$$

Proof. For any $\psi \in H^1(\Omega,\mathbb{C})$, we have $|\nabla_{-\sigma A}\overline{\psi}| = |\overline{\nabla_{\sigma A}\psi}| = |\nabla_{\sigma A}\psi|$. Thus, $\mu(-\sigma A) = \mu(\sigma A)$. Therefore, we may assume $\sigma > 0$. We shall show that for any $x_0 \in \Omega$, $\limsup_{\sigma\to+\infty} \frac{\mu(\sigma A)}{\sigma} \le \frac{1}{a(x_0)}$. Without loss of generality, we may assume $x_0 = 0$. Set $\delta = \frac{1}{\sqrt{\sigma}}$ and let $B_R \subset \Omega$ be a ball with radius R. Now, for any $\psi \in H^1(\Omega,\mathbb{C})$, let $\psi_\delta(x) = \psi(\delta x)$ and $A_\delta(x) = A(\delta x)/\delta$. Then

$$\frac{\mu(\sigma A)}{\sigma} \le \inf_{\psi\in H^1_0(B_R,\mathbb{C})} \frac{\int_{B_R} |\nabla_{\sigma A}\psi|^2}{\int_{B_R} a(x)|\psi|^2} = \inf_{\phi\in H^1_0(B_{R/\delta},\mathbb{C})} \frac{\int_{B_{R/\delta}} |\nabla_{A_\delta}\phi|^2}{\int_{B_{R/\delta}} a(\delta x)|\phi|^2}.$$

Using Lemma 2.3 and noting that $\nabla\xi(\delta x) = \delta\nabla\xi(x)$ and $\omega(\delta x) = \delta\omega(x)$, we have

$$A_\delta(x) = \nabla\chi_\delta(x) + \omega(x) + B_\delta(x),$$

where

$$\begin{aligned} \chi_\delta(x) &= \frac{1}{\delta}A(0)\cdot x + \xi(x) + \delta\zeta(x), \\ B_\delta(x) &= -\frac{\delta}{2}|x|^2 \operatorname{curl}^2 A(0) + \frac{1}{\delta}D(\delta x), \\ |B_\delta(x)| &\le \frac{\delta}{2}|\operatorname{curl}^2 A(0)||x|^2[1+o(R)] \qquad \text{in} \ \ B_{R/\delta}. \end{aligned}$$

Therefore, using gauge invariance property, we have

$$|\nabla_{A_\delta} e^{i\chi_\delta}\phi|^2 \le (1+\lambda)|\nabla_\omega\phi|^2 + \frac{(1+\lambda)\delta^2}{4\lambda}|\operatorname{curl}^2 A(0)|^2(1+o(R))^2|x|^4|\phi|^2,$$

where $0 \le \lambda \le 1$. Hence

$$\frac{\mu(\sigma A)}{\sigma} \le \inf_{\phi \in H_0^1(B_{R/\delta},\mathbb{C})} \frac{1}{\int_{B_{R/\delta}} a(\delta x)|\phi|^2} \Big\{ (1+\lambda) \int_{B_{R/\delta}} |\nabla_\omega \phi|^2 + \frac{(1+\lambda)\delta^2}{4\lambda}(1+o(R))^2 |\mathrm{curl}^2 A(0)|^2 \int_{B_{R/\delta}} |x|^4 |\phi|^2 \Big\}.$$

Now, choose $\phi = \phi_m = u\eta_m$ with an eigenfunction $u(x) = u(|x|) = \exp(-|x|^2/4)$ and a smooth cutoff function η_m whose support is in B_m such that $\eta_m = 1$ on $B_{m/2}$. Then, for a fixed R and all small δ,

$$\frac{\mu(\delta A)}{\sigma} \le \frac{1}{\int_{B_m} a(\delta x)|\phi_m|^2} \Big\{ (1+\lambda) \int_{B_m} |\nabla_\omega \phi_m|^2 + \frac{(1+\lambda)\delta^2}{4\lambda}(1+o(m))^2 |\mathrm{curl}^2 A(0)|^2 \int_{B_m} |x|^4 |\phi_m|^2 \Big\}.$$

First, letting σ tend to $+\infty$ for a fixed $m > 1$ and $\lambda \in (0,1)$ in the above inequality, we get

$$\limsup_{\sigma \to +\infty} \frac{\mu(\sigma A)}{\sigma} \le \frac{1+\lambda}{\int_{B_m} a(0)|\phi_m|^2} \int_{B_m} |\nabla_\omega \phi_m|^2.$$

Then, sending λ to 0 for a fixed m and m to $+\infty$ sequentially and using Lemma 2.1, we obtain

$$\limsup_{\sigma \to +\infty} \frac{\mu(\sigma A)}{\sigma} \le \frac{\int_{\mathbb{R}^2} |\nabla_\omega u|^2}{a(0) \int_{\mathbb{R}^2} |u|^2} = \frac{1}{a(0)}\alpha(1) = \frac{1}{a(0)}.$$

This completes the proof of Lemma 3.1.

Lemma 3.2. *Under the same assumptions of Theorem* 3.1*, we have*

$$\limsup_{\sigma \to \infty} \frac{\mu(\sigma A)}{|\sigma|} \le \beta_0 \min_{x \in \partial\Omega} \frac{1}{a(x)}, \tag{3.5}$$

where β_0 is the positive constant given in Lemma 2.1.

Proof. Using the same argument as in the proof of Lemma 3.1, the conclusion of Lemma 3.2 follows.

Lemma 3.3. *Under the assumptions of Theorem* 3.1*, we have*

$$\liminf_{\sigma \to \infty} \frac{\mu(\sigma A)}{|\sigma|} \ge \min \Big\{ \min_{x \in \Omega} \frac{1}{a(x)}, \beta_0 \min_{x \in \partial\Omega} \frac{1}{a(x)} \Big\}. \tag{3.6}$$

Proof. For $\sigma > 0$, denote $\delta = \frac{1}{\sqrt{\sigma}}$. Let ψ^δ be the minimizer of variational problem (3.1), then ψ^δ satisfies (3.2). Without loss of generality, let $\max_{x \in \overline{\Omega}} |\psi^\delta(x)| = 1$ and denote the maximum point of $|\psi^\delta|$ by x^δ. Now, assume that $\{\sigma_k\}$ is a given sequence such that $\sigma_k \to +\infty$. Then, choose a subsequence σ_{k_j} such that

$$x^{\delta_{k_j}} \to x^0 \qquad \text{and} \qquad \frac{\mu(\sigma_{k_j} A)}{\sigma_{k_j}} \to d$$

for some non-negative number d. For the simplicity of the notation, we denote σ_{k_j} by σ.

Case 1. $x^0 \in \Omega$. We shall show $d \geq \min_{x\in\Omega} \frac{1}{a(x)}$. Let $\Omega_\delta = (\Omega - x^\delta)/\delta$, $\psi_\delta(x) = \psi^\delta(x^\delta + \delta x)$ and $A_\delta(x) = (1/\delta)A(x^\delta + \delta x)$. Note that $\operatorname{curl} A_\delta(x) = 1$. Then, using (3.2), we verify that ψ_δ satisfies

$$-\nabla^2_{A_\delta}\psi_\delta = \frac{\mu(\sigma A)}{\sigma} a(x^\delta + \delta x)\psi_\delta \qquad \text{in} \ \ \Omega_\delta \tag{3.7}$$

and $|\psi_\delta(0)| = 1 = \|\psi_\delta\|_{L^\infty}$. We have that $\{\psi_\delta\}$ converges in $H^1_{\text{loc}}(\mathbb{R}^2)$ up to gauge transformations. Passing to a subsequence we may assume that $|\psi_\delta|$ converges in $L^2_{\text{loc}}(\mathbb{R}^2)$ as $\delta \to 0$. On the other hand, using the relation from Lemma 2.3 that $A_\delta(x) = \nabla\chi_\delta(x) + \omega_\delta(x) + B_\delta(x)$, where $\chi_\delta(x) = \frac{1}{\delta}A(x^\delta)x + \xi(x) + \delta\zeta(x)$ and setting $\psi_\delta(x) = \exp(i\chi_\delta)\phi_\delta(x)$ to (3.7), we obtain

$$-\nabla^2_\omega \phi_\delta = \frac{\mu(\sigma A)}{\sigma} a(x^\delta + \delta x)\phi_\delta + f_\delta(x), \tag{3.8}$$

where $f_\delta(x) = -[i \operatorname{div} B_\delta + 2\omega \cdot B_\delta + |B_\delta|^2]\phi_\delta - iB_\delta(x)\cdot\nabla\phi_\delta$. Moreover, since $|\nabla_{\omega+B_\delta}\phi_\delta| = |\nabla_{A_\delta}\psi_\delta|$ and $|\nabla\phi_\delta|^2 \leq |\nabla_{\omega+B_\delta}\phi_\delta + i(\omega+B_\delta)\phi_\delta|^2 \leq 2|\nabla_{\omega+B_\delta}\phi_\delta|^2 + 2|(\omega+B_\delta)\phi_\delta|^2$, $\{|\nabla\phi_\delta|\}$ is also uniformly bounded in L^2_{loc}. Passing to another subsequence we have $\phi_\delta \to \phi_0$ weakly in H^1_{loc} and strongly in L^2_{loc}. But since $\operatorname{div} B_\delta(x) = (\operatorname{div} A)(x^\delta + \delta x) - (\operatorname{div} A)(x^\delta) \to 0$ and $|B_\delta(x)| \leq C\delta|x|$, we have $f_\delta \to 0$ in L^2_{loc}. Therefore the limiting function ϕ_0 satisfies

$$-\nabla^2_\omega\phi_0 = d\, a(x^0)\phi_0 \qquad \text{in} \ \ \mathbb{R}^2 \tag{3.9}$$

with $|\phi_0(x)| \leq 1$. Using the fact that ϕ_0 is smooth, which results from (3.9) via Theorem 4.1 in [18] and denoting $\widehat{\phi}_\delta(x) = \phi_\delta(x) - \phi_0(x)$, we get from (3.8) and (3.9) that

$$-\nabla^2_\omega\widehat{\phi}_\delta = da(x^0)\widehat{\phi}_\delta + \hat{f}_\delta \tag{3.10}$$

with $\hat{f}_\delta = f_\delta + [\frac{\mu(\sigma A)}{\sigma}a(x^\delta + \delta x) - d\, a(x^0)]\phi_\delta \to 0$ in L^2_{loc} and $\widehat{\phi}_\delta \to 0$ in L^2_{loc}. Now, applying Lemma 4.2 in [18] to (3.10), we have $|\nabla_\omega\widehat{\phi}_\delta| \to 0$ in L^2_{loc}. But since $|\nabla\widehat{\phi}_\delta|^2 \leq 2|\nabla_\omega\widehat{\phi}_\delta|^2 + 2|\omega\widehat{\phi}_\delta|^2$, we also have $|\nabla\widehat{\phi}_\delta| \to 0$ in L^2_{loc} so that

$$\widehat{\phi}_\delta \to 0 \qquad \text{in} \ \ H^1_{\text{loc}}. \tag{3.11}$$

Denote $\omega = (\omega^1, \omega^2)$ and $\nabla_{\omega^j} = \partial_j - i\omega^j$. Then, applying Theorem 4.1 in [18] again to (3.10), we get

$$\nabla_{\omega^j}\nabla_{\omega^k}\widehat{\phi}_\delta \to 0 \qquad \text{in} \ \ L^2_{\text{loc}}. \tag{3.12}$$

Therefore, from (3.11) and (3.12), we conclude that $\partial_j\partial_k\widehat{\phi}_\delta \to 0$ in L^2_{loc}, which in turn leads to the result that $\widehat{\phi}_\delta \to 0$ in H^2_{loc}. Now, we apply Sobolev embedding Theorem and conclude that $\widehat{\phi}_\delta \to 0$ in C^α_{loc}, viz., $\phi_\delta \to \phi_0$ in C^α_{loc}. In particular, we get $\phi_0(0) = \lim_{\delta\to 0}\phi_\delta(0) = 1$. Therefore, ϕ_0 is a nonzero bounded and smooth solution of (3.9) in $\mathbb{R}^2$ and thus from Lemma 2.1 we have $da(x^0) \geq \alpha(1) = 1$. Hence, $d \geq \frac{1}{a(x^0)} \geq \min_{x\in\Omega}\frac{1}{a(x)}$, which is the conclusion of this case.

Case 2. $x^0 \in \partial\Omega$. We only have to prove $d \geq \beta_0 \min_{x\in\partial\Omega}\frac{1}{a(x)}$. Using Lemma 2.4 and the same argument as in the proof of case 1, we conclude that $da(x_0) \geq \beta(1) = \beta_0$ and hence $d \geq \beta_0\frac{1}{a(x_0)} \geq \beta_0 \min_{x\in\partial\Omega}\frac{1}{a(x)}$. This completes the proof of the case 2 and the lemma.

Proof of Theorem 3.1. From Lemmas 3.1 and 3.2, we have

$$\limsup_{\sigma\to\infty}\frac{\mu(\sigma A)}{|\sigma|} \leq \min\Big\{\min_{x\in\Omega}\frac{1}{a(x)}, \beta_0\min_{x\in\partial\Omega}\frac{1}{a(x)}\Big\}, \tag{3.13}$$

Similarly, from Lemmas 3.3, we have

$$\liminf_{\sigma\to\infty} \frac{\mu(\sigma A)}{|\sigma|} \geq \min\Big\{ \min_{x\in\Omega} \frac{1}{a(x)}, \beta_0 \min_{x\in\partial\Omega} \frac{1}{a(x)} \Big\}, \tag{3.14}$$

Hence, combining (3.17) and (3.18), the conclusion (3.3) in Theorem 3.1 follows, viz.,

$$\lim_{\sigma\to\infty} \frac{\mu(\sigma A)}{|\sigma|} = \min\Big\{ \min_{x\in\Omega} \frac{1}{a(x)}, \beta_0 \min_{x\in\partial\Omega} \frac{1}{a(x)} \Big\}.$$

§ 4. Estimates of the Upper Critical Field

Recall (1.5) that

$$G(\psi, A) = \frac{1}{2}\int_\Omega |\nabla_{\sigma\kappa A}\psi|^2 + (\sigma\kappa)^2|\operatorname{curl} A - 1|^2 + \frac{\kappa^2}{2}(a(x) - |\psi|^2)^2,$$

and set

$$c(\kappa,\sigma) = \inf_{(\psi,A)\in H^1(\Omega,\mathbb{C})\times H^1(\Omega,\mathbb{R}^2)} G(\psi, A),$$

where $H^1(\Omega,\mathbb{C})$ is the Sobolev space of all complex-valued functions and $H^1(\Omega,\mathbb{R}^2)$ is the Sobolev space of all vector-valued functions. It is known that there exists a unique smooth vector field F on $\overline{\Omega}$ such that

$$\operatorname{curl} F = 1, \quad \operatorname{div} F = 0 \quad \text{in } \Omega \qquad \text{and} \qquad F\cdot\nu = 0 \quad \text{on } \partial\Omega.$$

Note that $(0, F)$ is a trivial critical point of the functional $G(\psi, A)$ with

$$G(0, F) = \frac{\kappa^2}{4}\int_\Omega a^2(x).$$

If

$$c(\kappa,\sigma) < \frac{\kappa^2}{4}\int_\Omega a^2(x),$$

then $G(\psi, A)$ has a non-trivial minimizer and we have the following lemma.

Lemma 4.1. *The functional $G(\psi, A)$ defined by* (1.5) *has a non-trivial minimizer on $H^1(\Omega,\mathbb{C})\times H^1(\Omega,\mathbb{R}^2)$ provided*

$$\mu(\sigma\kappa F) < \kappa^2. \tag{4.1}$$

On the other hand, if $G(\psi, A)$ has a non-trivial minimizer (ψ, A), then $\mu(\sigma\kappa A) < \kappa^2$.

Proof. Let $\psi_0(x)$ satisfy $\|\psi_0\|_{L^\infty(\Omega)} = 1$ and

$$\mu(\sigma\kappa F) = \inf_{\psi\in H^1(\Omega,\mathbb{C})} \frac{\int_\Omega |\nabla_{\sigma\kappa F}\psi|^2}{\int_\Omega a(x)|\psi|^2} = \frac{\int_\Omega |\nabla_{\sigma\kappa F}\psi_0|^2}{\int_\Omega a(x)|\psi_0|^2}.$$

Then

$$\int_\Omega |\nabla_{\sigma\kappa F}\psi_0|^2 = \mu(\sigma\kappa F)\int_\Omega a(x)|\psi_0|^2.$$

If $\mu(\sigma\kappa F) < \kappa^2$, we may choose $\delta_0 > 0$ such that $\mu(\sigma\kappa F) < \kappa^2 - \delta_0$. Define

$$\psi_0 = \frac{\kappa}{\sqrt{2a_0\delta_0}}\psi_1,$$

then

$$G(\psi_1, F) \le -\frac{\delta_0 a_0}{2}\int_\Omega |\psi_1|^2 + \frac{\kappa^2}{4}\|\psi_1\|^2_{L^\infty}\int_\Omega |\psi_1|^2 + \frac{\kappa^2}{4}\int_\Omega a^2 < \frac{\kappa^2}{4}\int_\Omega a^2.$$

Hence

$$c(\kappa,\sigma) \le G(\psi_1, F) < \frac{\kappa^2}{4}\int_\Omega a^2,$$

which implies that $G(\psi, A)$ has a non-trivial minimizer on $H^1(\Omega,\mathbb{C}) \times H^1(\Omega,\mathbb{R}^2)$. On the other hand, if G has a non-trivial minimizer (ψ, A), then

$$G(\psi, A) \le \frac{\kappa^2}{4}\int_\Omega a^2,$$

which in turn implies that

$$\int_\Omega |\nabla_{\sigma\kappa A}\psi|^2 + \frac{\kappa^2}{2}(a - |\psi|^2)^2 \le \frac{\kappa^2}{2}\int_\Omega a^2.$$

Hence

$$\int_\Omega |\nabla_{\sigma\kappa A}\psi|^2 \quad < \quad \kappa^2\int_\Omega a|\psi|^2,$$

and thus from the definition of $\mu(\sigma\kappa A)$, we have

$$\mu(\sigma\kappa A) \quad \le \quad \frac{\int_\Omega |\nabla_{\sigma\kappa A}\psi|^2}{\int_\Omega a|\psi|^2} < \kappa^2.$$

The proof of this lemma is complete.

Next, define the quantity $\sigma_*(\kappa)$ as

$$\sigma_*(\kappa) = \min\{\sigma > 0 : \mu(\sigma\kappa F) = \kappa^2\}. \tag{4.2}$$

Then the following lemma gives an estimate of $\sigma_*(\kappa)$ for large κ.

Lemma 4.2. *Let $\sigma^*(\kappa)$ and $\sigma_*(\kappa)$ be defined as in* (1.7) *and* (4.2)*, respectively. Then we have*

$$\sigma^*(\kappa) \ge \sigma_*(\kappa) = \frac{\kappa}{\gamma_0} + o(\kappa) \qquad as \quad \kappa \to +\infty. \tag{4.3}$$

Proof. Since we know from (4.1) that G has a non-trivial minimizer when $0 \le \sigma < \sigma_*(\kappa)$, we have $\sigma^*(\kappa) \ge \sigma_*(\kappa)$. On the other hand, since $\mu(\sigma_*\kappa F) = \kappa^2$ and $\lim_{\sigma\to\infty}\frac{\mu(\sigma F)}{|\sigma|} = \gamma_0$, we get $\frac{\kappa}{\sigma_*} = \frac{\mu(\sigma_*\kappa F)}{\sigma_*\kappa} \to \gamma_0$ as $\kappa \to +\infty$ and thus the conclusion of this lemma follows.

Hereafter, we consider two arbitrary sequences $\{\kappa\}$ and $\{\sigma\}$ such that κ, $\sigma \to +\infty$ and

$$\sigma < \sigma^*(\kappa), \quad \lim_{\kappa\to+\infty}\frac{\kappa}{\sigma} = \gamma, \qquad \text{where} \;\; 0 \le \gamma \le \gamma_0. \tag{4.4}$$

For the simplicity of the notation, we set $\varepsilon = 1/\sqrt{\sigma\kappa}$. Then from (4.4) we see that $\kappa^2 \le (\gamma + o(1))/\varepsilon^2$. Now, rewrite the functional G as

$$G_\varepsilon(\psi, A) = \frac{1}{2}\int_\Omega |\nabla_{(1/\varepsilon^2)A}\psi|^2 + \frac{1}{\varepsilon^4}|\text{curl}A - 1|^2 + \frac{\kappa^2}{2}(a - |\psi|^2)^2.$$

Denote the minimizers by $(\psi^\varepsilon, A^\varepsilon)$, then, $(\psi^\varepsilon, A^\varepsilon)$ satisfies

$$\begin{aligned} &-\nabla^2_{(1/\varepsilon^2)A}\psi = \kappa^2(a - |\psi|^2)\psi && \text{in} \quad \Omega, && (4.5)\\ &\operatorname{curl}^2 A = \varepsilon^2(\psi, \nabla_{(1/\varepsilon^2)A}\psi) && \text{in} \quad \Omega, && (4.6)\\ &(\nabla_{(1/\varepsilon^2)A}\psi)\cdot\nu = 0, \quad \operatorname{curl}(A - F) = 0 && \text{on} \quad \partial\Omega. && (4.7)\end{aligned}$$

Here $(\psi^\varepsilon, A^\varepsilon)$ is called a minimal solution of (4.5)–(4.7). Due to the gauge invariance of the Ginzburg-Landau equation (4.5)–(4.7), we may always assume that

$$\operatorname{div} A^\varepsilon = 0 \quad \text{in} \quad \Omega, \qquad A^\varepsilon\cdot\nu = 0 \quad \text{on} \quad \partial\Omega. \tag{4.8}$$

We need the following estimates.

Lemma 4.3. *Let $(\psi^\varepsilon, A^\varepsilon)$ be a minimal solution of (4.5)–(4.8). Then*

$$\|\psi^\varepsilon\|_{L^\infty(\Omega)} \le 1, \quad \|\nabla_{(1/\varepsilon^2)A^\varepsilon}\psi^\varepsilon\|_{L^2(\Omega)} \le \frac{C}{\varepsilon}, \quad \|A^\varepsilon - F\|_{H^1(\Omega)} \le C\varepsilon. \tag{4.9}$$

Lemma 4.4. *Let $(\psi^\varepsilon, A^\varepsilon)$ be a minimal solution of (4.5)–(4.8). Then, we have*

$$\|\nabla_{(1/\varepsilon^2)A^\varepsilon}\psi^\varepsilon\|_{H^1(\Omega)} \le \frac{C}{\varepsilon^2}. \tag{4.10}$$

Moreover, for any $1 < p < \infty$ and $0 < \alpha < 1$,

$$\|A^\varepsilon\|_{W^{2,p}(\Omega)} \le C(p), \qquad \|A^\varepsilon\|_{C^{1,\alpha}(\Omega)} \le C(\alpha), \tag{4.11}$$

where the constants C, $C(p)$, and $C(\alpha)$ are independent of ε.

Proof of Theorem 1.1. By (4.9) and (4.11), we have, passing to a subsequence if necessary,

$$\begin{aligned} &A^\varepsilon \to A^0 \qquad \text{in} \quad C^{1,\alpha}(\Omega) \qquad \text{as} \quad \varepsilon \to 0, && (4.12)\\ &\operatorname{curl} A^0 = \operatorname{curl} F = 1. && (4.13)\end{aligned}$$

Denote

$$\mu\Big(\frac{1}{\varepsilon^2}A^\varepsilon\Big) = \inf_{\psi\in H^1(\Omega,\mathbb{C})} \frac{\int_\Omega |(i\nabla - \frac{1}{\varepsilon^2}A^\varepsilon)\psi|^2}{\int_\Omega a(x)|\psi|^2}.$$

Using (4.12) and by the method used to prove Theorem 3.1, we have

$$\begin{aligned}\lim_{\varepsilon\to 0}\varepsilon^2\mu((1/\varepsilon^2)A^\varepsilon) &= \lim_{\varepsilon\to 0}\varepsilon^2\mu((1/\varepsilon^2)A^0)\\ &= \min\Big\{\min_{x\in\Omega}\frac{1}{a(x)}, \beta_0\min_{x\in\partial\Omega}\frac{1}{a(x)}\Big\} = \gamma_0. \end{aligned} \tag{4.14}$$

On the other hand, from Lemma 4.1 we have $\mu((1/\varepsilon^2)A^\varepsilon) < \kappa^2$. Hence

$$\frac{\kappa}{\sigma} > \varepsilon^2\mu((1/\varepsilon^2)A^\varepsilon). \tag{4.15}$$

Combining (4.4), (4.14) and (4.15), we have

$$\gamma_0 \ge \gamma = \lim\frac{\kappa}{\sigma} \ge \lim_{\varepsilon\to 0}\varepsilon^2\mu((1/\varepsilon^2)A^\varepsilon) = \gamma_0.$$

Therefore, $\gamma_0 = \gamma$. Now, we show that $\sigma^* = \frac{\kappa}{\gamma_0} + o(\kappa)$ as $\kappa \to +\infty$. In fact, if this conclusion is not true, by Lemma 4.2, there exists a sequence $\kappa_n \to +\infty$ such that $\sigma^*(\kappa_n) > \sigma_*(\kappa_n)$ as $n \to \infty$. Thus, we may take a sequence $\{\sigma_n\}$ satisfying $\sigma^*(\kappa_n) > \sigma_n > \sigma_*(\kappa_n)$ as $\kappa_n \to \infty$. Hence

$$\frac{\kappa_n}{\sigma_n} < \gamma_0(1 + o(1)) \qquad \text{as} \quad n \to \infty. \tag{4.16}$$

Passing to a subsequence, we have $\lim\limits_{k\to\infty} \frac{\kappa_{n_k}}{\sigma_{n_k}} = \gamma$, and thus by (4.16), we get $\gamma < \gamma_0$. But, by the above argument, we have $\gamma = \gamma_0$. This yields a contradiction. Hence, $\sigma^* = \frac{\kappa}{\gamma_0} + o(\kappa)$ as $\kappa \to +\infty$, and the proof of Theorem 1.1 is complete.

§ 5 . Nucleations

In this section, we shall prove that the order parameter concentrates at the maximum points of $a(x)$. More detailed information can be obtained by considering the following two cases:

(I) $\gamma_0 = \min\limits_{x\in\Omega} \dfrac{1}{a(x)} < \beta_0 \min\limits_{x\in\partial\Omega} \dfrac{1}{a(x)}$,

(II) $\gamma_0 = \beta_0 \min\limits_{x\in\partial\Omega} \dfrac{1}{a(x)} < \min\limits_{x\in\Omega} \dfrac{1}{a(x)}$.

We will show that if (I) holds, then the interior nucleation occurs, and if (II) holds then the boundary nucleation occurs. This is a new phenomenon that has never been stated before. Let $x^\varepsilon \in \overline{\Omega}$ be a maximum point of $|\psi^\varepsilon|$ and denote $\lambda_\varepsilon = \|\psi^\varepsilon\|_{L^\infty(\Omega)} = |\psi^\varepsilon(x^\varepsilon)|$. Now, passing to a subsequence, we may assume $\lim\limits_{\varepsilon\to 0} x^\varepsilon = x^0$. Define

$$\varphi_\varepsilon(y) = \exp\Big(-\frac{i}{\varepsilon} A^\varepsilon(x^\varepsilon)\cdot y\Big)\psi^\varepsilon(x^\varepsilon + \varepsilon y).$$

Theorem 5.1. *Under the assumptions of* (4.4), *we have*

$$\operatorname{curl} A^\varepsilon \to 1 \qquad \text{in} \quad C^\alpha(\Omega), \tag{5.1}$$

$$\|\psi^\varepsilon\|_{L^\infty(\Omega)} \to 0. \tag{5.2}$$

On $\overline{\Omega} \setminus (\Omega_m \cup (\partial\Omega)_m)$, it holds that

$$\frac{\psi^\varepsilon(x)}{\|\psi^\varepsilon\|_{L^\infty(\Omega)}} \to 0 \qquad \text{as} \quad \varepsilon \to 0. \tag{5.3}$$

Moreover, if (I) *holds, then $x^0 \in \Omega_m$, $\frac{1}{a(x^0)} = \min\limits_{x\in\Omega} \frac{1}{a(x)}$, and* (5.3) *holds on $\overline{\Omega} \setminus \Omega_m$. After passing to a subsequence, we have*

$$\frac{\varphi_\varepsilon}{\|\psi^\varepsilon\|_{L^\infty(\Omega)}} \to \phi_0 \qquad \text{in} \quad C^{2,\alpha}_{\mathrm{loc}},$$

where $\phi_0 \exp(-i\eta)$ is an eigenfunction of (2.3) *with $h = 1$ and η satisfies $\triangle\eta = 0$ in $\mathbb{R}^2$.*

If (II) *holds, then $x^0 \in (\partial\Omega)_m$, $\frac{1}{a(x^0)} = \min\limits_{x\in\partial\Omega} \frac{1}{a(x)}$, and* (5.3) *holds on $\overline{\Omega} \setminus (\partial\Omega)_m$. After straightening a portion of the boundary around x^0, $\varphi_\varepsilon/\|\psi^\varepsilon\|_{L^\infty(\Omega)}$ converges to $\widetilde{\phi}_0$, where $\widetilde{\phi}_0 \exp(-i\eta)$ is an eigenfunction of* (2.5)–(2.6) *with $h = 1$ and $\triangle\eta = 0$ in $\mathbb{R}^2_+$.*

Several lemmas will be given first and then Theorem 5.1 will be proven. We consider two cases.

Case 1. $\lim_{\varepsilon\to 0} \operatorname{dist}(x^\varepsilon, \partial\Omega)/\varepsilon = +\infty$.

Case 2. $\operatorname{dist}(x^\varepsilon, \partial\Omega) \le C\varepsilon$.

In the case 1, we set $\Omega_\varepsilon = (\Omega - x^\varepsilon)/\varepsilon$ and define

$$A_\varepsilon(y) = \frac{1}{\varepsilon}[A^\varepsilon(x^\varepsilon + \varepsilon y) - A^\varepsilon(x^\varepsilon)], \quad F_\varepsilon(y) = \frac{1}{\varepsilon}[F(x^\varepsilon + \varepsilon y) - F(x^\varepsilon)].$$

Noting $\operatorname{curl} A_\varepsilon(y) = (\operatorname{curl} A^\varepsilon)(x^\varepsilon + \varepsilon y)$, we find that $(\varphi_\varepsilon, A_\varepsilon)$ satisfies the following equation

$$-\nabla^2_{A_\varepsilon}\varphi_\varepsilon = \frac{\kappa}{\sigma}(a(x^\varepsilon + \varepsilon y) - |\varphi_\varepsilon|^2)\varphi_\varepsilon \qquad \text{in} \quad \Omega_\varepsilon, \tag{5.4}$$

$$\operatorname{curl}^2(A_\varepsilon - F_\varepsilon) = \varepsilon^2(\varphi_\varepsilon, \nabla_{A_\varepsilon}\varphi_\varepsilon) \qquad \text{in} \quad \Omega_\varepsilon, \tag{5.5}$$

$$(\nabla_{A_\varepsilon}\varphi_\varepsilon)\cdot\nu = 0, \quad \operatorname{curl}(A_\varepsilon - F_\varepsilon) = 0 \qquad \text{on} \quad \partial\Omega_\varepsilon. \tag{5.6}$$

First, it is easy to check the following lemma.

Lemma 5.1. *If the case* 1 *holds, then*

$$\|\varphi_\varepsilon\|_{C^{2,\alpha}(B_R)} \le C(R), \quad \|A_\varepsilon - F_\varepsilon\|_{C^{2,\alpha}(B_R)} \le C(R). \tag{5.7}$$

Lemma 5.2. *If the Case* 1 *holds, then* (5.2) *holds.*

Proof. In fact, as $\varepsilon \to 0$ we have $x^\varepsilon \to x^0$ and since

$$A_\varepsilon(y) = \frac{1}{\varepsilon}[A^\varepsilon(x^\varepsilon + \varepsilon y) - A^\varepsilon(x^\varepsilon)] = \nabla A^\varepsilon(x^\varepsilon)y + O(\varepsilon^\alpha |y|^{1+\alpha}),$$

we have $A_\varepsilon(y) \to \nabla A^0(x^0)y$ in $C^\alpha_{\text{loc}}(\mathbb{R}^2)$ as $\varepsilon \to 0$. Therefore, $A_\varepsilon - F_\varepsilon \to \nabla A^0(x^0)y - \nabla F(x^0)y$ in $C^\alpha_{\text{loc}}(\mathbb{R}^2)$. By (4.12), (4.13) and (5.7), we have that the convergence is actually in $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2)$ and

$$A_\varepsilon \to \nabla A^0(x^0)y = \omega(y) + \nabla\eta \qquad \text{in} \quad C^{1,\alpha}_{\text{loc}}(\mathbb{R}^2) \qquad \text{as} \quad \varepsilon \to 0,$$

where $\omega(y) = \frac{1}{2}(-y_2, y_1)$. Here

$$\eta(y) = \frac{1}{2}[\partial_1 A^0_1(x^0)y_1^2 + (\partial_2 A^0_1(x^0) + \partial_1 A^0_2(x^0))y_1 y_2 + \partial_2 A^0_2(x^0)y_2^2]$$

and $\triangle\eta = 0$ in $\mathbb{R}^2$ since $\operatorname{div} A^0(x^0) = 0$. By (5.7), passing to a subsequence, we have $\varphi_\varepsilon \to \varphi_0$ in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)$ as $\varepsilon \to 0$. Sending $\varepsilon \to 0$ in (5.4) and using (4.4) and the above relation, we have $-\nabla^2_\omega\varphi_0 = \gamma(a(x^0) - |\varphi_0|^2)\varphi_0$ in $\mathbb{R}^2$. Let $\psi_0 = \varphi_0/\sqrt{a(x^0)}$, then

$$-\nabla^2_\omega\psi_0 = \gamma a(x^0)(1 - |\psi_0|^2)\psi_0 \qquad \text{in} \quad \mathbb{R}^2.$$

Noting that

$$\begin{aligned} a(x^0)\gamma \le a(x^0)\gamma_0 &= a(x^0)\min\left\{\min_{x\in\Omega}\frac{1}{a(x)}, \beta_0 \min_{x\in\partial\Omega}\frac{1}{a(x)}\right\} \\ &= a(x^0)\min_{x\in\Omega}\frac{1}{a(x)} = a(x^0)\frac{1}{\max_{x\in\Omega} a(x)} \le 1, \end{aligned}$$

and Lemma 2.2, we see that $\psi_0 = 0$. Thus, $\varphi_0 = 0$. So, $\varphi_\varepsilon \to 0$ in $C^{2,\alpha}_{\text{loc}}(\mathbb{R}^2)$ as $\varepsilon \to 0$. Hence, we have

$$\lambda_\varepsilon = \|\psi^\varepsilon\|_{L^\infty(\Omega)} = |\psi^\varepsilon(x^\varepsilon)| = |\varphi_\varepsilon(0)| \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

This completes the proof of (5.2).

Lemma 5.3. *If the case* 1 *holds, then* $x^0 \in \Omega_m$*,* $\gamma_0 = \min\limits_{x\in\Omega} \frac{1}{a(x)}$*. Moreover,* (5.3) *holds on* $\overline{\Omega} \setminus \Omega_m$ *and* $\frac{\varphi_\varepsilon}{\lambda_\varepsilon} \to \phi_0$ *in* $C^{2,\alpha}_{\rm loc}$ *holds for a subsequence, where* $\phi_0(y)\exp(-i\eta)$ *satisfies*

$$-\nabla^2_\omega \phi = \phi \qquad \text{in} \ \ \mathbb{R}^2. \tag{5.8}$$

Proof. Let $\phi_\varepsilon = \varphi_\varepsilon/\lambda_\varepsilon$. Then ϕ_ε satisfies

$$-\nabla^2_{A_\varepsilon}\phi_\varepsilon = \frac{\kappa}{\sigma}(a(x^\varepsilon+\varepsilon y) - {\lambda_\varepsilon}^2|\phi_\varepsilon|^2)\phi_\varepsilon \qquad \text{in} \quad \Omega_\varepsilon, \tag{5.9}$$

$$(\nabla_{A_\varepsilon}\phi_\varepsilon)\cdot\nu = 0 \qquad \text{on} \quad \partial\Omega_\varepsilon. \tag{5.10}$$

By standard Schauder estimates, we have $\|\phi_\varepsilon\|_{C^{2,\alpha}(B_R)} \le C(R)$. Passing to a subsequence, we have that $\phi_\varepsilon \to \phi_0$ in $C^{2,\alpha}_{\rm loc}(\mathbb{R}^2)$. Let $\widehat{\phi}_0 = \phi_0\exp(-i\eta)$. We let $\varepsilon \to 0$ in (5.9), using (4.4), then

$$-\nabla^2_\omega \widehat{\phi}_0 = \gamma a(x^0)\widehat{\phi}_0 \qquad \text{in} \ \ \mathbb{R}^2. \tag{5.11}$$

Noting $|\widehat{\phi}_0(0)| = \lim\limits_{\varepsilon\to0}|\phi_\varepsilon(0)| = 1$, we see that $\widehat{\phi}_0 \neq 0$. Recall that the first eigenvalue of (5.11) is 1. Hence, $\gamma a(x^0) \ge 1$. On the other hand,

$$\gamma a(x^0) \le \gamma_0 a(x^0) = a(x^0)\min\Big\{\min_{x\in\Omega}\frac{1}{a(x)}, \beta_0\min_{x\in\partial\Omega}\frac{1}{a(x)}\Big\} = a(x^0)\min_{x\in\Omega}\frac{1}{a(x)} \le 1.$$

So $\gamma a(x^0) = \gamma_0 a(x^0) = 1$. Therefore, $\gamma = \gamma_0$, $\frac{1}{a(x^0)} = \min\limits_{x\in\Omega}\frac{1}{a(x)} = \gamma_0$, $x^0 \in \Omega_m$ and $\phi_\varepsilon \to \phi_0$ in $C^{2,\alpha}_{\rm loc}$, where $\widehat{\phi}_0 = \phi_0\exp(-i\eta)$ is the bounded eigenfunction of (2.3) corresponding to the first eigenvalue 1 with $h = 1$. Hence, (5.8) holds. Fix $\bar{x} \in \overline{\Omega}\setminus\Omega_m$. Replacing x^ε by $\bar{x}$ and repeating the above argument we have that (5.3) holds on $\overline{\Omega}\setminus\Omega_m$. This completes the proof of this lemma.

Now we consider the Case 2. In this case, $x^0 \in \partial\Omega$ and $\mathrm{dist}\,(x^\varepsilon, \partial\Omega) \le C\varepsilon$. Choose $\bar{x}^\varepsilon \in \partial\Omega$ such that $\mathrm{dist}(x^\varepsilon, \bar{x}^\varepsilon) = \mathrm{dist}(x^\varepsilon, \partial\Omega)$. Let $\mathcal{F}_\varepsilon$ be the diffeomorphism which straightens a portion of the boundary around $\bar{x}^\varepsilon$ as defined in Section 2 with $\mathcal{F}_\varepsilon(0) = \bar{x}^\varepsilon$. Denote $y^\varepsilon = {\mathcal{F}_\varepsilon}^{-1}(x^\varepsilon)$ and $z^\varepsilon = y^\varepsilon/\varepsilon$. Note that z^ε is bounded. Passing to a subsequence, we may assume $z^\varepsilon \to z^0$. Note that $\mathcal{F}_\varepsilon$ depends on ε, but their domains contain a ball B_{R_0} with R_0 independent of ε, and both $\mathcal{F}_\varepsilon$ and $\det D\mathcal{F}_\varepsilon$ are uniformly smooth on this ball. For the simplicity of the notation, we denote $\mathcal{F}_\varepsilon$ by $\mathcal{F}$, and $\det D\mathcal{F}_\varepsilon$ by g. Let $\overline{A}^\varepsilon = [gA^\varepsilon\cdot e_1]e_1 + [A^\varepsilon\cdot e_2]e_2$ be the vector field associated with A^ε. Note that $\overline{A}^\varepsilon(0) = A^\varepsilon(\bar{x}^\varepsilon)$ and $\overline{A}^\varepsilon(0)\cdot e_2 = 0$. Define

$$\widetilde{\varphi}_\varepsilon(y) = \exp(-i\chi_\varepsilon)\psi^\varepsilon(\varepsilon y), \tag{5.12}$$

where $\chi_\varepsilon = \frac{1}{\varepsilon}y_1 A^\varepsilon(\bar{x}^\varepsilon)\cdot e_1$. The same argument as in the proof of Lemma 5.3 gives

Lemma 5.4. *If the case* 2 *holds, then* (5.2) *holds,* $x^0 \in (\partial\Omega)_m$*,* $\gamma_0 = \beta_0\min\limits_{\partial\Omega}\frac{1}{a(x)}$*, and* (5.3) *holds on* $\overline{\Omega}\setminus(\partial\Omega)_m$*, and*

$$\frac{\widetilde{\varphi}_\varepsilon}{\|\psi^\varepsilon\|_{L^\infty(\Omega)}} \to \widetilde{\phi}_0 \qquad \text{in} \ \ C^{2,\alpha}_{\rm loc}(\mathbb{R}^2_+), \tag{5.13}$$

where $\widetilde{\phi}_0\exp(-i\eta)$ *satisfies*

$$-\nabla^2_\omega\phi = \beta_0\phi \quad \text{in} \ \ \mathbb{R}^2_+, \qquad (\nabla_\omega\phi)\cdot\nu = 0 \quad \text{on} \ \ \partial\mathbb{R}^2_+. \tag{5.14}$$

Proof of Theorem 5.1. (5.1) follows from (4.12) and (4.13). (5.2) and (5.3) follow from Lemma 5.2 and Lemma 5.4. Other conclusions of this theorem come also from Lemma 5.3 and Lemma 5.4 and the proof of this theorem is complete.

Proof of Theorem 1.2. This is the consequence of Theorem 5.1.

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