Chin. Ann. Math.
25B:2(2004),199–206.

THE LAW OF THE ITERATED LOGARITHM OF THE KAPLAN-MEIER INTEGRAL AND ITS APPLICATION***

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Abstract

For right censored data, the law of the iterated logarithm of the Kaplan-Meier integral is established. As an application, the authors prove the law of the iterated logarithm for weighted least square estimates of randomly censored linear regression model.

Keywords Law of the iterated logarithm, Kaplan-Meier integral.
2000 MR Subject Classification 60B12, 60F15

§ 1. Introduction

Let Y be a random variable and $\mathbf{Z}' = (Z_1, Z_2, \cdots, Z_r)$ a random vector of covariables. Assume that $(Y_i, \mathbf{Z}_i')$, $i = 1, 2, \cdots, n$, are independent sample of the $(r+1)$-variate vector $(Y, \mathbf{Z}')$ and $F_n(y, \mathbf{z})$ is the empirical distribution of $F(y, \mathbf{z}) = P(Y \le y, \mathbf{Z} \le \mathbf{z})$. For any $F(dx, d\mathbf{z})$ integrable function $g(x, \mathbf{z})$, the law of the iterated logarithm (LIL) states that with probability one, the random sequence

$$LLn\Big(\int g(y,\mathbf{z})F_n(dy,d\mathbf{z}) - \int g(y,\mathbf{z})F(dy,d\mathbf{z})\Big), \qquad n = 3, 4, \cdots, \tag{1.1}$$

is relatively compact and the set of its limit points coincides with

$$[-\sqrt{\mathrm{Var}(g(Y,\mathbf{Z}))}, \sqrt{\mathrm{Var}(g(Y,\mathbf{Z}))}\ ],$$

where $LLn = \sqrt{n/2 \log\log n}$. The objective of the present paper is to prove the LIL result under random censorship. Under random censorship, rather than $(Y_i, \mathbf{Z}_i), i = 1, \cdots, n$, the variable of interest, we observe

$$U_i = \min(Y_i, C_i), \quad \mathbf{Z}_i, \quad \delta_i = I[Y_i \le C_i], \qquad i = 1, \cdots., n, \tag{1.2}$$

where $I[A]$ is the indicator function of A and $\{C_i\}$ is another independent and identically distributed (iid.) sequence independent of the $\{Y_i\}$ sequence. Let

$$U_{(1)} \le U_{(2)} \le \cdots \le U_{(n)}$$

Manuscript received March 25, 2003.
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***Project supported by the National Natural Science Foundation of China (No.10231030) and the Research Fund for the Doctoral Program of Higher Education.

be the ordered statistics of $(U_1, U_2, \cdots, U_n)$. Define $\delta_{(k)} = \delta_i$ when $U_{(k)} = U_i$. The nonparametric maximum likelihood estimator for $G(x) = P(C_1 \le x)$ is the well-known Kaplan-Meier estimate given by

$$G_n(x) = 1 - \prod_{i=1}^{n} \Big(1 - \frac{1-\delta_{(i)}}{n-i+1}\Big)^{I[U_{(i)} \le x]}, \tag{1.3}$$

where an empty product is set equal to one. Let $F_n^*(y, \mathbf{z})$ be the empirical distribution defined by

$$F_n^*(y, \mathbf{z}) = \frac{1}{n} \sum_{j=1}^{n} I[U_j \le y, \mathbf{Z}_j \le \mathbf{z}].$$

Here for any two vectors $\mathbf{a} = (a_1, \cdots, a_r)'$, $\mathbf{b} = (b_1, \cdots, b_r)'$, the inequality $\mathbf{a} \le \mathbf{b}$ means that $a_i \le b_i, i = 1, \cdots, r$. The joint distribution $F(y, \mathbf{z})$ is estimated by

$$F_n(y, \mathbf{z}) = \int_{u \le y} \int_{\mathbf{w} \le \mathbf{z}} \frac{1}{1 - G_n(u-)} F_n^*(du, d\mathbf{w}). \tag{1.4}$$

Random censorship typically comes up in the analysis of lifetime data. The consistency and the central limit theorem of (1.4) was established by [1] and [2], respectively. But the LIL result has not been established. In Theorems 2.1, 2.2 and 2.3 of the present paper, we prove the LIL for (1.4). As an application, in Theorem 3.1, we prove the LIL for weighted least square estimates (WLSE) of randomly censored linear regression model.

§ 2. Main Results

Let (a_F, b_F) be the range of F defined by

$$a_F = \inf\{x : F(x) > 0\} \quad \text{and} \quad b_F = \sup\{x : F(x) < 1\}.$$

Let a_G and b_G be similarly defined for the distribution G. Under random censorship, only

$$F(x \wedge b_G, \mathbf{z}), \qquad G(x \wedge b_F)$$

can be estimated, where $x \wedge v = \min(x, v)$. We shall assume throughout the paper that $b_F \le b_G$. Under this condition $F(x \wedge b_G, \mathbf{z}) = F(x, \mathbf{z})$.

Condition A. *Let F and G be continuous. Assume that for some $a \in [0, 1)$, $(1 - G(x))^a = O(1 - F(x))$ as $x \to b_F$ in the case of $b_F = b_G$, and in the case of $b_F < b_G$, assume that*

$$\int_{-\infty}^{b_F} dG(x)/(1 - F(x)) < \infty.$$

Condition B. *Assume that $(Y_i, \mathbf{Z}_i, C_i)$, $i = 1, 2, \cdots$, are iid. copies of the random vector $(Y, \mathbf{Z}, C)$ and $P(Y \le C | Y, \mathbf{Z}) = P(Y \le C | Y)$.*

Condition A was used by [3] to prove the LIL for Kaplan-Meier estimate G_n. Condition B was used by [1] and makes the censored model flexible enough to allow for a dependence between $\mathbf{Z}$ and C.

For any measurable function $g(x, \mathbf{z})$, define

$$M_n(g) = \int g(x, \mathbf{z}) F_n(dx, d\mathbf{z}) - \int g(x, \mathbf{z}) F(dx, d\mathbf{z}). \tag{2.1}$$

Here and after we use $\int$ for $\int_{-\infty}^{\infty}$.

Theorem 2.1. *Assume Conditions* A *and* B *hold. Let* $g(x, \mathbf{z})$ *be a measurable function satisfying*

$$\int \frac{g^k(x, \mathbf{z})}{1 - G(x)} F(dx, d\mathbf{z}) < \infty, \qquad k = 1, 2. \tag{2.2}$$

Then as $n \to \infty$, *with probability one*

$$\limsup_{n\to\infty} LLnM_n(g) = \sigma, \qquad \liminf_{n\to\infty} LLnM_n(g) = -\sigma, \tag{2.3}$$

where

$$\sigma^2 = E\frac{g^2(Y, \mathbf{Z})}{\overline{G}(Y)} - (Eg(Y, \mathbf{Z}))^2 - E\frac{\psi^2(C)}{\overline{F}(C)\overline{G}^2(C)}, \tag{2.4}$$

$$\overline{F} = 1 - F, \quad \overline{G} = 1 - G, \quad \psi(s) = E\{g(Y, \mathbf{Z})I[Y \geq s]\}. \tag{2.5}$$

Remark 2.1. If $P(C \geq Y) = 1$, then (2.2) reduces to $Eg^2(Y, \mathbf{Z}) < \infty$ and σ^2 reduces to $\mathrm{Var}(g(Y, \mathbf{Z}))$.

Theorem 2.2. *Under the conditions of Theorem* 2.1, *with probability one, the sequence* $\{LLnM_n(g)\}$ *is relatively compact and the set of its limit points coincides with* $[-\sigma, \sigma]$.

For $i = 1, 2, \cdots, r$, suppose that $g_i = g_i(x, \mathbf{x})$ satisfies (2.2). Define $\mathbf{g} = (g_1, g_2, \cdots, g_r)'$ and

$$\mathbf{M}_n = (M_n(g_1), M_n(g_1), \cdots, M_n(g_r))'. \tag{2.6}$$

Let $\Sigma = (\sigma_{ij})$ be an $r \times r$ nonnegative definite matrix with its (i, j) elements defined by

$$\begin{aligned} \sigma_{i,j} &= E\frac{g_i(Y, \mathbf{Z})g_j(Y, \mathbf{Z})}{\overline{G}(Y)} - Eg_i(Y, \mathbf{Z})Eg_j(Y, \mathbf{Z}) - E\frac{\psi_i(C)\psi_j(C)}{\overline{F}(C)\overline{G}^2(C)}, \\ \psi_j(s) &= E\{g_j(Y, \mathbf{Z})I[Y \geq s]\}. \end{aligned} \tag{2.7}$$

Let R^r be the r-dimentional Euclidian space equipped with Euclidian norm $\| \cdot \|$. We have the following Hartman-Wintner- Strassen's law of the iterated logarithm.

Theorem 2.3. *For each* $g_i(x, \mathbf{z})$, *suppose the conditions of Theorem* 2.1 *hold. If the matrix* $\Sigma = AA'$ *is positively definite, then with probability one the sequence* $\{LLn\mathbf{M}_n\}$ *is relatively compact and the set of its limit points coincides with*

$$K = \{A\mathbf{x} \ ; \ \| \mathbf{x} \| \leq 1, \ \mathbf{x} \in R^r\}. \tag{2.8}$$

To prove the theorems we need some preliminaries. Let

$$F^*(x, \mathbf{z}) = P(U_1 \leq x, \mathbf{Z}_1 \leq \mathbf{z}, \delta_1 = 1) \tag{2.9}$$

be the joint distribution of the censored random vector $(U_i, \mathbf{Z}_i, \delta_i = 1)$. Then it can be derived that (see [2])

$$F^*(x, \mathbf{z}) = \int_{y \leq x} \int_{\mathbf{w} \leq \mathbf{z}} \overline{G}(y-) F(dy, d\mathbf{w}). \tag{2.10}$$

It follows immediately from (2.10) that the joint distribution of $(Y, \mathbf{Z})$ is given by

$$F(x, \mathbf{z}) = \int_{y \leq x} \int_{\mathbf{w} \leq \mathbf{z}} \frac{1}{\overline{G}(y-)} F^*(dy, d\mathbf{w}). \tag{2.11}$$

For any $b_m < b_F$, let $g_m(x, \mathbf{z})$ be the restriction of $g(x, \mathbf{z})$ on $(-\infty, b_m]$ defined by $g_m(x, \mathbf{z}) = g(x, \mathbf{z}) I[x \le b_m]$. Define $H(x) = P(U_1 \le x)$, $\overline{H} = 1 - H$,

$$\mu_m = \int g_m(x, \mathbf{z}) F(dx, d\mathbf{z}), \quad \psi_m(s) = \int_{x \ge s} g_m(x, \mathbf{z}) F(dx, d\mathbf{z}). \tag{2.12}$$

Introduce

$$\begin{aligned} \beta_n &= (\log n/n)^{3/4}, \qquad H^0(s) = P(U_1 \le s, \delta_1 = 0), \\ T_k(u) &= \frac{(1-\delta_k) I[U_k \le u]}{\overline{H}(U_k)} - \int_{-\infty}^{u} \frac{I[s \le U_k]}{\overline{H}^2(s)} dH^0(s), \\ \xi_k(b_m) &= \frac{g_m(U_k, \mathbf{Z}_k)\delta_k}{\overline{G}(U_k)}, \quad S_k(b_m) = E[\xi_k(b_m) T_j(U_k) \big| Y_j, C_j, \mathbf{Z}_j]. \end{aligned}$$

Lemma 2.1. *Suppose the conditions of Theorem* 2.1 *hold. Then with probability one*

$$\limsup_{n \to \infty} LLn M_n(g_m) = \sigma_m, \quad \liminf_{n \to \infty} LLn M_n(g_m) = -\sigma_m, \tag{2.13}$$

where

$$\sigma_m^2 = E \frac{g_m^2(Y, \mathbf{Z})}{\overline{G}(Y)} - (E g_m(Y, \mathbf{Z}))^2 - E \frac{\psi_m^2(C)}{\overline{F}(C) \overline{G}^2(C)} \to \sigma^2 \qquad as \ \ b_m \to b_F. \tag{2.14}$$

Proof. By Lemma 3.2, (3.11) and the proof of Theorem 3.4 in [2], we have

$$M_n(g_m) = \frac{1}{n} \sum_{k=1}^{n} [\xi_k(b_m) - E\xi_k(b_m) + S_k(b_m)] + O(\beta_n), \qquad \text{a.s.}, \tag{2.15}$$

where the random variables $\xi_k(b_m) - E\xi_k(b_m) + S_k(b_m)$, $i = 1, 2, \cdots$, are iid. with mean zero and variance σ_m^2 given in (2.14). By dominated convergence theorem we get $\sigma_m^2 \to \sigma^2$ as $b_m \to b_F$. The result (2.13) follows from LIL of iid. random variables.

Lemma 2.2. *Under Condition* A*,*

$$\limsup_{n \to \infty} LLn \sup_{x \le U_{(n)}} |G(x) - G_n(x)| = M, \qquad a.s., \tag{2.16}$$

where G_n is the Kaplan-Meier estimator defined by (1.3) *and M is defined by*

$$M = \sup_{x \le \ b_F} \left\{ \overline{G}(x) \int_{-\infty}^{x} \frac{dG}{\overline{G}^2 \overline{F}} \right\} < \infty. \tag{2.17}$$

Proof. See Theorem 1 and Corollary 3 of [3].

Lemma 2.3. (cf. [1]) *Under the conditions of Theorem* 2.1*, for any nonnegative and measurable $g(x, \mathbf{z})$, as $n \to \infty$,*

$$\lim_{n \to \infty} \int g(x, \mathbf{z}) F_n(dx, d\mathbf{z}) \to \int g(x, \mathbf{z}) F(dx, d\mathbf{z}), \qquad a.s.$$

Proof of Theorem 2.1. Let $b_m < b_F$ and $g'_m(x,\mathbf{z}) = g_m(x,\mathbf{z})I[x > b_m]$. Using (1.4) and (2.11) we get

$$\begin{aligned}
|M_n(g) - M_n(g_m)| &= \Big| \int g'_m(x,\mathbf{z})F_n(dx,d\mathbf{z}) - \int g'_m(x,\mathbf{z})F(dx,d\mathbf{z})\Big| \\
&\le \Big| \int g'_m(x,\mathbf{z})\frac{F_n^*(dx,d\mathbf{z})}{\overline{G}_n(x)} - \int g'_m(x,\mathbf{z})\frac{F_n^*(dx,d\mathbf{z})}{\overline{G}(x)}\Big| \\
&\quad + \Big| \int \frac{g'_m(x,\mathbf{z})}{\overline{G}(x)}(F_n^*(dx,d\mathbf{z}) - F^*(dx,d\mathbf{z}))\Big| \\
&\le \sup_{x \le U_n} |G(x) - G_n(x)| \int \frac{|g'_m(x,\mathbf{z})|}{\overline{G}(x)} F_n(dx,d\mathbf{z}) \\
&\quad + \Big| \int \frac{g'_m(x,\mathbf{z})}{\overline{G}(x)}\big(F_n^*(dx,d\mathbf{z}) - F^*(dx,d\mathbf{z})\big)\Big| \\
&\equiv \eta_1(n,m) + \eta_2(n,m). \qquad (2.18)
\end{aligned}$$

Now by Lemmas 2.2 and 2.3, we get almost surely,

$$A_m \equiv \limsup_{n\to\infty} LLn\eta_1(n,m) = M \int \frac{|g'_m(x,\mathbf{z})|}{\overline{G}(x)} F(dx,d\mathbf{z}). \qquad (2.19)$$

Note that $\eta_2(n,m)$ is the sample mean of the iid random variables

$$\frac{g'_m(U_i,\mathbf{W_i})}{\overline{G}(U_i)} - E\Big(\frac{g'_m(U_i,\mathbf{W_i})}{\overline{G}(U_i)}\Big), \qquad i = 1,\cdots,n,$$

each with variance less than

$$B_m^2 \equiv E\Big(\frac{g'_m(U,W)}{\overline{G}(U)}\Big)^2 = \int \Big(\frac{g'_m(x,\mathbf{z})}{\overline{G}(x)}\Big)^2 F^*(dx,d\mathbf{z}) = \int \frac{(g'_m(x,\mathbf{z}))^2}{\overline{G}(x)} F(dx,d\mathbf{z}), \qquad \text{a.s.}$$

By LIL for sample mean of iid. random variables, we get

$$\limsup_{n\to\infty} LLn|\eta_2(n,m)| = B_m, \qquad \text{a.s.} \qquad (2.20)$$

Combining (2.18), (2.19) and (2.20), we get

$$\limsup_{n\to\infty} LLn|M_n(g) - M_n(g_m)| = \limsup_{n\to\infty} LLn|M_n(g'_m)| \le A_m + B_m, \qquad \text{a.s.} \qquad (2.21)$$

It is seen from the definition of $g'(x,\mathbf{z})$ that $\lim_{m\to\infty}(A_m + B_m) = 0$.

Now, Theorem 2.1 follows from Lemma 2.1, $b_m \to b_F$ and

$$M_n(g_m) - |M_n(g) - M_n(g_m)| \le M_n(g) \le M_n(g_m) + |M_n(g) - M_n(g_m)|. \qquad (2.22)$$

Proof of Theorem 2.2. The result will follow from Theorem 2.1 and

$$\limsup_{n\to\infty} |LLnM_n(g) - LL_{n-1}M_{n-1}(g)| = 0, \qquad \text{a.s.} \qquad (2.23)$$

But using (2.15), (2.18) and the fact that for any random variable X and positive ε,

$$E|X^2|/\varepsilon^2 = \int_0^\infty P(|X| \ge \sqrt{t}\varepsilon)dt \ge \sum_{k=0}^\infty P(|X| \ge \sqrt{k}\varepsilon),$$

we get

$$
\begin{aligned}
&\limsup_{n\to\infty} |LLnM_n(g) - LL_{n-1}M_{n-1}(g)| \\
&\le \limsup_{n\to\infty} |LLn[M_n(g) - M_{n-1}(g)]| \\
&\le \limsup_{n\to\infty} |LLn[(M_n(g_m) - M_{n-1}(g_m)] + LLn[M_n(g'_m) - M_{n-1}(g'_m)]| \\
&\le \limsup_{n\to\infty} \left|\frac{LLn}{n}[\xi_n(b_m) - E\xi_n(b_m) + S_n(b_m)]\right| + 2A_m + 2B_m \\
&\to 0 \qquad \text{as} \quad m \to \infty.
\end{aligned}
$$

To prove Theorem 2.3, we need another lemma.

Lemma 2.4. *Let $\{\xi_n\}$ be an iid. sequence with $E\xi_1 = 0$, $E\xi_1^2 = 1$, and independent of $\{M_n\}$. Under the conditions of Theorem 2.1, for any constants a and b, with probability one, the sequence $\{LLn(aM_n(g) + b\xi_n)\}$ is relatively compact and the set of its limit points coincides with*

$$[-\sqrt{a^2\sigma^2 + b^2}, \sqrt{a^2\sigma^2 + b^2}\,].$$

Proof. The result follows from (2.22), (2.15) and LIL of iid. random sequence.

Proof of Theorem 2.3. For convenience, suppose all the random variables are defined on a complete probability space $(\Omega, \mathcal{F}, P)$. By Theorem 2.2, there is a full event Ω_0 with $P(\Omega_0) = 1$, such that on Ω_0 each $\{M_n(g_j)\}$ is relatively compact and the set of its limit points coincides with $[-\sqrt{\sigma_{jj}}, \sqrt{\sigma_{jj}}]$. In the following, all the proofs will be based on Ω_0. It is easy to check that

$$K = \{A\mathbf{x}\ ;\ \ \|\ \mathbf{x}\ \| \le 1,\ \mathbf{x} \in R^r\} = \{\mathbf{x} \in R^r\ ;\ \mathbf{a}'\mathbf{x} \le \sqrt{\mathbf{a}'\Sigma\mathbf{a}}\ \text{ for all } \mathbf{a} \in R^r\}.$$

Let $C(LLn\mathbf{M}_n)$ be the set of limit points of the random sequence $\{LLn\mathbf{M}_n\}$. Firstly, we prove $C(LLn\mathbf{M}_n(g)) \subset K$. From Theorem 2.1, we see for any $\mathbf{a} \in R^r$ and $\omega \in \Omega_0$,

$$\limsup_{n\to\infty} LLn\mathbf{a}'\mathbf{M}_n = \limsup_{n\to\infty} LLnM_n(\mathbf{a}'\mathbf{g}) = \sqrt{\mathbf{a}'\Sigma\mathbf{a}}. \tag{2.24}$$

For any $\mathbf{x} \in K^c$, the complement of K, there is a vector $\mathbf{a} \in R^r$ such that $\mathbf{a}'\mathbf{x} > \sqrt{\mathbf{a}'\Sigma\mathbf{a}} + \varepsilon$, for some positive ε. If $\mathbf{x}$ is a limit point of $\{LLn\mathbf{M}_n\}$, then there is a subsequence of $\{LLn\mathbf{M}_n\}$ which converges to $\mathbf{x}$. Along this subsequence, $LLn\mathbf{a}'\mathbf{M}_n(\mathbf{g}) \to \mathbf{a}'\mathbf{x}$. But that is in contradiction with (2.24).

Now we prove $K \subset C(LLn\mathbf{M}_n)$. For any $\mathbf{y} = A\mathbf{x} \in K$, we have $\|\ \mathbf{x}\ \| \le 1$. Suppose $\|\ \mathbf{x}\ \| = 1$, firstly. For $\omega \in \Omega_0$ and $\varepsilon > 0$, from $C(LLn\mathbf{M}_n) \subset K$, we see that for n large enough,

$$\|\ LLnA^{-1}\mathbf{M}_n\ \| \le \sup_{\mathbf{y}\in K} \|\ A^{-1}\mathbf{y}\ \| + \varepsilon = \sup_{\|x\|\le 1} \|\ A^{-1}A\mathbf{x}\ \| + \varepsilon = 1 + \varepsilon.$$

And from Theorem 2.1, we get

$$\limsup_{n\to\infty} LLn\mathbf{y}'\Sigma^{-1}\mathbf{M}_n = \limsup_{n\to\infty} LLnM_n(\mathbf{y}'\Sigma^{-1}\mathbf{g}) = [\mathbf{y}'\Sigma^{-1}\Sigma(\Sigma)^{-1}\mathbf{y}]^{1/2} = \|\ A^{-1}\mathbf{y}\ \| = 1.$$

Hence, along a subsequence of $\{LLn\mathbf{y}'\Sigma^{-1}\mathbf{M}_n\}$ and for n large enough, $LLn\mathbf{y}'\Sigma^{-1}\mathbf{M}_n \ge 1 - \varepsilon$, and in this case we get

$$
\begin{aligned}
\|\ LLnA^{-1}\mathbf{M}_n - A^{-1}\mathbf{y}\ \|^2 &= \|\ LLnA^{-1}\mathbf{M}_n\ \|^2 + \|\ A^{-1}\mathbf{y}\ \|^2 - 2LLn\mathbf{y}'\Sigma^{-1}\mathbf{M}_n \\
&\le 1 + \varepsilon + 1 - 2 - 2\varepsilon \le 3\varepsilon.
\end{aligned}
$$

That means $\liminf_{n\to\infty} \parallel LLn\mathbf{M}_n - \mathbf{y} \parallel \to 0$.

For the case of $\parallel y \parallel = \theta < 1$, define

$$W_n = (LLn(A^{-1}\mathbf{M}_n)', LLn\xi_n)', \qquad n = 3, 4, \cdots,$$

where the iid. random sequence $\{\xi_n\}$ satisfies the conditions of Lemma 2.4. Let $\tilde{\mathbf{y}} = ((A^{-1}\mathbf{y})', (1-\theta^2)^{1/2})$, then $\parallel \tilde{\mathbf{y}} \parallel = 1$. By the preceding step and using Lemma 2.4, we can prove

$$\liminf_{n\to\infty} \parallel W_n - \tilde{\mathbf{y}} \parallel = 0, \qquad \text{a.s.}$$

It follows that $\liminf_{n\to\infty} \parallel LLn\mathbf{M}_n - \mathbf{y} \parallel \to 0$. This completes the proof of Theorem 2.3.

§ 3 . Application of Linear Regression Model

Assume that $(Y, \mathbf{Z})$ satisfies a linear regression model

$$Y = \mathbf{b}'\mathbf{Z} + \varepsilon, \tag{3.1}$$

where $\mathbf{b} = (b_1, \cdots, b_r)'$ is the vector of parameters of interest and the error ε is uncorrelated with $\mathbf{Z}$ with mean 0 and variance σ_ε^2. Let $(Y_i, \mathbf{Z}_i') = (Y_i, Z_{1i}, \cdots, Z_{ir})$ be iid. observations of $(Y, \mathbf{Z}')$.

Under random censorship, only the censored data (1.2) is observed. The WLSE of $\mathbf{b}$ is any vector $\hat{\mathbf{b}} = (\hat{b}_1, \hat{b}_2, \cdots, \hat{b}_r)'$ which minimizes the quadratic form

$$Q(\mathbf{b}) = \sum_{k=1}^{n} \frac{\delta_k}{\overline{G}_n(U_k-)} \left(U_k - b_1 Z_{k1} - \cdots - b_r Z_{kr}\right)^2. \tag{3.2}$$

The WLSE of the variance σ_ε^2 is defined by

$$\hat{\sigma}_\varepsilon^2 = \frac{1}{n} Q(\hat{\mathbf{b}}). \tag{3.3}$$

The censored regression problem has been used and studied by [4–6]. But the LIL result has not been established.

For $k = 0, 1, \cdots, r$, define $Z_{k0} = U_k$,

$$\hat{\mu}_{ij} = \int z_i z_j F_n(dz_0, d\mathbf{z}) = \frac{1}{n} \sum_{k=1}^{n} \frac{Z_{ki} Z_{kj} \delta_k}{1 - G_n(U_k-)}, \qquad 0 \le i, j \le r. \tag{3.4}$$

and $\hat{\mathbf{r}} = (\hat{\mu}_{01}, \cdots, \hat{\mu}_{0r})'$, $\widehat{\mathbf{\Gamma}} = (\hat{\mu}_{ij})_{i,j=1,\cdots,r}$. Write $Q(\mathbf{b})$ in the form of

$$Q(\mathbf{b}) = \sum_{k=1}^{n} \left(\frac{U_k \delta_k}{\sqrt{\overline{G}_n(U_k-)}} - b_1 \frac{Z_{k1}\delta_k}{\sqrt{\overline{G}_n(U_k-)}} - \cdots - b_r \frac{Z_{kr}\delta_k}{\sqrt{\overline{G}_n(U_k-)}} \right)^2.$$

It is seen that the WLSE is such vector $\hat{\mathbf{b}}$ satisfying

$$\hat{\mathbf{r}} = \widehat{\mathbf{\Gamma}}\hat{\mathbf{b}} \quad \text{and} \quad \hat{\sigma}^2 = \mu_{00} - \hat{\mathbf{b}}'\hat{\mathbf{r}}. \tag{3.5}$$

Now define

$$g_j(y, \mathbf{z}) = z_j(y - \mathbf{b}'\mathbf{z}), \qquad h(y, \mathbf{z}) = (y - \mathbf{b}'\mathbf{z})^2,$$

$$\psi_j(s) = \int_{y \ge s} g_j(y, \mathbf{z}) dF(y, \mathbf{z}), \qquad \tilde{h}(s) = \int_{y \ge s} h(y, \mathbf{z}) dF(y, \mathbf{z}).$$

$$\sigma_{i,j} = E\frac{Z_i Z_j \varepsilon^2}{\overline{G}(Y)} - E\frac{\psi_i(C)\psi_j(C)}{\overline{F}(C)\overline{G}^2(C)}, \qquad \sigma_G^2 = E\frac{\varepsilon^4}{\overline{G}(Y)} - \sigma_\varepsilon^4 - E\frac{\tilde{h}^2(C)}{\overline{F}(C)\overline{G}^2(C)}. \tag{3.6}$$

Let $\mathbf{Z} = (Z_1, \cdots, Z_r)'$, $\Gamma = E(\mathbf{Z}\mathbf{Z}')$ and $\Sigma = (\sigma_{ij})_{r\times r}$.

Theorem 3.1. *In the model of* (3.1), *suppose* $\Sigma = AA'$ *and* Γ *are positively definite, and assume the conditions* A *and* B *for* F *and* G.

(I) *If* $E[Z_j^k \varepsilon^2/\overline{G}(Y)] < \infty$, $j = 1, 2, \cdots, r$, $k = 1, 2$, *then with probability one, the sequence* $\{LLn(\hat{\mathbf{b}} - \mathbf{b})\}$ *is relatively compact and the set of its limit points coincides with*

$$K_G = \{\Gamma^{-1} A\mathbf{x} \ ; \quad \| \mathbf{x} \| \le 1, \mathbf{x} \in R^r\}.$$

(II) *If* $E[\varepsilon^{2k}/\overline{G}(Y)] < \infty$, $k = 1, 2$, *then with probability one, the sequence* $\{LLn(\hat{\sigma}^2 - \sigma^2)\}$ *is relatively compact and the set of its limit points coincides with* $[-\sigma_G, \sigma_G]$.

Proof. The conditions entail that for all $i, j = 1, \cdots, r$,

$$\int \frac{|g_i(y,\mathbf{z}) g_j(y,\mathbf{z})|}{1 - G(y)} F(dy, d\mathbf{z}) = E\Big(\frac{|Z_i Z_j| \varepsilon^2}{1 - G(Y)}\Big) < \infty,$$
$$\int \frac{h^k(y,\mathbf{z})}{1 - G(y)} F(dy, d\mathbf{z}) = E\Big(\frac{\varepsilon^{2k}}{1 - G(Y)}\Big) < \infty, \qquad k = 1, 2.$$

Define

$$\mathbf{g}(y,\mathbf{z}) = (g_1(y,\mathbf{z}), \cdots, g_r(y,\mathbf{z}))'. \tag{3.7}$$

Applying Theorem 2.3 and for n large enough, we obtain

$$\begin{aligned}\hat{\mathbf{b}} - \mathbf{b} &= \widehat{\Gamma}^{-1}\hat{\mathbf{r}} - \Gamma^{-1}\mathbf{r} \\ &= \widehat{\Gamma}^{-1}(\Gamma - \widehat{\Gamma})\Gamma^{-1}\hat{\mathbf{r}} + \Gamma^{-1}(\hat{\mathbf{r}} - \mathbf{r}) \\ &= \Gamma^{-1}[(\Gamma - \widehat{\Gamma})\mathbf{b} + \hat{\mathbf{r}} - \mathbf{r}] + o(LLn) \\ &= \Gamma^{-1}\Big(\int \mathbf{g}(y,\mathbf{z})(F_n(dy,d\mathbf{z}) - F(dy,d\mathbf{z}))\Big) + o(LLn), \qquad \text{a.s.}\end{aligned}$$

and similarly

$$\hat{\sigma}^2 - \sigma_\varepsilon^2 = \hat{\mu}_{00} - \mu_{00} + \mathbf{b}'\mathbf{r} - \hat{\mathbf{b}}'\hat{\mathbf{r}} = \int h(y,\mathbf{z})(F_n(dy,d\mathbf{z}) - F(dy,d\mathbf{z})) + o(LLn), \qquad \text{a.s.}$$

Now the results follow from Theorem 2.3 and

$$\int g_j(y,\mathbf{z}) F(dy, d\mathbf{z}) = E[Z_j(Y - \mathbf{b}'\mathbf{Z})] = E[Z_j \varepsilon] = 0.$$

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