Chin. Ann. Math.
25B:2(2004),207–212.

ON THE TOPOLOGY, VOLUME, DIAMETER AND GAUSS MAP IMAGE OF SUBMANIFOLDS IN A SPHERE**

WU Bingye*

Abstract

In this paper, the author uses Gauss map to study the topology, volume and diameter of submanifolds in a sphere. It is proved that if there exist ε, $1 \geq \varepsilon > 0$ and a fixed unit simple p-vector a such that the Gauss map g of an n-dimensional complete and connected submanifold M in S^{n+p} satisfies $\langle g, a\rangle \geq \varepsilon$, then M is diffeomorphic to S^n, and the volume and diameter of M satisfy $\varepsilon^n \mathrm{vol}(S^n) \leq \mathrm{vol}(M) \leq \mathrm{vol}(S^n)/\varepsilon$ and $\varepsilon\pi \leq \mathrm{diam}(M) \leq \pi/\varepsilon$, respectively. The author also characterizes the case where these inequalities become equalities. As an application, a differential sphere theorem for compact submanifolds in a sphere is obtained.

Keywords Gauss map, Volume, Diameter, Differential sphere theorem
2000 MR Subject Classification 53C42, 53B30

§ 1. The Main Results

The Gauss map of minimal surfaces or submanifolds with parallel mean curvature vector in spheres has been extensively studied. The basic idea is to obtain the rigidity results by sufficiently restricting the size of the image of Gauss map (see e.g., [1, 2]).

In this paper we shall use Gauss map to study the topology, volume and diameter of submanifolds in spheres. As is well known, there are different definitions for Gauss map. In this paper, the Gauss map is defined by the normal bundle of submanifolds. Our first result is the following

Theorem 1.1. *Let $\psi : M \to S^{n+p} \subset R^{n+p+1}$ be an isometric immersion of an n-dimensional complete and connected Riemannian manifold into unit $(n+p)$-sphere S^{n+p}. If there exist ε, $1 \geq \varepsilon > 0$ and a fixed unit simple p-vector a in $\wedge^p(R^{n+p+1})$ such that the Gauss map g of M satisfies $\langle g, a\rangle \geq \varepsilon$, then M is diffeomorphic to S^n, and the volume and diameter of M satisfy*

$$\varepsilon^n \mathrm{vol}(S^n) \leq \mathrm{vol}(M) \leq \mathrm{vol}(S^n)/\varepsilon \tag{1.1}$$

and

$$\varepsilon\pi \leq \mathrm{diam}(M) \leq \pi/\varepsilon, \tag{1.2}$$

Manuscript received March 13, 2003. Revised July 26, 2003.
*Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China.
E-mail: wby@mail.zjnu.net.cn
**Project supported by the Fund of the Education Department of Zhejiang Province of China (No.20030707).

respectively. Moreover, $\mathrm{vol}(M) = \varepsilon^n \mathrm{vol}(S^n)$ *or* $\mathrm{diam}(M) = \varepsilon\pi$ *holds if and only if* $\psi(M)$ *is a totally umbilical n-sphere with radius* ε*, while* $\mathrm{vol}(M) = \mathrm{vol}(S^n)/\varepsilon$ *or* $\mathrm{diam}(M) = \pi/\varepsilon$ *holds if and only if* $\varepsilon = 1$*, and* $\psi(M)$ *is a totally geodesic n-sphere.*

The second result of this paper is a differential sphere theorem from the viewpoint of submanifold geometry for compact spherial submanifolds which is established by means of Gauss map. We note that some topological sphere theorem have been obtained from the point of view of submanifold geometry (see [3–5]). In order to state our result, let us first introduce a notation. Let $x_0 \in M$ be a point in M, and $C : M \to R$ a function on M. We say that $\mathrm{Ric}_{x_0}^{\min} > C$ if the Ricci curvature of M satisfies

$$\mathrm{Ric}(\gamma'(t)) > C(\gamma(t))$$

for any minimal normal geodesic $\gamma : [a, b] \to M$ starting from x_0. Our differential sphere theorem is

Theorem 1.2. *Let* $\psi : M \to S^{n+p}$ *be an isometric immersion of an n-dimensional compact Riemannian manifold M into* S^{n+p}*. Assume that for some* $x_0 \in M$*, the Ricci curvature of M satisfies*

$$\mathrm{Ric}_{x_0}^{\min} > n - 1 + \frac{n\pi|H|}{2d} - \frac{\pi^2}{4d^2} \tag{1.3}$$

and $n|H| < \pi/d$*, where H is the mean curvature vector of M and* $d = \mathrm{diam}(M)$*. Then M is diffeomorphic to* S^n*.*

§ 2. The Gauss Map

Let $\psi : M \to S^{n+p} \subset R^{n+p+1}$ be an isometric immersion of an n-dimensional Riemannian manifold into unit $(n + p)$-sphere S^{n+p}, and $e_1, \cdots, e_{n+p+1}$ the local frame field of R^{n+p+1} such that, when restricted on M, $e_1, \cdots, e_n$ are tangent to M and $e_{n+1} = \psi$ is the position vector of M. The Gauss map g of M assigns each point $x \in M$ the oriented normal space of M in S^{n+p} at x, which can be considered as an element of the Grassmannian manifold $G(p, n + p + 1)$ by moving parallelly to the origin of R^{n+p+1}. As is well known, we can regard $G(p, n + p + 1)$ as a compact submanifold of Euclidean space $\wedge^p(R^{n+p+1})$. Locally the Gauss map

$$g : M \to G(p, n + p + 1) \subset \wedge^p(R^{n+p+1})$$

can be written as $g = e_{n+2} \wedge \cdots \wedge e_{n+p+1}$. Since g is a unit simple p-vector, it must lie in the unit hypersphere S^N of $\wedge^p(R^{n+p+1})$, where $N = \begin{pmatrix} n+p+1 \\ p \end{pmatrix} - 1$. It is convenient to consider the Gauss map g as a map from M to S^N:

$$g = e_{n+2} \wedge \cdots \wedge e_{n+p+1} : M \to S^N.$$

For simple vectors $g = e_{n+2} \wedge \cdots \wedge e_{n+p+1}, a = a_{n+2} \wedge \cdots \wedge a_{n+p+1}$, the inner product is defined as usual by $\langle g, a\rangle = \det(\langle e_\alpha, a_\beta\rangle)$. Here and in the sequel we use the following convention on the ranges of indies:

$$1 \le i, j, \cdots \le n + 1; \qquad n + 2 \le \alpha, \beta \le n + p + 1.$$

We say that the image of Gauss map g is contained in an open hemisphere if there exists a fixed p-vector $a = a_{n+2} \wedge \cdots \wedge a_{n+p+1}$ such that $\langle g, a\rangle > 0$.

Let $* : \wedge^p(R^{n+p+1}) \to \wedge^{n+1}(R^{n+p+1})$ be the de Rham-Hodge star operator on $\wedge^p(R^{n+p+1})$. It is well known that $*$ is an isometric isomorphism between $\wedge^p(R^{n+p+1})$ and $\wedge^{n+1}(R^{n+p+1})$, that is, for any $a, b \in \wedge^p(R^{n+p+1})$, we have $\langle a, b\rangle = \langle *a, *b\rangle$.

§ 3 . The Proof of Theorems

We shall complete the proof of Theorems 1.1 and 1.2 in this section. Let us first prove Theorem 1.1. Suppose that $\psi : M \to S^{n+p} \subset R^{n+p+1}$ be an isometric immersion of an n-dimensional complete and connected Riemannian manifold into S^{n+p} whose Gauss map g satisfies $\langle g, a\rangle \geq \varepsilon$ for some ε, $1 \geq \varepsilon > 0$ and a fixed unit simple p-vector $a = a_{n+2} \wedge \cdots \wedge a_{n+p+1} \in \wedge^p(R^{n+p+1})$. We extend $a_{n+2}, \cdots, a_{n+p+1}$ to an orthonormal basis $a_1, \cdots, a_{n+p+1}$ of R^{n+p+1}, and define the projection $\Pi : M \to S_a^n$ by

$$\Pi(p) = \frac{1}{\sqrt{\sum\limits_i \langle \psi(p), a_i\rangle^2}} \sum_i \langle \psi(p), a_i\rangle a_i, \qquad \forall p \in M, \tag{3.1}$$

where $S_a^n = \{x \in S^{n+p} : \langle x, a_\alpha\rangle = 0,\ n+2 \leq \alpha \leq n+p+1\}$ is the totally geodesic n-sphere determined by a. A straightforward computation shows that

$$d\Pi(X) = \frac{1}{\Big(\sum\limits_i \langle \psi, a_i\rangle^2\Big)^{3/2}} \sum_{i,j} (\langle \psi, a_j\rangle^2 \langle X, a_i\rangle - \langle \psi, a_j\rangle\langle X, a_j\rangle\langle \psi, a_i\rangle) a_i$$

for every tangent vector field X of M, and consequently

$$\begin{aligned} |d\Pi(X)|^2 &= \frac{1}{\Big(\sum\limits_i \langle \psi, a_i\rangle^2\Big)^2} \sum_{i,j} (\langle \psi, a_i\rangle^2 \langle X, a_j\rangle^2 - \langle \psi, a_i\rangle\langle X, a_i\rangle\langle \psi, a_j\rangle\langle X, a_j\rangle) \\ &= \frac{1}{\Big(\sum\limits_i \langle \psi, a_i\rangle^2\Big)^2} \sum_{i<j} \langle X \wedge \psi, a_i \wedge a_j\rangle^2. \end{aligned} \tag{3.2}$$

Since $\langle g, a\rangle \geq \varepsilon$, we have

$$\begin{aligned} \varepsilon \leq \langle g, a\rangle &= \langle *g, *a\rangle = \langle e_1 \wedge \cdots \wedge e_n \wedge \psi, a_1 \wedge \cdots \wedge a_{n+1}\rangle \\ &= \sum_i (-1)^{n+1+i} \langle \psi, a_i\rangle \langle e_1 \wedge \cdots \wedge e_n, a_1 \wedge \cdots \hat{a}_i \wedge \cdots \wedge a_{n+1}\rangle \\ &\leq \sqrt{\sum_i \langle \psi, a_i\rangle^2 \sum_i \langle e_1 \wedge \cdots \wedge e_n, a_1 \wedge \cdots \hat{a}_i \wedge \cdots \wedge a_{n+1}\rangle^2} \\ &\leq \sqrt{\sum_i \langle \psi, a_i\rangle^2}, \end{aligned}$$

where $\hat{a}_i$ denotes removing the factor a_i. Hence

$$\sum_i \langle \psi, a_i\rangle^2 \geq \langle g, a\rangle^2 \geq \varepsilon^2. \tag{3.3}$$

On the other hand, by the Laplace expansion theorem for determinant, we have

$$
\begin{aligned}
\varepsilon &\le \langle e_1\wedge\cdots\wedge e_n\wedge\psi, a_1\wedge\cdots\wedge a_{n+1}\rangle\\
&= \sum_{i<j}(-1)^{i+j+1}\langle e_n\wedge\psi, a_i\wedge a_j\rangle\langle e_1\wedge\cdots\wedge e_{n-1}, a_1\wedge\cdots\hat{a}_i\wedge\cdots\wedge\hat{a}_j\wedge\cdots\wedge a_{n+1}\rangle\\
&\le \sqrt{\sum_{i<j}\langle e_n\wedge\psi, a_i\wedge a_j\rangle^2\sum_{i<j}\langle e_1\wedge\cdots\wedge e_{n-1}, a_1\wedge\cdots\hat{a}_i\wedge\cdots\wedge\hat{a}_j\wedge\cdots\wedge a_{n+1}\rangle^2}\\
&\le \sqrt{\sum_{i<j}\langle e_n\wedge\psi, a_i\wedge a_j\rangle^2}. \qquad (3.4)
\end{aligned}
$$

Now we want to estimate $|d\Pi(X)|^2$. Without loss of generality, we can assume that $X = |X|e_n$. Then by (3.4) we have $\sum_{i<j}\langle X\wedge\psi, a_i\wedge a_j\rangle^2 \ge \varepsilon^2|X|^2$, which together with (3.2) and (3.3) yields

$$\varepsilon^2|X|^2 \le |d\Pi(X)|^2 \le |X|^2/\varepsilon^2. \qquad (3.5)$$

It follows from (3.5) that Π is a local diffeomorphism. Since $\langle\,,\,\rangle$ is a complete Riemannian metric on M, the same holds for the homothetic metric $\widetilde{\langle\,,\,\rangle} = \varepsilon^2\langle\,,\,\rangle$. Then, (3.5) means that the map $\Pi : (M, \widetilde{\langle\,,\,\rangle}) \to (S^n_a, \langle\,,\,\rangle)$ increases the distance. If a map, from a connected complete Riemannian manifold M_1 into another Riemannian manifold M_2 of the same dimension, increases the distance, then it is a covering map and M_2 is complete [6, Chapter VIII, Lemma 8.1]. Hence Π is a covering map, but S^n_a being simply connected this means that Π is in fact a global diffeomorphism between M and S^n_a. Hence, M is diffeomorphic to S^n.

Now we want to prove (1.2). Let $x, y \in S^n_a$ be antipodal points such that $\mathrm{dist}_{S^n_a}(x, y) = \mathrm{diam}(S^n_a) = \pi$, and $x' = \Pi^{-1}(x) \in M$, $y' = \Pi^{-1}(y) \in M$. Suppose that $\gamma : [a, b] \to M$ is the minimal geodesic joining x' and y'. Then from (3.5), we have

$$
\begin{aligned}
\pi = \mathrm{diam}(S^n_a) &\le L(\Pi\circ\gamma) = \int_a^b |d\Pi(\gamma'(t))|dt\\
&\le \frac{1}{\varepsilon}\int_a^b |\gamma'(t)|dt = \frac{1}{\varepsilon}L(\gamma) \le \frac{1}{\varepsilon}\mathrm{diam}(M). \qquad (3.6)
\end{aligned}
$$

Hence, we have $\mathrm{diam}(M) \ge \varepsilon\pi$, and when $\mathrm{diam}(M) = \varepsilon\pi$, (3.3) becomes an equality. Then it follows from the proof of (3.3) that $\sum_i\langle e_1\wedge\cdots\wedge e_n, a_1\wedge\cdots\hat{a}_i\wedge\cdots\wedge a_{n+1}\rangle^2 = 1$, which means that constant vectors $a_{n+2}, \cdots, a_{n+p+1}$ are normal to M. It is clear that in this case $\psi(M)$ is a totally umbilical n-sphere with radius ε. Conversely, if $\psi(M)$ is a totally umbilical n-sphere with radius ε, we certainly have $\mathrm{diam}(M) = \varepsilon\pi$. On the other hand, assume that $x_0, y_0 \in M$ be two points so that $\mathrm{dist}_M(x_0, y_0)=\mathrm{diam}M$, and $x'_0 = \Pi(x_0) \in S^n_a, y'_0 = \Pi(y_0) \in S^n_a$. Let $\alpha : [a, b] \to S^n_a$ be the minimal geodesic connecting x'_0 and y'_0. Then

$$\mathrm{diam}(M) \le L(\Pi^{-1}\circ\alpha) = \int_a^b |d\Pi^{-1}(\alpha'(t))|dt \le \frac{1}{\varepsilon}\int_a^b |\alpha'(t)|dt = \frac{1}{\varepsilon}L(\alpha) \le \frac{\pi}{\varepsilon}, \qquad (3.7)$$

and it is easy to see that all inequalities become equalities if and only if $\varepsilon = 1$, and $\psi(M)$ is the totally geodesic n-sphere. Thus (1.2) is proved.

As for the inequalities (1.1), using the diffeomorphism $\Pi : M \to S^n_a$ we know that

$$\mathrm{vol}(S^n) = \int_{S^n_a} dS = \int_M \Pi^*(dS), \qquad (3.8)$$

where dS stands for the volume element of S^n_a. Note that (3.1) can also be written as

$$\Pi(p) = \frac{1}{\sqrt{1-\sum_\alpha \langle \psi(p), a_\alpha\rangle^2}}\Big(\psi(p) - \sum_\alpha \langle \psi(p), a_\alpha\rangle a_\alpha\Big), \qquad \forall\, p \in M, \tag{3.9}$$

and consequently

$$d\Pi(X) = \frac{1}{\sqrt{1-\sum_\alpha \langle \psi, a_\alpha\rangle^2}} X + \sum_\alpha (\ )_\alpha a_\alpha + (\ )\psi \tag{3.10}$$

for any tangent vector field X of M. By using (3.9) and (3.10), it follows that

$$\begin{aligned}
\Pi^*(dS)(X_1, \cdots, X_n) &= \det(d\Pi(X_1), \cdots, d\Pi(X_n), a_{n+2}, \cdots, a_{n+p+1}, \Pi)\\
&= \frac{1}{\Big(1-\sum_\alpha \langle \psi, a_\alpha\rangle^2\Big)^{(n+1)/2}} \det(X_1, \cdots, X_n, a_{n+2}, \cdots, a_{n+p+1}, \psi)\\
&= \frac{\langle g, a\rangle}{\Big(\sum_i \langle \psi, a_i\rangle^2\Big)^{(n+1)/2}} \det(X_1, \cdots, X_n, e_{n+2}, \cdots, e_{n+p+1}, \psi)\\
&= \frac{\langle g, a\rangle}{\Big(\sum_i \langle \psi, a_i\rangle^2\Big)^{(n+1)/2}} dM(X_1, \cdots, X_n)
\end{aligned}$$

for tangent vector fields $X_1, \cdots, X_n$ of M, where dM is the volume element of M. In other words,

$$\Pi^*(dS) = \frac{\langle g, a\rangle}{\Big(\sum_i \langle \psi, a_i\rangle^2\Big)^{(n+1)/2}} dM. \tag{3.11}$$

From (3.3), (3.8) and (3.11) we see that

$$\mathrm{vol}(S^n) = \int_M \frac{\langle g, a\rangle}{\Big(\sum_i \langle \psi, a_i\rangle^2\Big)^{(n+1)/2}} dM \le \int_M \frac{1}{\langle g, a\rangle^n} dM \le \int_M \frac{1}{\varepsilon^n} dM \le \frac{1}{\varepsilon^n}\mathrm{vol}(M),$$

and so $\mathrm{vol}(M) \ge \varepsilon^n \mathrm{vol}(S^n)$. We can argue as above to deduce out that $\mathrm{vol}(M) = \varepsilon^n \mathrm{vol}(S^n)$ if and only if $\psi(M)$ is a totally umbilical n-sphere with radius ε. Similarly

$$\mathrm{vol}(S^n) \ge \int_M \langle g, a\rangle dM \ge \varepsilon \mathrm{vol}(M)$$

and all inequalities become equalities if and only if $\varepsilon = 1$, and $\psi(M)$ is the totally geodesic n-sphere. Therefore, (1.1) is proved, and we complete the proof of Theorem 1.1.

In the following, let us prove Theorem 1.2. It is easy to know from Theorem 1.1 that if the Gauss map g of M satisfies $\langle g, a\rangle > 0$, then M is diffeomorphic to S^n. This is equivalent to say that when the image of the Gauss map $g : M \to S^N$ is contained in a geodesic ball of S^N with radius $\rho < \pi/2$, then M is diffeomorphic to S^n. We will prove Theorem 1.2 by showing that under the assumption of Theorem 1.2, the image of Gauss map must be contained in a geodesic ball of S^N with radius $\rho < \pi/2$. Let $y \in M$ be an arbitrary point, and $\gamma : [a, b] \to M$ be the minimal normal geodesic joining x_0 and y. Then $g \circ \gamma : [a, b] \to S^N$ is a curve in S^N connecting $g(x_0)$ and $g(y)$. The length of $g \circ \gamma$ is

$$L = \int_a^b \Big| \frac{d}{dt} g \circ \gamma(t) \Big| dt. \tag{3.12}$$

We have

$$\begin{aligned} \frac{d}{dt} g \circ \gamma(t) &= \frac{d}{dt}(e_{n+2}(\gamma(t)) \wedge \cdots \wedge e_{n+p+1}(\gamma(t))) \\ &= D_{\gamma'(t)} e_{n+2} \wedge e_{n+3} \wedge \cdots \wedge e_{n+p+1} + \cdots + e_{n+2} \wedge \cdots \wedge e_{n+p} \wedge D_{\gamma'(t)} e_{n+p+1} \\ &= -A_{n+2}(\gamma'(t)) \wedge e_{n+3} \wedge \cdots \wedge e_{n+p+1} - \cdots - e_{n+2} \wedge \cdots \wedge e_{n+p} \wedge A_{n+p+1}(\gamma'(t)), \end{aligned}$$

where D is the standard connection on R^{n+p+1} and A_α is the Weingarten transformation on TM with respect to e_α. Therefore we get

$$\Big| \frac{d}{dt} g \circ \gamma(t) \Big|^2 = \sum_\alpha |A_\alpha(\gamma'(t))|^2. \tag{3.13}$$

On the other hand, from the Gauss equation it is clear that

$$\begin{aligned} \mathrm{Ric}(\gamma'(t)) &= n - 1 + \sum_\alpha \mathrm{tr}(A_\alpha) \langle A_\alpha(\gamma'(t)), \gamma'(t) \rangle - \sum_\alpha |A_\alpha(\gamma'(t))|^2 \\ &\leq n - 1 + n|H| \sqrt{\sum_\alpha |A_\alpha(\gamma'(t))|^2} - \sum_\alpha |A_\alpha(\gamma'(t))|^2, \end{aligned}$$

and so,

$$\Big| \frac{d}{dt} g \circ \gamma(t) \Big| = \sqrt{\sum_\alpha |A_\alpha(\gamma'(t))|^2} \leq \frac{n|H| + \sqrt{n^2|H|^2 - 4\mathrm{Ric}(\gamma'(t)) + 4(n-1)}}{2}. \tag{3.14}$$

Combining (1.3), (3.12) and (3.14) together with the assumption that $n|H| < \pi/d$, we finally get

$$L \leq \int_a^b \frac{n|H| + \sqrt{n^2|H|^2 - 4\mathrm{Ric}(\gamma'(t)) + 4(n-1)}}{2} dt < \int_a^b \frac{\pi}{2d} dt \leq \frac{\pi}{2d} \cdot d = \frac{\pi}{2}. \tag{3.15}$$

Since $y \in M$ is arbitrary, (3.15) means that the Gauss map image of M lies in a geodesic ball of S^N centered at $g(x_0)$ with radius $< \pi/2$, so we are done.

References

[1] Fischer-Colbrie, D., Some rigidity theorems for minimal submanifolds of the sphere, *Acta Math.,* **145**(1980), 29–46.

[2] Wang, C. P., On the Gauss Map of Submanifolds in R^n and S^n, Lecture Notes in Math., **1255**, 1987, 109–129.

[3] Shiohama, K. & Xu, H., The topological sphere theorem for complete submanifolds, *Compositio Math.,* **107**(1997), 221–232.

[4] Hasanis, T. & Vlachos T., Ricci curvature and minimal submanifolds, *Pacific J. Math.,* **197**(2001), 13–24.

[5] Vlachos, T., A sphere theorem for odd-dimensional submanifolds of spheres, *Proc. Amer. Math. Soc.,* **130**(2002), 13–24.

[6] Kobayashi, S. & Nomizu K., Foundations of Differential Geometry, Vol. II, Interscience, New York, 1969.